Matemática Contemporânea, Vol. 5, 15–28
http://doi.org/10.21711/231766361993/rmc52

COBORDISM AND SINGULARITIES

Oskar Burlet

Cobordism groups have been invented by René Thom in the early fifties to solve in particular the problem of representability of homology classes in manifolds. R. Thom [1] .

But the methods, introduced a little bit earlier by Pontrijagin in a special case, proved to be powerful in several other situations.

In the present case we look at embeddings and immersions of closed n-dimensional manifolds into euclidean space $\mathbb{R}^{n+k}$; k is called the codimension of the embedding.

There are different equivalence relations among embeddings which have been considered.

Isotopy: Two embeddings (V, f_0) and (V, f_1) of V into $\mathbb{R}^{n+k}$ are isotopic if there exists a differentiable family of embeddings $f_t : V \to \mathbb{R}^{n+k}$, $t \in [0,1]$, connecting f_0 to f_1.

Regular Homotopy: Similar to the preceding one, regular homotopy is an equivalence relation among immersions. The family $f_t : V \to \mathbb{R}^{n+k}, t \in [0,1]$, has to be a differentiable family of immersions connecting f_0 to f_1.

Observe that in these equivalence relations the source manifold V remains fixed. This will not be the case for cobordism.

Cobordism: Two embeddings (immersions) (V_0, f_0) and (V_1, f_1) are cobordant if there exists a compact manifold W with boundary $\partial W = V_0 \amalg V_1$, the disjoint union of V_0 and V_1 , and a relative embedding (immersion).

$$(W, \partial V) \xrightarrow{f} (\mathbb{R}^{n+k} \times I, \mathbb{R}^{n+k} \times \partial I), \quad \text{where} \quad I = [0,1],$$

such that

$$f \mid V_0 = f_0 : V_0 \longrightarrow \mathbb{R}^{n+k} \times (0) = \mathbb{R}^{n+k}$$

and

$$f \mid V_1 = f_1 : V_1 \longrightarrow \mathbb{R}^{n+k} \times (1) = \mathbb{R}^{n+k} .$$

Notice that $f : W \longrightarrow \mathbb{R}^{n+k} \times I$ has to meet the boundary transversally and $\partial W = f^{-1}(\mathbb{R}^{n+k} \times \partial I)$.

(W, f) is called a cobordism between (V_0, f_0) and (V_1, f_1).

or example if $V = V_0 = V_1$ and f_0 is isotopic to f_1 , (V_0, f_0) and (V_1, f_1) are cobordant. A cobordism is given by $W = V \times I$ and $f : W \longrightarrow \mathbb{R}^{n+k} \times I$ such that $f(x,t) = (f_t(x), t)$ where f_t , $t \in I$, is an isotopy joining f_0 to f_1 .

Hence cobordism is an equivalence relation among embeddings, cruder than isotopy and in which the source manifold may change.

An equivalence class is called a cobordism class. Let us denote by Emb_n^k the set of cobordism classes of embeddings of closed n-dimensional manifolds into $\mathbb{R}^{n+k}$. Similarly Imm_n^k denotes the set of cobordism classes of immersions of closed n-dimensional manifolds into $\mathbb{R}^{n+k}$.

To study cobordism classes we need only consider embeddings up to isotopy and immersions up to regular homotopy. Therefore disjoint union defines an abelian group structure on Emb_n^k and Imm_n^k.

In fact if $[V, f], [V', f'] \in \mathrm{Emb}_n^k$ we can assume, up to isotopy, that $f(V) \subset \{x_1 < 0\}$ and $f'(V') \subset \{x_1 > 0\}$. Then $[V, f] + [V', f'] \stackrel{\text{def}}{=} [V \amalg V', f \amalg f']$. Where "$\amalg$" denotes disjoint union. The neutral element will be $[\phi, i]$, $\phi \stackrel{i}{\subset} \mathbb{R}^{n+k}$. The inverse of $[V, f]$ is given by $[V, \sigma f]$ where σ is a reflection in a hyperplane not meeting $f(V)$.

In the same way we can define an abelian group structure on Imm_n^k.R. Wells [2]

While groups of isotopy classes and regular homotopy classes are rather difficult to deal with, cobordism groups turn out to be easier to handle. In fact the Thom-Pontrijagin construction identifies Emb_n^k with $\pi_{n+k}(MO(k))$ and Imm_n^k with $\pi_{n+k}^s(MO(k))$ where $MO(k)$ is the Thom space of the universal $O(k)$ vector bundle and $\pi_\star^s$ denotes stable homotopy in dimension $\star$.

Recall that if ξ is a vector bundle over a space X , with orthogonal structure, we may consider $BE(\xi)$ the set of elements of length less or equal to one in the total space $E(\xi)$ of the bundle ξ. The boundary $SE(\xi)$ of $BE(\xi)$ is the set of elements of length one in $E(\xi)$. It is called the associated sphere bundle. The Thom space of ξ is defined to be $M(\xi) = BE(\xi)/_{SE(\xi)}$.

If ξ is a differentiable vector bundle $M(\xi)$ will be a manifold except at the base point $SE/_{SE}$ and it will contain X as a submanifold with normal bundle ξ .

The universal $O(k)$ bundle is just the canonical vector bundle on the Grassmannian $BO(k,N)$ of k-dimensional subspaces of $\mathbb{R}^N$, N large. We denote this bundle by λ_k. Its fiber over $P \in BO(k,N)$ is the vector space P, a subspace of $\mathbb{R}^N$. Let us write $BO(k)$ instead of $BO(k,N)$ and $MO(k)$ instead of $M(\lambda_k)$.

There is a natural mapping

$$\pi_{n+k}MO(k) \longrightarrow \mathrm{Emb}_n^k$$

which sends $[\varphi] \in \pi_{n+k}MO(k)$ to the cobordism class of the submanfolds $\varphi^{-1}(BO(k)) \subset S^{n+k}$.

Here $\varphi : (S^{n+k},\star) \longrightarrow (MO(k),\star)$ is chosen to be a representative of $[\varphi]$ transverse to $BO(k) \subset MO(k)$. The map $\pi_{n+k}MO(k) \longrightarrow \mathrm{Emb}_n^k$ is a homomorphism and its inverse is given by the Thom-Pontrijagin construction [1]. Hence

$$\pi_{n+k}MO(k) = \mathrm{Emb}_n^k \ .$$

It turns out that the most interesting applications concern often embeddings with some special structure in the normal bundle. For example an orientation or a complex structure. This means that the structure group of the normal bundle is reduced to a closed subgroup G of $O(k)$. Equivalently the classifying map of the normal bundle — denoted by ν or ν_f — has a well defined lift, up to homotopy, into BG the classifying space for G bundles.

In this case the corresponding cobordism group is denoted by $Emb_n^k(G)$ and the Thom-Pontrijagin construction identifies it to $\pi_{n+k}MG$ where MG is the Thom space of the universal G-vector bundle.

Examples:

1. The cobordism group of embeddings with a trivialization of the normal bundle, so called framed embeddings, is identified with $\pi_{n+k}(S^k)$. Pontrijagin [3]

2. The cobordism group of embeddings of closed n-dimensional manifolds V in $\mathbb{R}^{n+k+\ell}$, with a normal ℓ-frame is identified with $\pi_{n+k+\ell}(S^\ell \wedge MO(k))$.

   Smale-Hirsch theory shows that regular homotopy classes of immersions in codimension $k+\ell$ with a normal ℓ-frame are the same as regular homotopy classes of immersions in codimension k, $k \geq 1$. But if $k+\ell \geq n+2$ any immersion of V in codimension k is regularly homotopic to an embedding in codimension $k+\ell$ with a normal ℓ-frame.

   Moreover if two embeddings in codimension $k+\ell$, with normal ℓ-frame, are cobordant as immersions with normal ℓ-frame they are also cobordant as embeddings with normal ℓ-frame.

   Hence
   $$\mathrm{Imm}_n^k = \pi_{n+k+\ell}(S^\ell \wedge MO(k)), \quad k+\ell \geq n+2 \ .$$

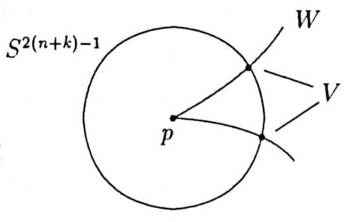
   But the last group is just stable homotopy of $MO(k)$ in dimension $n+k$. That is $\mathrm{Imm}_n^k = \pi_{n+k}^s MO(k)$.

   The mapping $\mathrm{Emb}_n^k \to \mathrm{Imm}_n^k$ which considers an embedding as an immersion can be identified with the suspension homomorphism
   $$\pi_{n+k}(MO(k)) \xrightarrow{E} \pi_{n+k}^s(MO(k)) \ .$$

   The same construction holds for cobordisms with structure a general closed subgroup $G \subset O(k)$.

3. This example will give a link of these groups to singularity theory.

   Let W^n be an affine complex variety in $\mathbb{C}^{n+k}$ with isolated singularity at a point p.



The intersection of W with the boundary of a small ball centered at p will be a closed submanifold $V \subset S^{2(n+k)-1}$ with a $U(k)$ structure in the normal bundle. V is called the link of the singularity. The corresponding cobordism class is a well defined element in $\pi_{2(n+k)-1}MU(k)$.

A necessary condition for the smoothability of the singularity is that the link should be cobordant to 0. Smoothable means that there is an algebraic flat one parameter family W_t of affine complex varieties, t varying in a neighborhood of $0 \in \mathbb{C}$, such that $W_0 = W$ and W_t is smooth if $t \neq 0$.

For example $W = \left\{ \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \in \mathbb{C}^6 \mid \text{rank} \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \leq 1 \right\}$.

W is of complex codimension 2 in $\mathbb{C}^6$ and $0 \in \mathbb{C}^6$ is an isolated singular point.

The link V of W is a 7-dimensional submanifold of S^{11} with complex structure in the normal bundle. Its cobordism class is an element of $\pi_{11}MU(2)$ which turns out to be non zero. This has been proved first by R. Thom.

In the unstable case, where $m > 4k-1$, the homotopy groups $\pi_m MU(k)$ are difficult to compute (E. Thomas & E. Rees [4]). But the ranks of these groups, or more precisely the groups $\pi_m MU(k) \otimes Q$, obtained by tensoring $\pi_m MU(k)$ with the rational numbers, are easily determined (O. Burlet [5]).

Their direct sum, as Lie algebra, with brackets the Whitehead product, is an extension of Q by a free Lie algebra on infinitely many generators. Hence there are lots of non zero elements, even rationally. But in any case there remains the problem of finding invariants which give a way to test if a given element is non zero. Such invariants have been defined for links of affine varieties W which are cones over projective varieties. But not all elements in the cobordism group are represented by such varieties.

For instance $\pi_{22}MU(2) \otimes Q$ is isomorphic to Q but no non zero element is representable by the link of the cone on a projective variety (H. Shiga [6]).

We will now indicate how one can define another type of invariants, than those of Thom, for testing elements in $\pi_{n+2k}MU(k)$.

Recall that the cohomology ring $H^\star(BU(k);\mathbf{Z})$ is the polynomial algebra $\mathbf{Z}[c_1,\dots,c_k]$ on the Chern classes $c_i \in H^{2i}(BU(k);\mathbf{Z})$, $i = 1,2\cdots k$, of the bundle λ_k.

Let $\alpha = (\alpha_1,\dots,\alpha_{k-1})$ and $\beta = (\beta_1,\dots,\beta_{k-1})$ be multi-indices and $c^\alpha = c_1^{\alpha_1}\cdots c_{k-1}^{\alpha_{k-1}}$, $c^\beta = c_1^{\beta_1}\cdots c_{k-1}^{\beta_{k-1}}$ the corresponding monomials in the Chern classes. Write $|\alpha| = 2\alpha_1 + 4\alpha_2 + \cdots + 2(k-1)\alpha_{k-1}$.

Then $c^\alpha \in H^{|\alpha|}(BU(k);\mathbf{Z})$ and there is a map φ_α, unique up to homotopy, from $BU(k)$ into the Eilenberg-MacLane space $K(\mathbf{Z},|k|)$ such that $\varphi_\alpha^\star(L) = c^\alpha$, $L \in H^{|\alpha|}(K(\mathbf{Z},|\alpha|);\mathbf{Z})$ beeing the fundamental class of $K(\mathbf{Z},|\alpha|)$. We note K_1 the space $K(\mathbf{Z},|\alpha|)$.

In the same way c^β corresponds to a map

$$\varphi_\beta : BU(k) \longrightarrow K(\mathbf{Z},|\beta|) := K_2 \ .$$

For any integer N and some integerm we have the following diagram, commutative up to homotopy:

$$\begin{array}{ccccc}
BU(k)^N_{\alpha\beta} & \xrightarrow{\tilde{\varphi}_{\alpha\beta}} & E^N_{\alpha\beta} & \xrightarrow{\tilde{m}} & (K_1 \vee K_2)^N \\
\Big\downarrow \tilde{p} & & \Big\downarrow p & & \Big\downarrow \\
BU(k)^N & \xrightarrow{\varphi_{\alpha\beta}} & (K_1 \times K_2)^N & \xrightarrow{m} & (K_1 \times K_2)_N
\end{array}$$

In this diagram $E_{\alpha\beta} \xrightarrow{p} K_1 \times K_2$ is the principal fibration associated to the cohomology class $L_1 \times L_2 \in H^{|\alpha|+|\beta|}(K_1 \times K_2; \mathbf{Z})$, where L_ℓ is the fundamental class of K_ℓ, $\ell = 1, 2$. The map $\varphi_{\alpha\beta}$ is the product mapping $\varphi_\alpha \times \varphi_\beta$ and $BU(k)_{\alpha\beta}$ is the fiber product of $\varphi_{\alpha\beta}$ and p. The map m denotes multiplication by the integer m on each component K_i of $K_1 \times K_2$. Notice that all space involved can be realized as simplicial complexes and X^N denotes the N skeleton of the simplicial complex X.

Now except the cohomology class $L_1 \times L_2$ the map $K_1 \vee K_2 \to K_1 \times K_2$ has only torsion k-invariants. Hence for some integer m, depending on N, there will be a map $\tilde{m}$ lifting $m.p$ restricted to $E^N_{\alpha\beta}$. The maps $\tilde{p}$ and $\tilde{\varphi}_{\alpha\beta}$ are canonically defined by the fiber product.

Up to $(n+1)$-homotopy type $BU(k)_{\alpha\beta}$ and $K_1 \vee K_2$ are manifolds, even a connected sum in the case of $K_1 \vee K_2$.

Let $\mathrm{Emb}_n U(k)_{\alpha\beta}$ denote the group of cobordism classes of embeddings of closed n-manifolds in $\mathbb{R}^{n+2k}$ with complex structure in the normal bundle and a lifting of the classifying map into $BU(k)_{\alpha\beta}$.

Let $\lambda_k^{\alpha\beta}$ be the bundle induced from λ_k by the fibration

$$BU(k)_{\alpha\beta} \to BU(k) \quad \text{and} \quad MU(k)_{\alpha\beta} \quad \text{its Thom space.}$$

Then by the Thom-Pontrijagin construction

$$\mathrm{Emb}_n U(k)_{\alpha\beta} \simeq \pi_{n+2k} MU(k)_{\alpha\beta} \ .$$

Moreover by a result of Thom [1] there exists an integer $\ell > 0$ such that the cohomology classes ℓL_1 and ℓL_2 are dual to submanifolds in K_1 and K_2 hence to disjoint submanifolds in $K_1 \vee K_2$.

Making $\tilde{m} \cdot \tilde{\varphi}_{\alpha\beta}$ transverse to these submanifolds, by a homotopy, and taking their preimages, we have two disjoint submanifolds V_j, $j = 1, 2$, in $BU(k)_{\alpha\beta}$ which are dual respectively to $m \cdot \ell \tilde{p}^* c^\alpha$ and $m \cdot \ell \tilde{p}^* c^\beta$.

Hence for $[V, f] \in \mathrm{Emb}_n U(k)_{\alpha\beta}$ we have the following situation:

$$\begin{array}{ccc} & & BU(k)_{\alpha\beta} \supset V_1, V_2 \\ & \overset{\tilde{\nu}}{\nearrow} & \downarrow \\ V & \xrightarrow{\ \nu\ } & BU(k) \end{array}$$

disjoint submanifolds of codimension $|\alpha|$ and $|\beta|$ respectively.

Here $\tilde{\nu}$ is the classifying map for the normal bundle to the embedding $f : V \to S^{n+2k}$.

We may assume $\tilde{\nu}$ transverse to V_1 and V_2.

It follows that $\tilde{\nu}^{-1}(V_1) \overset{\text{def}}{=} C_\alpha$ and $\tilde{\nu}^{-1}(V_2) \overset{\text{def}}{=} C_\beta$ are disjoint submanifolds of V dual to ℓmc^α and ℓmc^β respectively.

Observe that the fiber of $\tilde{p}$ is an Eilenberg-MacLane space $K(\mathbf{Z}, |\alpha| + |\beta| - 1)$ and the only obstruction for lifting a map $\varphi : V \to BU(k)$ to $BU(k)_{\alpha\beta}$ is the cohomology class $\varphi^*(c^{\alpha+\beta})$.

Remark: From the construction it is clear that if $[V_0, f_0] = [V_1, f_1]$ in $\mathrm{Emb}_n U(k)_{\alpha\beta}$ and if (W, f) is a cobordism joining (V_0, f_0) to (V_1, f_1) with the required restrictions on the normal bundle, the disjoint submanifolds C^0_α, C^0_β for (V_0, f_0) and C^1_α, C^1_β for (V_1, f_1) are cobordant $C^0_\alpha \sim C^1_\alpha$ and $C^0_\beta \sim C^1_\beta$ by disjoint cobordisms embedded in W. The image of these cobordisms by f will give disjoint chains in $S^{n+2k} \times I$ whose boundaries are respectively $C^0_\alpha \amalg C^1_\alpha$ and $C^0_\beta \amalg C^1_\beta$.

For $[V, f] \in \mathrm{Emb}_n U(k)_{\alpha\beta}$ define $L_{\alpha\beta}(V, f) \in \mathbf{Z}$ to be the linking number of $f(C_\alpha)$ and $f(C_\beta)$ in S^{n+2k}.

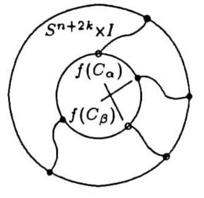
Proposition: *$L_{\alpha\beta}(V, f)$ depends only on the cobordism class of (V, f) in $\mathrm{Emb}_n U(k)_{\alpha\beta}$ and the map $\mathrm{Emb}_n U(k)_{\alpha\beta} \xrightarrow{L_{\alpha\beta}} \mathbf{Z}$ is a homomorphism into $\mathbf{Z}$.*

Observe that $L_{\alpha\beta}$ can be non zero only if

$$n - |\alpha| + n - |\beta| = n + 2k - 1\ , \quad \text{that is} \quad n - 2k + 1 = |\alpha| + |\beta|\ .$$

The proof of the proposition relies on the definition of the linking number as intersection number of chains which bound $f(C_\alpha)$ and $f(C_\beta)$ in D^{n+2k+1}.

By the remark above this intersection number will not change if we modify (V, f) by a cobordism alowed for $\mathrm{Emb}_n U(k)_{\alpha\beta}$.



Example: Consider mappings $f : S^8 \longrightarrow MU(3)$ and $g : S^{10} \longrightarrow MU(3)$ such that $f^\star(Uc_1)$ and $g^\star(Uc_2)$ are non zero. Here $U \in H^6(MU(3);\mathbf{Z})$ denotes the Thom class of the canonical bundle over $BU(3)$.

To f corresponds a submanifold V_1 of dimension 2 in D^8 with normal Chern class $c_1 \neq 0$. Similarly there corresponds to g a submanifold V_2 of dimension 4 in D^{10} with normal Chern class $c_2 \neq 0$.

Then $V = V_1 \times S^9 \amalg S^7 \times V_2$ will be a manifold of dimension 11 and $F : V \longrightarrow S^{17}$ defined as the mapping $i_1 \times id \amalg id \times i_2$ from $V_1 \times S^9 \amalg S^7 \times V_2$ to $D^8 \times S^9 \cup S^7 \times D^{10} = \partial(D^8 \times D^{10}) = S^{17}$, will be an embedding. Here i_1, i_2 are the inclusions of V_1, V_2 into the interior of D^8 and D^{10} respectively.

By construction $L_{\alpha\beta}(V, F)$ is defined if $\alpha = (1, 0)$ and $\beta = (0, 1)$ and moreover $L_{\alpha\beta}(V, F) \neq 0$.

Notice that if $[V, f] \in \pi_{n+2k}MU(k)$ is an element such that in V the class $\nu^\star c^{\alpha+\beta}$ is zero then $[V, f]$ is in the image of the natural homomorphism

$$h : \pi_{n+2k}MU(k)_{\alpha\beta} \longrightarrow \pi_{n+2k}MU(k)$$

induced by $\tilde{p} : BU(k)_{\alpha\beta} \longrightarrow BU(k)$, see page 7.

Theorem: *The kernel of the homomorphism*

$$h : \pi_{n+2k}MU(k)_{\alpha\beta} \longrightarrow \pi_{n+2k}MU(k)$$

is a finite abelian group.

Proof. Since $(Kerh) \otimes Q = Ker(h \otimes id)$ and $kerh$ is finite abelian if $(Kerh) \otimes Q = 0$, if will be sufficent to show that

$$h \otimes id : \pi_{n+2k} MU(k)_{\alpha\beta} \otimes Q \longrightarrow \pi_{n+2k} MU(k) \otimes Q \ .$$

is injective.

The fiber $K(\mathbf{Z}, |\alpha| + |\beta| - 1)$ of $BU(k)_{\alpha\beta} \xrightarrow{\tilde{p}} BU(k)$ is a rational homotopy sphere $S^{|\alpha|+|\beta|-1}$ and the Gysin sequence of the fibration shows that $p^\star : H^\star(BU(k); Q) \to H^\star(BU(k)_{\alpha\beta}; Q)$ is surjective with kernel the ideal generated by $c^{\alpha+\beta}$. Indeed, $H^\star(BU(k); Q)$ is a polynomial algebra.

As in [5] we have the following commutative diagram, where the lines are exact.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_\star(\bigvee_{\alpha' \in A'} S_{\alpha'}) \otimes Q & \longrightarrow & \pi_\star MU(k)_{\alpha\beta} \otimes Q & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \pi_\star(\bigvee_{\alpha' \neq 0} S_{\alpha'}) \otimes Q & \longrightarrow & \pi_\star MU(k) \otimes Q & \longrightarrow & Q & \longrightarrow & 0
\end{array}$$

Here α' is a multi index $(\alpha'_1, \ldots, \alpha'_{k-1})$, $A' = \left\{\alpha' \mid c^{\alpha'} \text{ is not divisible by } c^{\alpha+\beta}\right\}$ and $S_{\alpha'}$ is a sphere of dimension $|\alpha'|$.

The vertical arrow on the left is injective because $\bigvee_{\alpha' \in A'} S_{\alpha'}$ is a retract of $\bigvee_{\alpha' \neq 0} S_{\alpha'}$.

By the five Lemma the middle vertical arrow is also injective. □

Corollary: *If $[V, f] \in \pi_{n+2k} MU(k)$ is in the image of the homomorphism $\pi_{n+2k} MU(k)_{\alpha\beta} \to \pi_{n+2k} MU(k)$ — that is if the classifying map of the normal bundle of a representative, (V, f), lifts to $BU(k)_{\alpha\beta}$ — then $L(V, f)$ is defined and if it's non zero we have $[V, f] \neq 0$.*

Example: Let $W^n \subset \mathbb{C}^{n+k}$ be a complex affine variety with isolated singular point at $0 \in \mathbb{C}^{n+k}$.

Assume that W admits a desingularization $\overline{W} \xrightarrow{\pi} W$ of $0 \in W$ with residue variety $Z = \pi^{-1}(0)$ of $\dim_{\mathbb{C}} Z < n-k$. Recall that by definition $\pi/\overline{W} - Z$ is a homeomorphism onto $W - (0)$. Then we can show that the classifying map for the normal bundle of its link $V \subset S^{2(n+k)-1}$ lifts to $BU(k)_{\alpha\beta}$ whenever $|\alpha| + |\beta| \geq 2(n-k)$. To see this consider the classifying map $\overline{\nu}$ for the stable normal bundle of $\overline{W}$, say $\overline{\nu} : \overline{W} \to BU(k+m)$. Because $\overline{W} - Z$ is homeomorphic to $W - (0)$, $\overline{\nu} \mid \overline{W} - Z$ is homotopic to $\sigma \cdot \nu \mid W - (0)$ where ν is the classifying map for the normal bundle of $W - (0) \subset \mathbb{C}^{n-k}$ and $\sigma : BU(k) \to BU(k+m)$ is the suspension induced by the natural inclusion of $U(k)$ into $U(k+m)$.

By restriction to a sufficently small neighborhood of 0 in $\mathbb{C}^{n+k}$ we can assume that Z is a deformation retract of $\overline{W}$. That means that for any map ψ : $BU(k+m) \to K_1 \times K_2$, the map $\psi \cdot \overline{\nu}$ lifts to $E_{\alpha\beta}$ if $|\alpha| + |\beta| \geq 2(n-k)$ just because all obstructions vanish trivially. Moreover this lift is unique up to homotopy.

Take $\psi : BU(k+m) \to K_1 \times K_2$ such that $\psi \cdot \sigma = \varphi_{\alpha\beta}$. This is always possible because the first k Chern classes correspond under the homomorphism induced by the suspension $\sigma : BU(k) \to BU(k+m)$. Then $\psi \cdot \overline{\nu}$ is homotopic to $\psi \cdot \sigma\nu = \varphi_{\alpha\beta}$. Hence if $\psi \cdot \overline{\nu}$ lifts to $E_{\alpha\beta}$ the same holds for $\varphi_{\alpha\beta} \cdot \nu$. But this is equivalent to a lifting of ν into $BU(k)_{\alpha\beta}$. It follows that for $V \underset{i}{\subset} S^{2(n+k)-1}$ the invariant $L_{\alpha\beta}(V, i)$ is well defined. As before V is the link of the singulalrity.

Let us look at the example of the Segre cone.

$$W = \left\{ \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \in \mathbb{C}^6 \mid \operatorname{rank} \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \leq 1 \right\}$$

$\dim_{\mathbb{C}} W = 4$, and W has an isolated singularity at $0 \in \mathbb{C}^6$.

We describe a desingularization as follows:

$\overline{W} = E\,3\lambda$ the total space of the vector bundle which is the sum of three copies of the canonical line bundle λ on the complex projective line $\mathbb{P}^1$.

$$\begin{array}{ccccc}
\overline{W} & & & & \\
\| & & & & \\
E\,3\lambda & \xrightarrow{\tilde{\Delta}} & E\lambda \times E\lambda \times E\lambda & \longrightarrow & \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 \\
\downarrow & & \downarrow & & \\
\mathbb{P}^1 & \xrightarrow[\Delta]{} & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 & &
\end{array}$$

3λ is induced from the vector bundle $E\lambda \times E\lambda \times E\lambda \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by the diagonal map $\Delta(x) = (x, x, x)$. By definition $E\lambda = \{(x, v) \in \mathbb{P}^1 \times \mathbb{C}^2 \mid \mathrm{v} \in \mathrm{X}\}$.

Therefore we have a canonical projection of $E\lambda$ onto $\mathbb{C}^2$ which is a homeomorphism of the complement of the zero section onto $\mathbb{C}^2 - (0)$.

We define the mapping $E\lambda \times E\lambda \times E\lambda \longrightarrow \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2$ as the product of three times this projection. Its composition with $\tilde{\Delta}$ is a map $\pi : \overline{W} \to \mathbb{C}^6$ whose image is W. More precisely

$$\overline{W} = \{([x, y], u_1, v_1, u_2, v_2, u_3, v_3) \in \mathbb{P}^1 \times \mathbb{C}^6 \mid \mathrm{yv_i} = \mathrm{xu_i} \;\; \mathrm{i} = 1, 2, 3\}$$

and $\pi([x, y], u_1, v_1, \ldots, u_3, v_3) = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$. Observe that $\pi^{-1}(0) = \mathbb{P}^1$ and outside of $\mathbb{P}^1$ π is a homeomorphism.

So in this case the residue variety is $\mathbb{P}^1 = \mathrm{Z}$. Take $\alpha = (1)$, $\beta = (1)$ then we have $|\alpha| + |\beta| = 4$. But $n - k = 2$, hence $|\alpha| + |\beta| \geq 2(n - k)$ and $\dim_{\mathbb{C}} Z = 1 < n - k$. The invariant $L_{\alpha\beta}$ is well defined. From $T\overline{W} \approx T\mathbb{P}^1 \oplus 3\lambda$ we find easily that the first Chern class of the stable normal bundle of $\overline{W}$ is $c^1(\overline{\nu}) = 5t$ where $t \in H^2(\mathbb{P}^1; \mathbf{Z})$ is the orientation class of $\mathbb{P}^1$. It is dual to a point in $\mathbb{P}^1$.

This implies that $c^1(\overline{\nu})$ is dual to 5 fibers of $\overline{W} \to \mathbb{P}^1$ or equivalently to 5 fibers in the associated sphere bundle $S\,3\lambda$. But $\pi : S\,3\lambda \to V \subset S^{11}$ describes exactly the embedding of the link of W. Any two fibers of $S\,3\lambda$ are 5-dimensional spheres whose images by π are linked in S^{11}. In fact these images bound 6-dimensional discs in D^{12} which intersect exactly in the point $0 \in D^{12} \subset \mathbb{C}^6$. Moreover the intersection is transverse because they are the graphs of two linear maps of the type $\ell = \begin{pmatrix} \frac{x}{y} & 0 & 0 \\ 0 & \frac{x}{y} & 0 \\ 0 & 0 & \frac{x}{y} \end{pmatrix}$ and $\ell' = \begin{pmatrix} \frac{x'}{y'} & 0 & 0 \\ 0 & \frac{x'}{y'} & 0 \\ 0 & 0 & \frac{x'}{y'} \end{pmatrix}$ with $\frac{x}{y} \neq \frac{x'}{y'}$. Here $\frac{x}{y}$ and $\frac{x'}{y'}$ correspond to the points in $\mathbb{P}^1$ on which the fibers are considered. Hence the intersection number is $+1$.

More generally consider $r + s$ distinct fibers of $S\,3\lambda \rightarrow \mathbb{P}^1$ and chose the union of r of them to be the manifold C_1 and the union of the remaining ones to be C_2. Then the linking number of $\pi(C_1)$ and $\pi(C_2)$ in S^{11} is $s \cdot r$.

In our example the disjoint manifolds we have to consider are dual to some positive multiple of $c_1(\nu)$. That is they consist in bunches of fibers. It follows that the linking number of their images in S^{11} is non zero and therefore $[V, i]$ is non zero in $\pi_{11}MU(2)$.

It would be interesting to find examples of isolated singular points of varieties which are not of the type of the cone on a projective variety and whose link is non cobordant to zero.

References

[1] R. Thom, *Quelques propriétés globales des variétés differentiables*, Comm. Math. Helv. 28 (1954), 17-86.

[2] R. Wells, *Cobordism of immersions*, Topology 5, (1966), 281-293.

[3] L. Pontrijagin, *Smooth manifolds and their applications in homotopy theory*, Trudy Mat. instituta, (1955), V. 45.

[4] E. Rees and E. Thomas, *Cobordism obstructions to deforming isolated singularities*, Math. Ann. 232 (1978), 33-53.

[5] O. Burlet, *Cobordismes de plongements et produits homotopiques*, Comm. Math. Helv. 46 (1971), 277-288.

[6] H. Shiga, *Notes on links of complex isolated singular points*, Kodai Math. J. 3 (1980), 44-47.

Institut of Mathèmatiques
Université de Lausanne
CH - 1015 Lausanne - Dorigny
Suice