*Matemática Contemporânea, Vol. 5, 29–51*
http://doi.org/10.21711/231766361993/rmc53


# STABILITY AND GENERICITY OF COMPOSITION OF MAPS

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**Abstract**

With the purpose of obtaining a unified viewpoint on some of the applications of singularity theory, we consider compositions of smooth maps $X \xrightarrow{g} Y \xrightarrow{f} Z$, and study the relation between their stability and that of $g$ (resp. $f$) when $f$ (resp. $g$) is fixed (under convenient restrictions on the maps $f$ and $g$). We also study the analogous problem when one of the maps is substituted by a family of smooth maps.

## 1 Introduction

In various contexts of the applications of the singularity theory, eg. stability of caustics [6], Generic Geometry [7], [8], weak transversality [2], stability of the cut-locus in a Riemannian manifold [3], one has to analyze a situation which could be generalized as follows:

Given a composition of $C^\infty$-maps

$$X \xrightarrow{g} Y \xrightarrow{f} Z$$

find the relations between the $\mathcal{G}$-stability ($\mathcal{G}$- genericity) of the map $f \circ g$ and the $\mathcal{G}'$-stability ($\mathcal{G}'$- genericity) of the map $g$ when $f$ is fixed, or viceversa, of the map $f$ when $g$ is fixed. Here $\mathcal{G}$ and $\mathcal{G}'$ are appropriate diffeomorphism groups, which for most of the applications purposes we may take as subgroups of $\mathcal{A}$ or $\mathcal{K}$.

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1990 Mathematics Subject Classification Primary 58C27.

†Work partially supported by CNPq and CAPES.

Looking at the literature, we see that for each particular case, the group $\mathcal{G}$ is well defined, usually $\mathcal{A}(X,Z)$ as in [7], [11] or $\mathcal{K}(X,Z)$ as in [2], [8]. Whereas $\mathcal{G}'$ is not explicity given, but just suggested either by a concept of stability, or by a constructive process in the proofs in which the candidates for the orbits of $j^k\mathcal{G}'$ in $J^k(X,Y)$ or in $J^k(Y,Z)$ are sketched ([2],[8]). Indeed, the knowledge of $\mathcal{G}'$ is not needed in any explicit way for carrying out the proofs of most of the stability and genericity theorems of the kind we refer to. Nevertheless, we think that the introduction of such a group may give a more enlightening and unifying viewpoint on various of the results already known, apart from introducing some other new results that will be presented in this paper.

We shall define a group $\mathcal{G}'$ which will depend on $\mathcal{G}$ and $f$ or $g$ in each case, so it will be denoted by $\mathcal{G}_f$ or $\mathcal{G}_g$ respectively. This $\mathcal{G}_f$ (resp. $\mathcal{G}_g$) will be expected to be such that, for sufficiently well behaved fixed $f$ (resp. $g$), for instance a submersion (resp. an immersion), $\mathcal{G}_f$-stability of $g$ (resp. $\mathcal{G}_g$-stability of $f$) will imply $\mathcal{G}$-stability of $f \circ g$.

We shall consider the problem of stability in Section 1, and in Section 2 the problem of versality of the families obtained by composing a single map with a family of maps.

**Conventions, notations and basic facts:**

We impose here some general restrictions to the problem in consideration:

a) $X$ is a compact manifold;

b) $\dim X,\ \dim Z < \dim Y$ ;

c) the group $\mathcal{G}$ is either $\mathcal{A}(X,Z)$ or $\mathcal{K}(X,Z)$, although one expects that the obtained results can be adapted to others subgroups of $\mathcal{K}(X,Z)$.

All the maps and manifolds we consider in this work are smooth. The topologies on the function spaces are the corresponding Whitney $C^\infty$-topologies.

We denote by $\mathrm{Subm}^{\infty}(Y,Z)$ the subspace of all smooth submersions from $Y$ to $Z$ and by $\mathrm{Emb}^{\infty}(X,Y)$ that of all smooth embeddings from $X$ to $Y$.

We denote by $\mathcal{R}(M)$ the group of diffeomorphisms on a manifold $M$. Given $f \in C^{\infty}(X,Y)$, we understand by $\mathrm{Iso}(f)$ the isotropy subgroup of $f$ in $\mathcal{A}(X,Y) = \mathcal{R}(X)\times\mathcal{R}(Y)$ that is, $\mathrm{Iso}(f) = \{(h,h') \in \mathcal{A}(X,Y) : h' \circ f \circ h^{-1} = f\}$. By abuse of language we also call $\mathrm{Iso}(f)$ the subgroups $\{h \in \mathcal{R}(X) : \exists h' \in \mathcal{R}(Y)$ with $h' \circ f \circ h^{-1} = f\}$ or $\{h' \in \mathcal{R}(Y) : \exists h \in \mathcal{R}(X)$ with $h' \circ f \circ h^{-1} = f\}$ when no confusion can arise.

Given any group $\mathcal{G}$ of diffeomorphisms acting on a space $C^{\infty}(X,Y)$, we have induced group actions of the groups of germs of diffeomorphisms of $\mathcal{G}$ at appropriate points on the set $\mathcal{E}_{x,y}(X,Y)$ of germs of maps in $C^{\infty}(X,Y)$ at some point $x$ with fixed target $y$. The associated equivalence relations both among maps in $C^{\infty}(X,Y)$ and among germs in $\mathcal{E}_{x,y}(X,Y)$ will be denoted by the same symbol $\sim_{\mathcal{G}}$. Given $g, g' \in C^{\infty}(X,Y)$ we shall denote by $g_x$ the germ of $g$ at $x \in X$, and by $g'_{x'}$ , the germ of $g'$ at $x' \in X$. Then we shall write $g_x \sim_{\mathcal{G}} g'_{x'}$ , whenever there are manifold charts: $(\phi, U)$ for $X$ at $x$, $(\phi', U')$ for $X$ at $x'$, $(\psi, V)$ for $Y$ at $g(x)$ and $(\psi', V')$ for $Y$ at $g'(x')$ such that $\phi(x) = \phi'(x') = 0 \in \mathbf{R}^m$, $\psi(g(x)) = \psi'(g'(x')) = 0 \in \mathbf{R}^n$ and $\psi \circ g \circ \phi^{-1} \sim_{\mathcal{G}} \psi' \circ g \circ \phi'^{-1}$ as germs at 0 of maps from $\mathbf{R}^m$ to $\mathbf{R}^n$.

We also notice that the action of $\mathcal{G}$ on $C^{\infty}(X,Y)$ induces an action of the group $J^k\mathcal{G}$ of $k$-jets of elements of $\mathcal{G}$ on the space $J^k(X,Y)$ that will be denoted by $\sim_{\mathcal{G}}$ too.

## 2 Group actions and stability

Consider a composition of maps

$$\phi : X \xrightarrow{g} Y \xrightarrow{f} Z$$

and a group $\mathcal{G}$ acting on $C^{\infty}(X,Z)$. To say that $\phi = f \circ g$ is $\mathcal{G}$-stable means that the $\mathcal{G}$-orbit of $f \circ g$ is open in $C^{\infty}(X,Z)$. Or in other words, that there is a neighbourhood $\mathcal{N}$ of $\phi = f \circ g$ in $C^{\infty}(X,Z)$ such that $\forall \phi' \in \mathcal{N}$, $\exists \theta \in \mathcal{G}$ with $\phi' = \theta \star \phi$, where $\star$ denotes the action of $\mathcal{G}$ on $C^{\infty}(X,Z)$. This action defines an equivalence relation, $\sim_{\mathcal{G}}$, on $C^\infty(X,Z)$. And this induces in turn, for each fixed $f \in C^\infty(Y,Z)$ another equivalence relation, $\sim_f$ , on $C^\infty(X,Y)$, namely: $g \sim_f g' \Longleftrightarrow f \circ g \sim_{\mathcal{G}} f \circ g'$. The relation $\sim_f$ defines a stability criterion on $C^\infty(X,Y)$ studied by G. Wassermann [11]. It would be interesting to identify this stability criterion with the one associated to some group action on $C^\infty(X,Y)$.

In an analogous way, given a fixed $g \in C^\infty(X,Y)$, we can define a relation $\sim_g$ from $\sim_{\mathcal{G}}$ by requiring that $f \sim_g f' \Longleftrightarrow f \circ g \sim_{\mathcal{G}} f \circ g'$. There is also a stability criterion associated to $\sim_g$ on $C^\infty(Y,Z)$, that we would like to relate to the standard one defined by some group action on $C^\infty(Y,Z)$.

We are thus looking for groups, that we denote by $\mathcal{G}_f$ and $\mathcal{G}_g$ respectively, whose orbits coincide with the equivalence classes of the relations $\sim_f$ and $\sim_g$ respectively. We shall restrict our study to the cases $\mathcal{G} = \mathcal{A}, \mathcal{K}$.

The path that we shall follow consists in defining groups $\mathcal{G}_f$ and $\mathcal{G}_g$ and prove that they provide the correct setting for the local situation, when $g$ is an embedding and $f$ a submersion. For the global situation we shall have to content ourselves with some more restrictive results when $\mathcal{G} = \mathcal{A}$, namely:

$$\text{a)}\quad f \circ g \sim_{\mathcal{G}^*} f \circ g' \Longrightarrow g \sim_{\mathcal{G}_f} g' \Longrightarrow f \circ g \sim_{\mathcal{G}} f \circ g' \ ;$$

$$\text{b)}\quad f \circ g \sim_{\mathcal{G}^*} f' \circ g \Longrightarrow f \sim_{\mathcal{G}_g} f' \Longrightarrow f \circ g \sim_{\mathcal{G}} f' \circ g \ ;$$

where $\mathcal{G}^* = \{ (h,k) \in \mathcal{G} : k \text{ is in the component of } 1_Z \}$.

In order to get the global equivalence one should prove that given $\phi, \phi' \in C^\infty(X,Z) : \phi \sim_{\mathcal{G}} \phi' \Longleftrightarrow \phi \sim_{\mathcal{G}^*} \phi'$ , with, perhaps, some conditions on $\phi$ and $\phi'$, like $\phi$ and $\phi'$ being near enough. We leave this question unanswered.

For $\mathcal{G} = \mathcal{K}$, both the local and the global situation are solved when $g$ is an embedding and $f$ a submersion.

We finally get, for $\mathcal{G} = \mathcal{K}$,

$$f \circ g \text{ is } \mathcal{G}\text{-stable} \Longleftrightarrow f \text{ is } \mathcal{G}_g\text{-stable} \Longleftrightarrow g \text{ is } \mathcal{G}_f\text{-stable}.$$

## 2.1 Fixing the $2^{nd}$ map in the diagram

Consider a fixed map $f \in C^\infty(Y, Z)$. We define

$$\mathcal{G}_f = \{ (h, \ell) \in \mathcal{A}(X, Y) : h \in \mathcal{R}(X),\ \ell \in \mathrm{Iso}_{\mathcal{G}}(f) \} ,$$

where we distinguish two different cases:

a) For $\mathcal{G} = \mathcal{A}(X, Z)$, $\mathrm{Iso}_{\mathcal{G}}(f)$ is the usual isotropy subgroup of $f$ in $\mathcal{A}(Y, Z)$.

**Remark:** This definition is suggested by the following commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{g} & Y & \xrightarrow{f} & Z \\ h \downarrow & & \downarrow \ell & & \downarrow k \\ X & \xrightarrow{g'} & Y & \xrightarrow{f} & Z \end{array}$$

b) For $\mathcal{G} = \mathcal{K}(X, Z)$, as is usual in problems arising from the contact viewpoint we put $Z = \mathbf{R}^p$ and define, for $f$ a submersion,

$$\mathrm{Iso}_{\mathcal{G}}(f) = \{ \ell \in \mathcal{R}(Y) / \ell(f^{-1}(0)) = f^{-1}(0) \} .$$

Observe that the group $\mathcal{G}_f$ acts on $C^\infty(X, Y)$ in the obvious way and we have the following.

**Lemma 1**

a) *For* $\mathcal{G} = \mathcal{A}(X, Z) : g \sim_{\mathcal{G}_f} g' \Longrightarrow f \circ g \sim_{\mathcal{G}} f \circ g',\ \forall g, g' \in C^\infty(X, Y)$.

b) *For* $\mathcal{G} = \mathcal{K}(X, Z)$ *and* $f \in C^\infty(Y, Z)$ *being a submersion at every point of* $f^{-1}(0) \subset Y$ *:*

$$g \sim_{\mathcal{G}_f} g' \Longrightarrow f \circ g \sim_{\mathcal{G}} f \circ g',\ \forall g, g' \in C^\infty(X, Y).$$

**Proof.**

a) Follows easily.

b) If $g \sim_{\mathcal{G}_f} g'$, then there exists $(h, \ell) \in \mathcal{A}(X,Y)$ such that $g' = l \circ g \circ h^{-1}$. Therefore we can write locally:

$$I[f \circ \ell \circ g] = h^*(I[f \circ g']) \tag{1}$$

where $I[\,]$ denotes the local ideal generated by the map inside the brackets. Now from [9] it follows that the contact class of the composed map doesn't depend on the submersion $f$ or $f \circ \ell$. Consequently

$$I[f \circ \ell \circ g] = I[f \circ g] \tag{2}$$

Now (1) and (2) imply that $(f \circ g')_{x'} \sim_{\mathcal{K}} (f \circ g)_x$.

Using a standard argument of partitions of unity on $X$ we can obtain a global diffeomorphism $H : X \times \mathbf{R}^p \longrightarrow X \times \mathbf{R}^p$ such that the pair $(h, H)$ will give the $\mathcal{K}$-equivalence between $f \circ g$ and $f \circ g'$. □

In the case when $f$ is a submersion and $g$ is an immersion, Lemma 2 shows that the inverse implication of Lemma 1 holds either at germ level for $\mathcal{G}$=$\mathcal{A}$, $\mathcal{K}$ or globally for some special subgroups of $\mathcal{G}$.

**Lemma 2** *Let $f \in \mathrm{Subm}^\infty(Y,Z)$, then for all $g, g' \in \mathrm{Emb}^\infty(X,Y)$ we have:*

a) *For $\mathcal{G} = \mathcal{A}(X,Z)$:*

i) $(f \circ g)_x \sim_{\mathcal{G}} (f \circ g')_{x'} \Longrightarrow g_x \sim_{\mathcal{G}_f} g'_{x'}$ ;

ii) $f \circ g \sim_{\mathcal{G}^*} f \circ g' \Longrightarrow g \sim_{\mathcal{G}_f} g'$ .

b) *For $\mathcal{G} = \mathcal{K}(X,Z)$ :*

i) $(f \circ g)_x \sim_{\mathcal{K}} (f \circ g')_{x'} \Longrightarrow g_x \sim_{\mathcal{K}_f} g'_{x'}$ ;

ii) $f \circ g \sim_{\mathcal{C}} f \circ g' \Longrightarrow g \sim_{\mathcal{C}_f} g'$ ,

*where, as usual $\mathcal{C} = \mathcal{C}(X,Z)$ is defined as the subgroup* $\{(1,H) \in \mathcal{K}(X,Z)\}$.

**Proof.**

a) To prove i) one notice that the condition $(f\circ g)_x \sim_{\mathcal{G}} (f\circ g')_{x'}$ means that there exist local diffeomorphisms $h : (X,x) \to (X,x')$ and $k : (Z,z) \to (Z,z')$, where $(f\circ g)(x) = z$ and $(f\circ g')(x') = z'$, such that the following diagram commutes

$$\begin{array}{ccccc} (X,x) & \xrightarrow{g} & (Y,g(x)) & \xrightarrow{f} & (Z,z) \\ {\scriptstyle h}\downarrow & & & & \downarrow{\scriptstyle k} \\ (X,x') & \xrightarrow{g'} & (Y,g'(x')) & \xrightarrow{f} & (Z,z') \end{array} \tag{3}$$

We now need to find $(h',\ell) \in \mathcal{G}_f$ such that

$$\begin{array}{ccc} (X,x) & \xrightarrow{g} & (Y,g(x)) \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle \ell} \\ (X,x') & \xrightarrow{g'} & (Y,g'(x')) \end{array}$$

be a commutative diagram.

Observe that $f$ and $k\circ f$ are both local submersions at some neighbourhoods of $y' = g'(x')$ and $y = g(x)$ respectively. And thus there is a local diffeomorphism $\ell : (Y,y) \longrightarrow (Y,y')$ such that

$$k\circ f = f\circ \ell \tag{4}$$

We now define $h' \in \mathcal{R}(X)$ by conveniently modifying $h$ in such a way that $(h',\ell) \in \mathcal{G}_f$ (in a local sense). Firstly observe that from the commutativity of the diagram (3) it follows $f\circ g = k^{-1}\circ f\circ g'\circ h$, and from (4) we have $k\circ f\circ \ell^{-1} = f$. Therefore $f\circ g = f\circ \ell^{-1}\circ g'\circ h$. That is, in a small enough neighbourhood $N_z$ of $z$ in $Z$ we can write

$$g^{-1}(f^{-1}(c)) = h^{-1}\circ g'^{-1}\circ \ell(f^{-1}(c)),\ \forall c\in N_z\,,$$

i.e.

$$h(g^{-1}(f^{-1}(c)) = g'^{-1}(\ell(f^{-1}(c))),\ \forall c\in N_z\,,$$

So we can define a germ of diffeomorphism

$$T : (X,x) \longrightarrow (X,x')$$

as follows: given $a \in X$ sufficiently close to $x$, then $g(a) \in f^{-1}(c)$, for some $c \in N_z \subset Z$. Put $T(a) = (h^{-1} \circ g'^{-1} \circ \ell)(g(a))$.

The fact that $g'$ is an embedding ensures that $T$ is a germ of diffeomorphism at $x$.

If we take $h' = h \circ T$ it is not difficult to see that this is the local diffeomorphism satisfying the required conditions.

ii) We can use an analogous argument to part i) but assuming that $k$ is in the component of the identity, $1_Z$, on $Z$. Hence $k \circ f \sim_{\mathcal{R}(Y)} f$ and we can get a globally defined diffeomorphism $\ell \in \mathcal{R}(Y)$, for $g$ and $g'$ are embeddings.

b) The proof of i) can be found in [9]. The idea is the following: suppose first that $\dim X = \dim(f^{-1}(0))$. We can view $Y$ locally as $\mathbf{R}^k \times \mathbf{R}^p$ and $f : \mathbf{R}^k \times \mathbf{R}^p \longrightarrow \mathbf{R}^p$ as the usual projection.

We can also consider locally, $g(X)$ and $g'(X)$ as the graphs of the maps $\phi : \mathbf{R}^k \longrightarrow \mathbf{R}^p$ and $\phi' : \mathbf{R}^k \longrightarrow \mathbf{R}^p$ respectively. (Observe that for this particular case we have $X \cong f^{-1}(0) = \mathbf{R}^k$). Now, we know from the hypothesis that $\phi \sim_{\mathcal{K}} \phi'$. Hence any diffeomorphism $\ell \in \mathcal{R}(\mathbf{R}^k \times \mathbf{R}^p)$ taking $\phi$ to $\phi'$ will carry $g(X)$ onto $g'(X)$ and leave $f^{-1}(0)$ invariant. Therefore, for the equidimensional case we have proven that $(f \circ g)_x \sim_{\mathcal{K}} (f \circ g')_{x'}$ implies $g_x \sim_{\mathcal{K}_f} g'_{x'}$ . It also follows from this, that by taking $h = (g')^{-1} \circ \ell \circ g$ and $k = f \circ \ell$, we can obtain a diagram for the $\mathcal{K}_f$-equivalence.

The case $\dim X \neq \dim f^{-1}(0)$ reduces to the equidimensional one by considering a suspension of the smaller dimensional manifold. Montaldi [9] proves that the contact class is invariant by suspension.

ii) It follows from the compactness of $X$ and standard arguments of partitions of unity. □

Consider the map

$$f_* : C^\infty(X,Y) \longrightarrow C^\infty(X,Z)$$
$$g \longmapsto f\circ g \quad .$$

Lemma 1 above implies that

$$f_*(\mathcal{G}_f\text{-orbit of } g) \subset \mathcal{G}\text{-orbit of } f\circ g \ .$$

On the other hand from Lemma 2 it follows:

a) $f_*^{-1}$ ($\mathcal{A}^*$-orbit of $f\circ g$) $\subset \mathcal{A}_f$-orbit of $g$ .

b) $f_*^{-1}$ ($\mathcal{C}$-orbit of $f\circ g$) $\subset \mathcal{C}_f$-orbit of $g$ .

**Definitions.** *We say that $g$ is $\mathcal{G}_f$-stable if the $\mathcal{G}_f$-orbit of $g$ is open in $C^\infty(X,Y)$. We say that $\phi \in C^\infty(X,Z)$ is $\mathcal{G}$-stable if the $\mathcal{G}$-orbit of $\phi$ is open in $C^\infty(X,Z)$.*

**Theorem 1** *Let $f \in \mathrm{Subm}^\infty(Y,Z)$, then given any $g \in \mathrm{Emb}^\infty(X,Y)$, $g$ is $\mathcal{G}_f$-stable $\Longleftrightarrow$ $f\circ g$ is $\mathcal{G}$-stable.*

**Proof.** Consider the continuous map

$${}_r\Gamma^k_f : {}_rJ^k(X,Y) \longrightarrow {}_rJ^k(X,Z)$$
$${}_rj^k h(x) \longmapsto {}_rj^k(f\circ h)(x) \ ,$$

where $x = (x_1, \text{—}, x_r) \in X^{(r)}$ (see [5] for notation). The assumption that $f$ is a submersion ensures that ${}_r\Gamma^k_f$ is also a submersion, hence transversal to all the ${}_r\mathcal{G}^k$-invariant submanifolds of ${}_rJ^k(X,Z)$, $\forall k$, $\forall r$. Then, we have that given any ${}_r\mathcal{G}^k$-invariant submanifold $W$ of ${}_rJ^k(X,Z)$ :

$${}_rj^k(f\circ g) \pitchfork W \Longleftrightarrow {}_rj^k g \pitchfork ({}_r\Gamma^k_f)^{-1}(W) \ .$$

Suppose that $g$ is $\mathcal{G}_f$-stable and let $\phi = f\circ g$. Let $W_x \subset {}_rJ^k(X,Z)$ be the ${}_r\mathcal{G}^k$-orbit of ${}_rj^k\phi(x)$. Then we know from Lemma 1 that $({}_r\Gamma^k_f)^{-1}(W_x)$ must contain the ${}_r\mathcal{G}^k_f$-orbit, ${}_r\Omega^k_x$ of ${}_rj^k g(x)$. Now,

$$ {}_rj^kg \pitchfork {}_r\Omega^k_x \Longrightarrow {}_rj^kg \pitchfork ({}_r\Gamma^k_f)^{-1}(W_x) \Longrightarrow {}_r\Gamma^k_f \circ j^kg \pitchfork W_x, \forall r, \forall k . $$

Hence $\phi$ is $\mathcal{G}$-stable (see [6]).

Conversely, let $\phi$ be $\mathcal{G}$-stable. Then there is a neighbourhood $N_\phi$ of $\phi$ in $C^\infty(X, Z)$ contained in the $\mathcal{G}$-orbit of $\phi$ . Let $g'$ be an embedding sufficiently close to $g$ , such that $f \circ g' \in N_\phi$ .

Now, $N_\phi$ may be taken small enough such that $f \circ g' \sim_\mathcal{G} \phi$ where the diffeomorphism in the $\mathcal{G}$-equivalence is sufficiently close to the identity. Consequently, from Lemma 2 if follows that $g' \sim_{\mathcal{G}_f} g$ and hence $g$ is $\mathcal{G}_f$-stable.

Observe that for $\mathcal{G} = \mathcal{K}$ we consider $Z = \mathbf{R}^p$. Also notice that in this case the proof may be simplified by just considering transversality of jets instead of multijets. Moreover $\forall x$, $x' \notin (f \circ g)^{-1}(0)$, $j^k(f \circ g)(x)$ and $j^k(f \circ g')(x')$ are in the same $\mathcal{K}$-orbit. □

**Remark.** When $\mathcal{G} = \mathcal{A}$, it is possible to give a more direct proof for the necessary condition in Theorem 1, which is valid for all $g \in C^\infty(X, Y)$. In fact, let $\phi = f \circ g$ and $f_* : C^\infty(X, Y) \longrightarrow C^\infty(X, Z)$, $f_*(g) = f \circ g$. Being $f$ a submersion, $f_*$ is open; hence, given $\phi'$ sufficiently close to $\phi$, there exists $g'$ close to $g$ such that $\phi' = f \circ g'$. The rest of the argument follows easily from the $\mathcal{G}_f$ stability of $g$. Indeed , we think that the first proof presented above is more coherent with the purposes of this paper. Moreover, for $\mathcal{G} = \mathcal{K}$ it is not clear how to obtain a simpler proof as in the case $\mathcal{G} = \mathcal{A}$.

The sufficient implication in Theorem 1 may not remain true under the weaker hypothesis on the stability either of $g$ or $f$. In fact, let us consider the following situations:

(a) $g : \mathbf{R}^2 \longrightarrow \mathbf{R}^2, g(x, y) = (x, y^3 + xy)$, and $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$ the projection on the second factor;

(b) $g : \mathbf{R} \longrightarrow \mathbf{R}^2, g(t) = (t, 0)$, and $f : \mathbf{R}^2 \longrightarrow \mathbf{R}^2, f(x, y) = (x, y^2)$.

In both cases $f \circ g$ is $\mathcal{A}$-stable, but $g$ is not $\mathcal{G}_f$- stable.

## 2.2 Fixing the $1^{st}$ map in the diagram

We now fix $g \in C^\infty(X, Y)$. We also consider here two cases:

a) $\mathcal{G} = \mathcal{A}$, then we define

$$\mathcal{G}_g = \{ (\ell, k) \in \mathcal{G}(Y, Z) : \ell(g(X)) = g(X) \}$$

b) $\mathcal{G} = \mathcal{K}$, then we put

$$\mathcal{G}_g = \{ (\ell, H) \in \mathcal{G}(Y, \mathbf{R}^p) : \ell(g(X)) = g(X) \} ,$$

where each $(\ell, H)$ corresponds to a diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\iota} & Y \times \mathbf{R}^p & \xrightarrow{\pi_1} & Y \\ \ell \downarrow & & \downarrow H & & \downarrow \ell \\ Y & \xrightarrow{\iota} & Y \times \mathbf{R}^p & \xrightarrow{\pi_1} & Y \end{array}$$

and the map $\iota$ is the inclusion $\iota(y) = (y, 0)$.

**Lemma 3** *Given a* $1:1$ *immersion* $g$ *from* $X$ *to* $Y$ *and*

i) $\mathcal{G} = \mathcal{A}(X, Z)$, *then*

$$f \sim_{\mathcal{G}_g} f' \Longrightarrow f \circ g \sim_{\mathcal{G}} f' \circ g, \ \forall f, f' \in C^\infty(Y, Z) .$$

ii) $\mathcal{G} = \mathcal{K}(X, Z)$, $Z = \mathbf{R}^p$, *then*

$$f \sim_{\mathcal{G}_g} f' \Longrightarrow f \circ g \sim_{\mathcal{G}} f' \circ g, \ \forall f, f' \in C^\infty(Y, \mathbf{R}^p) .$$

**Proof.**

ii) If $\mathcal{G} = \mathcal{K}(X, \mathbf{R}^p)$ and $f \sim_{\mathcal{G}_g} f'$. This implies of $(\ell, H) \in \mathcal{K}(Y, \mathbf{R}^p)$, with $\ell(g(X)) = g(X)$ making commutative the diagram

$$\begin{array}{ccccc} Y & \xrightarrow{(id, f)} & Y \times \mathbf{R}^p & \xrightarrow{\pi_1} & Y \\ \downarrow \ell & & \downarrow H & & \downarrow \ell \\ Y & \xrightarrow{(id, f')} & Y \times \mathbf{R}^p & \xrightarrow{\pi_1} & Y \end{array}$$

and we define $h : X \to X$ by $h(x) = (g^{-1} \circ \ell \circ g)(x)$. Clearly $h$ is a diffeomorphism and the pair $(h, \tilde{H})$, where $\tilde{H} = (g^{-1} \times id) \circ \bar{H} \circ (g \times id)$, $\bar{H} = H \mid_{g(X) \times \mathbf{R}^p}$ defines the required $\mathcal{K}$-equivalence between $f \circ g$ and $f' \circ g$.

Analogous arguments adapted to $\mathcal{G} = \mathcal{A}(X, Z)$ prove part $i$). □

As in 2.1, we need additional conditions to obtain the inverse implication of Lemma 3:

**Lemma 4** *Let $g$ be a fixed* $1:1$ *immersion from $X$ to $Y$. Then the following holds*

i) *if* $\mathcal{G} = \mathcal{A}(X, Z) : f \circ g \sim_{\mathcal{G}^*} f' \circ g \Longrightarrow f \sim_{\mathcal{G}_g} f' \; \forall f, f' \in \mathrm{Subm}^{\infty}(Y, Z)$, *close enough;*

ii) *if* $\mathcal{G} = \mathcal{K}(X, Z)$, $Z = \mathbf{R}^p$ ,

a) $(f \circ g)_x \sim_{\mathcal{K}} (f' \circ g)_{x'} \Longrightarrow f_{g(x)} \sim_{\mathcal{G}_g} f'_{g(x')}$ , $\forall f, f' \in \mathrm{Subm}^{\infty}(Y, \mathbf{R}^p)$ .

b) $(f \circ g) \sim_{\mathcal{C}} (f' \circ g) \Longrightarrow f \sim_{\mathcal{C}_g} f'$, $\forall f, f'$ *sufficiently close submersions from $Y$ to* $\mathbf{R}^p$. *Here $\mathcal{C}_g$ is the obvious subgroup of $\mathcal{K}_g$* .

**Proof.**

i) Let $\mathcal{G} = \mathcal{A}(X, Z)$ and $\mathcal{G}^*$ as previously defined. If $f \circ g \sim_{\mathcal{G}^*} f' \circ g$ this means that there is a commutative diagram .

$$\begin{array}{ccccc} X & \xrightarrow{g} & Y & \xrightarrow{f} & Z \\ h \downarrow & & & & \downarrow k \\ X & \xrightarrow{g} & Y & \xrightarrow{f'} & Z \end{array}$$

with $h \in \mathcal{R}(X)$ and $k \in \mathcal{R}(Z)$ lying in the component of $1_Z$. Then provided that $f$ and $f'$ are close enough submersions we can assert that $k \circ f$ and $f'$ are right equivalent. That is, $\exists \ell \in \mathcal{R}(Y)$ such that $k \circ f = f' \circ \ell$, where this $\ell$ can be taken in the component of the identity. Now, $(\ell, k)$ is not necessarily an element of $\mathcal{G}_g$, for it is not clear that $\ell(g(X)) = g(X)$. We shall hence find another diffeomorphism $\ell' : Y \to Y$ satisfying:

1. $\ell'(\ell(g(X)) = g(X)$ ;

2. $\ell'$ carries the fibres of $f'$ into the fibres of $f'$ ; i.e. $\exists k' : Z \to Z$ diffeomorphism, such that $(\ell', k') \in \mathrm{Iso}(f')$ .

First observe that $f' \circ \ell \circ g \sim_{\mathcal{G}^*} f' \circ g$, for $1_Z \circ (f' \circ \ell \circ g) = (f' \circ g) \circ h$, with $(h, 1_Z) \in \mathcal{G}^*$. So from, Lemma 2 we conclude that $\ell \circ g \sim_{\mathcal{G}_{f'}} g$. That is, $\exists (h', \ell') \in \mathcal{G}_f$, such that $\ell' \circ (\ell \circ g) = g \circ h'$, with $\ell' \in \mathrm{Iso}(f')$; so there is a diffeomorphism $k' \in \mathcal{R}(Z)$ satisfying $k' \circ f' = f' \circ \ell'$.

Hence the pair $(\ell' \circ \ell, k' \circ k)$ is in $\mathcal{G}_g$ and $(k' \circ k) \circ f = f' \circ (\ell' \circ \ell)$. Consequently $f \sim_{\mathcal{G}_g} f'$.

ii) a) Follows from the main result in [9]. Observe that this proof becomes analogous to that of Lemma 2 b) i) by interchanging the roles of the 1 : 1 immersion $g$ and the submersion $f$ .

b) This proof is also similar to that of Lemma 2 b) ii) in which we interchange $f^{-1}(0)$ with $g(X)$ and $g'(X)$ with $(f')^{-1}(0)$. □

**Remark.** The corresponding result for $\mathcal{K}_g$ in above lemma can be obtained with the additional hypothesis that the diffeomorphism $h$ in the source of the $\mathcal{K}$-equivalence between $f \circ g$ and $f' \circ g$ be close to the identity.

In a similar way to Section 2.1 we may consider here a map

$$\begin{aligned} g^* : C^\infty(Y, Z) &\longrightarrow C^\infty(X, Z) \\ f &\longmapsto f \circ g \end{aligned}$$

which is continuous, for $X$ is compact (see [5, pg.49]).

Then from Lemma 3 we deduce that $g^*$ carries $\mathcal{G}_g$-orbits into $\mathcal{G}$-orbits. So the inverse image of a $\mathcal{G}$-orbit through $(g^*)^{-1}$ must be a union of $\mathcal{G}_g$-orbits.

Again the map $f$ is said to be $\mathcal{G}_g$-stable if its $\mathcal{G}_g$-orbit is open in $C^\infty(Y, Z)$.

**Theorem 2** *Let* $g \in \mathrm{Emb}^\infty(X, Y)$, *then*

$$f \textit{ is } \mathcal{G}_g\textit{-stable} \iff f \circ g \textit{ is } \mathcal{G}\textit{-stable} , \ \forall f \in \mathrm{Subm}^\infty(Y, Z) .$$

**Proof.** Denote by $J^k_{g(X)}(Y,Z)$ the subset of $k$-jets of maps from $Y$ to $Z$ with source in $g(X) \subset Y$. Analogously ${}_rJ^k_{g(X)}$ represents the multijets of such maps with sources in $(g(X))^{(r)} \subset Y^{(r)}$. We can define continuous maps

$$
\begin{aligned}
{}_r\Gamma^k_g : {}_rJ^k_{g(X)}(Y,Z) &\longrightarrow {}_rJ^k(X,Z) \\
(j^kh_1(y_1), - , j^kh_r(y_r)) &\longmapsto (j^k(h_1 \circ g)(g^{-1}(y)), - , j^k(h_r \circ g)(g^{-1}(y)))
\end{aligned}
$$

$\forall k$, $\forall r \geq 1$ .It can be seen that they are submersions.

Let us suppose that $f$ is $\mathcal{G}_g$-stable, then given any $y \in Y^{(r)}$ if we denote $\omega = {}_rj^kf(y)$ and $W_\omega$ is the ${}_r\mathcal{G}^k_g$-orbit of $\omega$ in ${}_rJ^k(Y,Z)$, $k \geq 1$, $r \geq 1$ we can write

$$
{}_rj^kf \pitchfork_\omega W_\omega \qquad (*)
$$

where ${}_rj^kf : Y^{(r)} \longrightarrow {}_rJ^k(Y,Z)$ is the multijet extension map of $f$ .

Let $\phi = f \circ g$ and $\Omega_\sigma$ be the ${}_r\mathcal{G}^k$-orbit of $\sigma = {}_rj^k\phi(x)$ with $x = g^{-1}(y)$, for some $y \in (g(X))^{(r)} \subset Y^{(r)}$. In order to prove that $\phi$ is $\mathcal{G}$-stable it is enough to see that ${}_rj^k\phi \pitchfork_\sigma \Omega_\sigma$ , for any $x \in X^{(r)}$, $\forall k$ , $\forall r \geq 1$ , (see [6]).

Observe that we can write ${}_rj^k\phi$ as the composition

$$
{}_rj^k\phi : X^{(r)} \xrightarrow{g^r} (g(X))^{(r)} \xrightarrow{{}_rj^kf|} {}_rJ^k_{g(X)}(Y,Z) \xrightarrow{{}_r\Gamma^k_g|} {}_rJ^k(X,Z) ,
$$

where the vertical bar on the right of a map denotes the appropriate restriction of the considered map.

Notice that ${}_rJ^k_{g(X)}(Y,Z)$ is a union of ${}_r\mathcal{G}^k_g$-orbits. In fact the orbits of $\mathcal{G}^k_g$ coincide with the $\mathcal{G}^k$-orbits outside from $J^k_{g(X)}(Y,Z)$ and, on the other hand a $k$-jet with source in $g(X)$ cannot be $\mathcal{G}^k_g$-equivalent to another $k$-jet with source off $g(X)$ .

Now, from (*) we have that

$$
T_y\, {}_rj^kf(T_xg^r(T_xX^{(r)})) + T_\omega W_\omega = T_\omega\, {}_rJ^k_{g(X)}(Y,Z) .
$$

where $x = (g^r)^{-1}(y)$ .

And by applying $T_\omega\, {}_r\Gamma^k_g$ on both sides we get

$$T_x\ {}_rj^k\phi(T_xX) + T_\omega\ {}_r\Gamma^k_g(T_\omega W_\omega) = T_\sigma\ {}_rJ^k(X,Z)\ .$$

From Lemma 3 it follows that ${}_r\Gamma^k_g(W_\omega) \subset \Omega_\sigma$ and hence we have

$$T_x\ {}_rj^k\phi(T_xX) + T_\sigma\Omega_\sigma = T_\sigma\ {}_rJ^k(X,Z)\ ,$$

i.e. ${}_rj^k\phi \pitchfork_\sigma \Omega_\sigma$, as we wanted to show.

Conversely, suppose that $\phi = f \circ g$ is $\mathcal{G}$-stable. This means that the $\mathcal{G}$-orbit of $\phi$ contains some open neighbourhood $N_\phi$ of $\phi$ . Let $f'$ be in a sufficiently small neighbourhood $V_f$ of $f$ such that $f'$ is a submersion too and $f' \circ g \in N_\phi$. Then $f' \circ g \sim_{\mathcal{G}} \phi$. In fact we can take $V_f$ small enough such that $f' \circ g \sim_{\mathcal{G}^*} \phi$. Then from Lemma 4 we can conclude that $f' \sim_{\mathcal{G}_g} f$. Hence $V_f$ is contained in the $\mathcal{G}_g$-orbit of $f$ and therefore this is an open $\mathcal{G}_g$-orbit. Consequently $f$ is $\mathcal{G}_g$-stable.

As in Thm. 1 we must remark that, for the case $\mathcal{G} = \mathcal{K}(X, \mathbf{R}^p)$, this proof can be simplified by just considering jet extensions instead of multijets. □

**Corollary 1** *Given* $f \in \mathrm{Subm}^\infty(Y,Z)$ *and* $g \in \mathrm{Emb}^\infty(X,Y)$ ,

$$f \textit{ is } \mathcal{G}_g\textit{-stable} \iff g \textit{ is } \mathcal{G}_f\textit{-stable.}$$

**Proof.** It follows immediately from Thms. 1 and 2. □

Following J.P. Dufour [4] we define a diagram $X \xrightarrow{g} Y \xrightarrow{f} Z$ to be stable iff there exists a neighbourhood $N$ of $(g,f)$ in $\mathcal{L} = C^\infty(X,Y) \times C^\infty(Y,Z)$ such that for any $(g',f') \in N$, $\exists(h,\ell,k) \in \mathrm{Diff}(X) \times \mathrm{Diff}(Y) \times \mathrm{Diff}(Z)$ making commutative the diagram

$$\begin{array}{ccccc} X & \xrightarrow{g} & Y & \xrightarrow{f} & Z \\ h\downarrow & & \downarrow\ell & & \downarrow k \\ X & \xrightarrow{g'} & Y & \xrightarrow{f'} & Z \end{array} \qquad (**)$$

**Corollary 2** *Given* $f \in \mathrm{Subm}^\infty(Y,Z)$ *and* $g \in \mathrm{Emb}^\infty(X,Y)$ , *the diagram* $X \xrightarrow{g} Y \xrightarrow{f} Z$ *is stable* $\Longleftrightarrow$ $f \circ g$ *is* $\mathcal{A}$*-stable.*

**Proof.** For the necessity, we shall show that the stability of the above diagram is equivalent to the $\mathcal{A}_g$ stability of $f$ and then from Thm.2 if follows that $f \circ g$ must be $\mathcal{A}$-stable.

If $X \xrightarrow{g} Y \xrightarrow{f} Z$ is stable, then we can find a neighbourhood $N$ of $(g, f)$ in $\mathcal{L}$, satisfying the requirement of the above definition. Observe that we can take $N$ to be of the form $N_g \times N_f$, with $N_g$ a neighbourhood of $g$ in $C^\infty(X, Y)$ and $N_f$ a neighbourhood of $f$ in $C^\infty(Y, Z)$. Now the commutativity of diagram (**) means that $g' = \ell \circ g \circ h^{-1}$ and $f' = k \circ f \circ \ell^{-1}$. But this implies that $f' \circ g' = k \circ (f \circ g) \circ h^{-1}$, that is $f' \circ g' \sim_{\mathcal{A}} f \circ g$.

With this we have proven that $\forall f' \in N_f$, $f' \sim_{\mathcal{A}_g} f$ and hence that $f$ is $\mathcal{A}_g$-stable.

Let's see the sufficiency. Suppose that $f \circ g$ is $\mathcal{A}$-stable. From Thm.1 we know that $g$ is $\mathcal{A}_f$-stable and hence there is a neighbourhood $V_g$ of $g$ in $C^\infty(X, Y)$ such that $\forall g' \in V_g : g' \sim_{\mathcal{A}_f} g$. In other words, we must have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{g} & Y & \xrightarrow{f} & Z \\ h\downarrow & & \downarrow \ell & & \downarrow k \\ X & \xrightarrow{g'} & Y & \xrightarrow{f} & Z \end{array} \qquad \text{with } h \in \mathrm{Diff}(X),\ \ell \in \mathrm{Diff}(Y) \text{ and } k \in \mathrm{Diff}(Z).$$

Now, given $g' \in V_g$, we must have that $g'$ is an embedding and it is also $\mathcal{A}_f$-stable. Hence from Corollary 1 it follows that $f$ is $\mathcal{A}_{g'}$-stable. Therefore we can find a neighbourhood $V_f$ of $f$ in $C^\infty(Y, Z)$ such that $\forall f' \in V_f$, $f' \sim_{\mathcal{A}_{g'}} f$, i.e., there is a commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{g'} & Y & \xrightarrow{f} & Z \\ h'\downarrow & & \downarrow \ell' & & \downarrow k' \\ X & \xrightarrow{g'} & Y & \xrightarrow{f'} & Z \end{array}$$

and from both diagrams we get

$$\begin{array}{ccccc} X & \xrightarrow{g} & Y & \xrightarrow{f} & Z \\ h' \circ h\downarrow & & \downarrow \ell' \circ \ell & & \downarrow k' \circ k \\ X & \xrightarrow{g'} & Y & \xrightarrow{f'} & Z \end{array}$$

which implies that $X \xrightarrow{g} Y \xrightarrow{f} Z$ is equivalent in the sense of Dufour [4] to $X \xrightarrow{g'} Y \xrightarrow{f'} Z \ \forall (g', f') \in V_g \times V_f$. The stability of the diagram then follows. □

# 3 Versality of families of composed maps

We shall study in this section families of compositions of the two following kinds:

a) $X \times U \xrightarrow{H} Y \xrightarrow{f} Z$, $H \in C^\infty(X \times U, Y)$, $f \in \mathrm{Subm}^\infty(Y, Z)$ ;

and

b) $X \times U \xrightarrow{h \times 1_U} Y \times U \xrightarrow{F} Z$, $h \in \mathrm{Imm}^\infty(X, Y)$, $F \in C^\infty(Y \times U, Z)$, $U$ being a parameter manifold in both cases and $1_U$ the identity map on $U$.

For case a) we have the following results:

a1) If $f$ is a fixed submersion and $H$ varies among the $C^\infty$-families of immersions from $X$ to $Y$, then

$$H \text{ is } \mathcal{G}_f\text{-versal} \iff f \circ H \text{ is } \mathcal{G}\text{-versal.}$$

a2) If $H$ is a fixed $C^\infty$-family of immersions and $\dim U$ is small enough (e.g. $\dim U \leq 6$ for $Z = \mathbf{R}$ and $\mathcal{G} = \mathcal{A}$ ) then $\exists$ residual subset $\mathcal{F}$ of $\mathrm{Subm}^\infty(Y, Z)$ such that $\forall f \in \mathcal{F}$, the family $f \circ H$ is $\mathcal{G}$-versal.

And the following holds for case b):

b1) If $h$ is a fixed immersion and $F$ varies among the $C^\infty$-families of submersions from $Y$ to $Z$, then

$$F \text{ is } \mathcal{G}_h\text{-versal} \iff F \circ (h \times 1_U) \text{ is } \mathcal{G}\text{-versal.}$$

b2) If $F$ is a fixed $C^\infty$-family of submersions from $Y$ to $Z$ and $\dim U$ is small enough, then $\exists$ residual subset $\mathcal{H} \subset \mathrm{Imm}^\infty(X, Y)$ such that $\forall h \in \mathcal{H}$, the family $F \circ (h \times 1_U)$ is $\mathcal{G}$-versal.

**Remark.** Actually just results a1) and b1) will be treated from our group actions viewpoint. Nevertheless, we have included a2) and b2) in order to exhibit a more complete picture of the possibilities that arise with these kinds of compositions. We give below the proofs of a1), a2) and b1). A proof of b2) for $\mathcal{G} = \mathcal{K}$ was first given in ( J. Montaldi [8] ) and for $\mathcal{G} = \mathcal{A}$ in ( G. Wassermann [11] ). In a recent paper Montaldi [10] proved b2) under the hypothesis that the fixed family $F$ is $\mathcal{G}$-versal, for a large class of groups $\mathcal{G}$, including $\mathcal{G} = \mathcal{A}$ and $\mathcal{K}$. The dual result of this extension of b2) would correspond to an extension of a2) for the case of a fixed versal family $H$. This seems to be true, but the methods used by Montaldi apparently do not apply in this case.

a1) Let $f$ be a fixed submersion from $Y$ to $Z$ and consider the compositions

$$\Phi : X \times U \xrightarrow{H} Y \xrightarrow{f} Z \ .$$

We denote,

$$\begin{aligned} j_1^k\Phi : X \times U &\longrightarrow J^k(X,Z) \\ (x,u) &\longmapsto j^k\Phi_u(x) \end{aligned}$$

and

$$\begin{aligned} j_1^k H : X \times U &\longrightarrow J^k(X,Y) \\ (x,u) &\longmapsto j^k H_u(x) \ . \end{aligned}$$

**Note.** When $\mathcal{G} = \mathcal{K}$ it would be enough to ask $f$ to be a submersion at $f^{-1}(0)$.

**Proposition 1** *Given any ($\mathcal{G}^k$-invariant) submanifold $S$ of $J^k(X,Z)$ the subset $\{ H \in C^\infty(X \times U, Y) : j_1^k\Phi \pitchfork S \}$ is residual in $C^\infty(X \times U, Y)$ with the Whitney $C^\infty$-topology.*

**Proof.** As in Theorem 1, Section 2, the mapping

$$\begin{aligned} \Gamma_f^k : J^k(X,Y) &\longrightarrow J^k(X,Z) \\ j^k h(x) &\longmapsto j^k(f \circ h)(x) \end{aligned}$$

is a submersion, $\forall k$, and $\Omega = (\Gamma^k_f)^{-1}(S)$ is a $\mathcal{G}_f$-invariant submanifold of $J^k(X,Y)$. Now,

$$j^k_1 H \pitchfork \Omega \iff \Gamma^k_f \circ j^k_1 H \pitchfork S$$

$$\iff j^k_1\Phi \pitchfork S \qquad (\text{ for } \Gamma^k_f \circ j^k_1 H = j^k_1\Phi)\ .$$

And the required result follows immediately from the following version of the Thom's transversality Theorem whose proof can be found in (Bruce [2]): Let $\Omega$ be a submanifold of the jet space $J^k(X,Y)$. There is a residual subset of smooth maps $H \in C^\infty(X\times U, Y)$ such that the jet extension $j^k_1 H : X\times U \longrightarrow J^k(X,Y)$ is transverse to $\Omega$. □

**Corollary 3** *Let $f : Y \to Z$ be a fixed submersion. Then, for any $C^\infty$-family of immersions $H$ from $X$ to $Y$ we have*

$$f\circ H \textit{ is a } \mathcal{G}\textit{-versal family} \iff H \textit{ is a } \mathcal{G}_f\textit{-versal family.}$$

**Proof.** It follows easily from Proposition 1 above together with Lemma 2 in Section 2 and the characterization of versality in terms of transversality to the orbits of the corresponding group actions. □

**Remarks.** Observe that in general given a $\mathcal{G}^k$-orbit $S$ in $J^k(X,Z)$, the submanifold $(\Gamma^k_f)^{-1}(S)$ may contain more than one $\mathcal{G}_f$-orbit. So when we consider $H : X\times U \to Y$ as a $C^\infty$-family of maps (not necessarily immersions) from $X$ to $Y$, we can only say that, when the dimension of the parameter manifold $U$ is small enough (see [11] for an analysis of the relevance of this dimension), there is a residual subset of families $H \in C^\infty(X\times U, Y)$ such that $f\circ H$ is $\mathcal{G}$-versal.

Notice that when $\mathcal{G} = \mathcal{K}$ the relevant orbits are those in $J^k_{f^{-1}(0)}(Y,Z)$, for the complement of this subset in $J^k(Y,Z)$ is a unique $\mathcal{K}$-orbit.

a2) Let $H \in C^\infty(X\times U, Y)$ be a fixed family of immersions from $X$ to $Y$.

**Proposition 2** *Given any ($\mathcal{G}^k$-invariant) submanifold $W$ of $J^k(X,Z)$, the subset $\tilde{R}_W = \{ f \in C^\infty(Y,Z) : j^k_1\Phi \pitchfork W\}$; with $\Phi$ as above, is residual in $C^\infty(Y,Z)$ with the Whitney $C^\infty$-topology.*

**Proof.** Let's define for each $u \in U$ a map

$$\Gamma_u^k : J^k_{H_u(X)}(Y,Z) \longrightarrow J^k(X,Z)$$
$$j^k h(H_u(a)) \longmapsto j^k(h \circ H_u)(a) \ .$$

Notice that these must be submersions $\forall u$, for all the maps $H_u$ are immersions (see proof of Theorem 2 in Section 2).

And hence we can define a map from the disjoint union $\bigsqcup_{u\in U} J^k_{H_u(X)}(Y,Z)$ to $J^k(X,Z)$, as

$$\Gamma^k : \bigsqcup_{u\in U} J^k_{H_u(X)}(Y,Z) \longrightarrow J^k(X,Z)$$
$$j^k h(H_u(a)) \longmapsto j^k(h \circ H_u)(a) \ ,$$

which is also a submersion.

Observe that $\bigsqcup_{u\in U} J^k_{H_u(X)}(Y,Z)$ is a submanifold of $J^k(Y,Z) \times U$ which is stratified by the pull-backs of the $\mathcal{G}$-orbits in $J^k(X,Z)$ by the submersions $\Gamma_u$, $u \in U$ .

Now, the jet extension map $j_1^k\Phi$ is also given by the composition

$$j_1^k\Phi : X \times U \xrightarrow{H} \bigsqcup_{u\in U} H_u(X) \xrightarrow{j^k f | \bigsqcup H_u(X)} \bigsqcup_{u\in U} J^k_{H_u(X)}(Y,Z) \xrightarrow{\Gamma} J^k(X,Z)$$
$$(x,u) \longmapsto H_u(x) \longmapsto j^k f(H_u(x)) \longmapsto j^k(f \circ H_u)(x).$$

Then given a $\mathcal{G}^k$-invariant submanifold $W$ in $J^k(X,Z)$, the pull-back $(\Gamma^k)^{-1}(W)$ is a submanifold of $\bigsqcup_{u\in U} j^k_{H_u(X)}(Y,Z)$. In fact it is a union of pull-backs of $W$ by the maps $\Gamma_u$. It is also a submanifold of $J^k(Y,Z) \times U$.

We can use at this point the following variation of the Thom's transversality theorem as given in [2]: If $C$ is a submanifold of $J^k(Y,Z) \times U$, then the subset

$$T_C = \{ f \in C^\infty(Y,Z) : (j^k f \times 1_U) \pitchfork C \}$$

is residual in $C^\infty(Y,Z)$ with the $C^\infty$-Whitney topology.

Now a similar argument to the one used along the proof of Theorem 2 in Section 2 shows that $\forall f \in T_C$, $j_1^k(f \circ H) \pitchfork W$ . And the required result follows from the fact that $T_C \subset \tilde{R}_W$. □

b1) Let $h$ be a fixed immersion 1 : 1 from $X$ to $Y$. If $F \in C^\infty(Y \times U, Z)$ is a family of submersions from $Y$ to $Z$ with parameters in $U$, we denote their composition by

$$\Phi : X \times U \xrightarrow{h\times 1_U} Y \times U \xrightarrow{F} Z \ .$$

The jet extensions of the families $F$ and $\Phi$ will be respectively written as $j_1^k F$ and $j_1^k \Phi$ .

**Proposition 3** *With $h$, $F$ and $\Phi$ as above and for any $\mathcal{G}^k$-invariant submanifold $S$ of $j^k(X, Z)$ we have that the subset $\{ F \in C^\infty(Y \times U, Z) : j_1^k \Phi \pitchfork S \}$ is residual in $C^\infty(Y \times U, Z)$ with the Whitney $C^\infty$-topology.*

**Proof.** The proof is similar to the proof of Theorem 2, in Section 2, and we omit it. □

**Corollary 4** *Let $h$ be a fixed injective immersion from $X$ to $Y$. Then for any smooth family of submersions $F$ from $Y$ to $Z$ we have.*

$$F \circ (h \times 1_U) \text{ is } G\text{-versal} \Longleftrightarrow F \text{ is } \mathcal{G}_h\text{-versal} \ .$$

**Proof.** It follows as a consequence of Proposition 3 above, Lemma 4 in Section 2, and the characterization of versality of families in terms of transversality to the orbits of the considered group actions. □

**Comments.**

1. J.W. Bruce obtains in [2] a result which is similar, in some sense to our Proposition 1 in Section 3. The difference resides in the fact that we study here not only the finite-singularity-type but the versality of the composition. On the other hand we restrict our attention to smooth families $H$ of immersions, whereas Bruce's result holds for arbitrary smooth families of maps.

   As a consequence of a2) we have that $\forall u \in U$ the local submanifolds $H_u(X)$ are weakly transversal in Bruce's sense to $f^{-1}(z)$, $\forall z \in Z$ , for all submersion $f \in \mathcal{F}$. Then all of the Bruce's considerations in [2, pg.116] apply. As a particular case we may consider the following

$$S^1 \times \mathbf{R}^p \xrightarrow{H} \mathbf{R}^3 \xrightarrow{f} \mathbf{R}$$

to conclude that, given any $p$-parameter family $H$ of curves in $\mathbf{R}^3 (p \leq 6)$, there exists a residual subset $\mathcal{F}$ in $\mathrm{Subm}^\infty(\mathbf{R}^3, \mathbf{R})$ such that all the surfaces $f^{-1}(t)$, $t \in \mathbf{R}$ , are tangent to each curve of the family at most at a finite set of points.

2. An extensive geometrical study of the implications of resul b2) for $\mathcal{G} = \mathcal{K}$ in low dimensions can be found in [8].

**Acknowledgement.** The authors would like to thank J.Montaldi for helpful discussions.

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