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THE EULER CHARACTERISTIC OF THE IMAGE OF A STABLE MAPPING FROM A CLOSED n-MANIFOLD TO A (2n-1)-MANIFOLD.

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1. Introduction

One of the themes in the global theory of singularities of mappings $f : N \to P$ between manifolds is to study the relationship between the topology of N, P and $f(N)$ in the case when $\dim N < \dim P$ ([3]).

Recently, there appeared a considerable progress in the local theory of singularities of mappings ([4],[5],[6],[7]) mainly due to the work of David Mond. In [4] a method has been introduced to compute the Euler characteristic of the image of a stable perturbation of an $\mathcal{A}$-finite map-germ. Here we shall apply this method to compute the Euler characteristic of the image of a stable mapping from a closed n-manifold to (2n-1)-manifold. As an application of our theorem, we determine the set of values of the Euler characteristic of the image of stable mappings from a closed n-manifold to a (2n-1)-manifold.

All mappings considered here are differentiable class C^∞, unless stated otherwise.

2. The main result

It is well known that a mapping $f : N \to P$ from an n-manifold to a $(2n-1)$-manifold is stable if and only if it is an immersion with normal crossings except at the isolated singularities of cross-caps ([8], fig.1). It follows that the number of cross-caps is finite and we denote it by $C(f)$. There also exist finitely many three-to-one points in $f(N)$ where three sheets of regular images meet in general positions. Such a point (fig.2) is called a triple point of f and the number of triple points is denoted by $T(f)$.

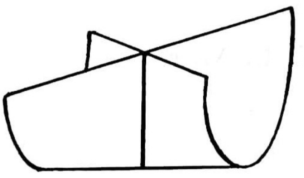

fig.1

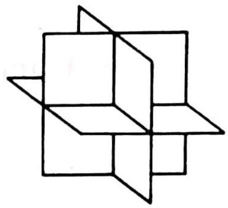

fig.2

We denote the Euler characteristic of a topological space X by $\chi(X)$. Our main result is the following:

Theorem 1 *(i)* $\chi(f(N)) = \chi(N) + T(f) + C(f)/2,$ *if* $n = 2$.

(ii) $\chi(f(N)) = \chi(N) + C(f)/2,$ *if* $n \geq 3$.

Proof: (i) Let us consider the following sets:

$D^2(f) = cl\{x \in N | \# f^{-1}f(x) \geq 2\},$

$D^3(f) = \{x \in D^2(f) | \# f^{-1}f(x) = 3\}$ and

$D^2(f,(2)) = \{x \in D^2(f) | \# f^{-1}f(x) = 1\},$

where clX denotes the topological closure of X.

By the characterization of stable mappings ([8]), $D^2(f)$ is a union of closed curves on the n-manifold N whose set of self-intersection is $D^3(f)$, which is the inverse image of triple points, and $D^2(f,(2))$ is the set of cross-cap points of f. It follows that these are immersed submanifolds of N with dim $D^2(f) = 1$ and dim $D^3(f) =$ dim $D^2(f,(2)) = 0$, if not empty.

In order to prove the theorem, we consider the following problem: find real numbers α, β, γ and δ such that

$$\chi(f(N)) = \alpha\chi(N) + \beta\chi(D^2(f)) + \gamma\chi(D^2(f,(2)) + \delta\chi(D^3(f)). \tag{1.1}$$

We shall solve this by a purely combinatorial method.

Initially we construct a triangulation K_f of the set $f(N)$ as follows: we start to triangulate $f(N)$ by including the image of $D^2(f,(2))$ and the image of $D^3(f)$ among the vertices of K_f. After this, we build up the one-skeleton $K_f^{(1)}$ of K_f so that the image of $D^2(f)$ is a subcomplex of $K_f^{(1)}$. We complete our procedure by constructing the 2-skeleton $K_f^{(2)}$ of K_f.

Since f and its restrictions to $D^2(f), D^2(f,(2))$ and $D^3(f)$ are proper and finite-to-one mappings, then we can pull back K_f to obtain triangulations for N, $D^2(f)$, $D^2(f,(2))$ and $D^3(f)$ respectively. Let C_i^X be the number of i-cells in X, where $X = f(N)$, N, $D^2(f)$, $D^2(f,(2))$ or $D^3(f)$. Then the equation (1.1) can be written as $\sum_i(-1)^i C_i^{f(N)} = \alpha\sum_i(-1)^i C_i^N + \beta\sum_i(-1)^i C_i^{D^2(f)} + \gamma\sum_i(-1)^i C_i^{D^2(f,(2))} + \delta\sum_i(-1)^i C_i^{D^3(f)}$, where $C_i^X = 0$ if $i > \dim X$. So, if we can find real numbers α, β, γ and δ such that

$$C_i^{f(N)} = \alpha C_i^N + \beta C_i^{D^2(f)} + \gamma C_i^{D^2(f,(2))} + \delta C_i^{D^3(f)} \tag{1.2}$$

for any i, then we have an answer for the problem. By the construction of the triangulation, we may concentrate on solving (1.2) in the case when $i = 0$. We remark that f is 3 to 1 over the points in the image of $D^3(f)$, 1 to 1 over the points in the image of $D^2(f,(2))$, 2 to 1 over the points in the image of $D^2(f) - (D^2(f,(2)) \cup D^3(f))$, and 1 to 1 over the points in the image of $N - D^2(f)$. It follows that the equation

$$C_0^{f(N)} = \alpha C_0^N + \beta C_0^{D^2(f)} + \gamma C_0^{D^2(f,(2))} + \delta C_0^{D^3(f)}$$

is equivalent to the system of linear equations:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 3 & 3 & 0 & 3 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}.$$

Then we have the solutions $\alpha = 1$, $\beta = -1/2$, $\gamma = 1/2$ and $\delta = -1/6$ so that

$$\chi(f(N)) = \chi(N) - \chi(D^2(f))/2 + \chi(D^2(f,(2)))/2 - \chi(D^3(f))/6. \qquad (1.3)$$

By definition, $\chi(D^2(f,(2)) = C(f)$ and $\chi(D^3(f)) = 3T(f)$. Since $D^2(f)$ is a union of closed curves on the surface N with $3T(f)$ crossings and circles, then we can triangulate it with $3T(f) + n$ 0-cells and $6T(f) + n$ 1-cells, where n is the number of circles. It follows that $\chi(D^2(f)) = -3T(f)$. Finally, substituting these on the equation (1.3), we get

$$\chi(f(N)) = \chi(N) + T(f) + C(f)/2.$$

This completes the proof of (i).
(ii) When $n \geq 3$ then $D^k(f) = \emptyset$, for any $k \geq 3$. So, following the same arguments as above we get

$$\chi(f(N)) = \chi(N) + C(f)/2.$$

3. An application

In this section we shall determine the set of values of the Euler characteristic of the image of stable mappings from a connected closed n-manifold to a $(2n-1)$-manifold as an application of the theorem.

We now define $\chi(N,P) = \{\chi(f(N)) \mid f : N \to P \text{ is stable }\}$. Then we have the following:

Proposition 2 *(1) Suppose that $n = 2$.*
(i) If N is not homeomorphic to the connected sum of a projective plane and an orientable surface, then

$$\chi(N,P) = \{n \in \mathbb{Z} | n \geq \chi(N)\}.$$

(ii) If N is homeomorphic to the connected sum of a projetive plane and an orientable surface, then

$$\chi(N,P) = \{n \in \mathbb{Z} | n \geq \chi(N) + 1\}.$$

(2) Suppose that $n \geq 3$, then

$$\chi(N, P) = \{n \in \mathbb{Z} | n \geq \chi(N)\}.$$

Proof: (1) (i) In this case we can always construct an immersion $f : N \to P$ with normal crossings without triple points. Then we have $\chi(f(N)) = \chi(N)$. We now define a stable mapping $g : D \to P$ by $g(x, y) = (x, y^2, yx^2 + y^3 - r^2 y)$ in suitable local coordinates, where D is a disc centred at the origin of $\mathbb{R}^2$ and r is any positive number smaller than the radius of D. Then g has two cross-caps (fig.3).

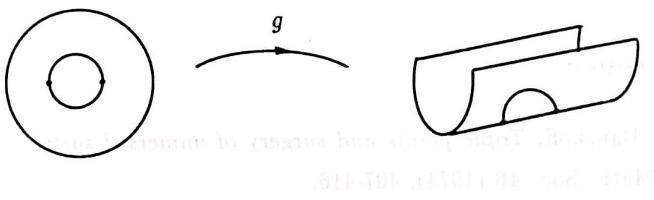

fig.3

If we consider the connected sum of f and g, then we obtain a stable mapping $f\#g : N \to P$ with $C(f\#g) = 2$ and $T(f\#g) = 0$. It follows that $\chi(f\#g(N)) = \chi(N) + 1$. By this procedure, we can construct a stable mapping $h : N \to P$ such that $\chi(h(N)) = n$, for any $n \geq \chi(N)$.

(ii) It is enough to consider the case when $N = \mathbb{P}^2$. In this case we cannot construct an immersion with normal crossings without triple points ([1]). If we consider $f(\mathbb{P}^2)$ as the Boy surface, then the number of triple points is 1 ([2]) and $\chi(f(\mathbb{P}^2)) = \chi(\mathbb{P}^2) + 1$. Now, by the same procedure as that of (i) above, we can get the result.

(2) By the immersion theorem ([8]), we have an immersion with normal crossings $f : N \to P$. Since $n \geq 3$, then f has no triple points. Then, if we use the

mapping

$$g: D^n \to P;\; g(x_1, ..., x_n) = (x_1, x_2^2, x_3, ...x_n, (x_1^2 + x_2^2 - r)x_2, x_1x_3, ..., x_1x_n)$$

in suitable local coordinates as in (1) (ii) above, we can complete the proof.

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