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DIFFERENTIAL FORMS AND VECTOR FIELDS WITH A MANIFOLD OF SINGULAR POINTS.

M. Zhitomiriskii*

Vector fields vanishing at each point of a submanifold form a codim ∞ set in the space of all vector fields on a manifold. Nevertheless, these vector fields appear typically in various problems, especially those related to singularities of differential forms. We give a few examples in Sections 1-4.

It is well known that a germ of a vector field at a singular isolated point is not stable with respect to the C^1-equivalence. Let $\tilde{V}_n^k$ be the space of vector fields on R^n vanishing on some (non-fixed) codimension k submanifold. Are there stable germs in $\tilde{V}_n^k$ (of course, if to consider only perturbations in $\tilde{V}_n^k$ in the definition of the stability)? The answer (surprisable from the first point of view) is yes: for certain conditions on (n, k) a generic germ of a vector field of $\tilde{V}_n^k$ is stable. We study stability and give normal forms in Sections 5-6 (Section 6 is devoted to the case $k = 1$ which leads to the classification of pairs of a vector field and a hypersurface). In Section 7 we give some open questions.

All objects considered below are assumed to be of class C^∞ unless it is explicitly mentioned otherwise.

1. Vector fields with a manifold of singular points related to typical singularities of closed differential 2-forms (see [M,R,AG,GT])

Let μ be a generic differential 2-form on a manifold M of dimension $2k$. Near a generic point $\alpha \in M$ the 2-form μ defines a symplectic structure since $\mu^k|_\alpha \neq 0$. The degeneration $\mu^k|_\alpha = 0$ holds on a hypersurface $S \subset M$ (S might be also a

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stratified codim 1 submanifold). At a generic point $\alpha \in S$ the kernel $Ker\mu|_\alpha$ is 2-dimensional, and the intersection of $Ker\mu|_\alpha$ and $T_\alpha S$ defines a direction in $T_\alpha S$. Therefore, we have a field of directions on S. This field of directions is singular at points $\beta \in S$ such that $Ker\mu|_\beta \subset T_\beta S$, or, equivalently $(\mu^{k-1}|_S)|_\beta = 0$. The points β with this property form a codim 2 submanifold $S_1 \subset S$. One can define a smooth field of directions in a neighbourhood of a point of S_1 coinciding with $Ker\mu|_\alpha \cap T_\alpha S$ at a generic point α. To define such a field of directions take a volume form δ on S and introduce a vector field v on S by a relation

$$\delta(v, Y_1, ..., Y_{2k-2}) = \mu^{k-1}(Y_1, ..., Y_{2k-2}),$$

where $Y_1, ..., Y_{2k-2}$ are arbitrary vector fields on S. It is clear that the vector field v vanishes at every point of S_1, and that its direction at every point $\alpha \notin S_1$ coincides with the intersection of $Ker\mu|_\alpha$ and $T_\alpha S$. So, v generates a field of directions on S, and has a codim 2 manifold of singular points. The field of directions generated by v is an invariant of the singularity (the field v depends on the choice of δ, the field of directions does not).

2. Vector fields with a manifold of singular points related to typical singularities of Pfaffian equations on an odd-dimensional manifold (see [M,Z1,Z2])

The following example can be considered as the odd-dimensional analogus of the previous one. We consider singularities of Pfaffian equations, i.e. differential 1-forms defined up to the multiplication by a non-vanishing function.

Let ω be a generic differential 1-form on a manifold M of dimension $2k+1$. Near a generic point $\alpha \in M$, ω defines a contact structure since $\omega(d\omega)^k|_\alpha \neq 0$. The degeneration $\omega(d\omega)^k = 0$ holds on a hypersurface $S \subset M$ (S might be also a stratified codim 1 submanifold). At a generic point $\alpha \in S$ the kernel $Ker\ \omega(d\omega)^{k-1}$ is 2-dimensional, and the intersection of $Ker\ \omega(d\omega)^{k-1}|_\alpha$ and $T_\alpha S$ defines a direction in $T_\alpha S$. Therefore, we have a field of directions on S. This field of directions is singular at points $\beta \in S$ such that $Ker\ \omega(d\omega)^{k-1}|_\beta \subset T_\beta S$

or, equivalently, $(\omega(d\omega)^{k-1}|_S)|_\beta = 0$. The points β with this property form a codim 2 submanifold $S_1 \subset S$. One can define a smooth field of directions in a neighbourhood of a point of S_1 coinciding with $Ker\ \omega(d\omega)^{k-1}|_\alpha \cap T_\alpha S$ at generic points $\alpha \in S$. To define such a field of directions take a volume form δ on S and introduce a vector field v on S by a relation

$$\delta(v, Y_1, ..., Y_{2k-1}) = (\omega(d\omega)^{k-1})|_S(Y_1, ..., Y_{2k-1}),$$

where $Y_1, ..., Y_{2k-1}$ are arbitrary vector fields on S. It is clear that the vector field v vanishes at every point of S_1, and that its direction at every point $\alpha \notin S_1$ coincides with the intersection of $Ker\ \omega(d\omega)^{k-1}|_\alpha$ and $T_\alpha S$. So, v generates a field of directions on S and has a codim 2 manifold of singular points. The field of directions generated by v is an invariant of the singularity of the Pfaffian equation $\omega = 0$ (the field v depends on the choice of δ, the field of directions does not).

3. Vector fields with a manifold of singular points related to typical singularities of Pfaffian equations on an even-dimensional manifold (see [M,Z1,Z3])

Given a generic differential 1-form ω on a manifold M of dimension $2k$, introduce a vector field v satisfying the following equation

$$\delta(v, Y_1, ..., Y_{2k-1}) = \omega(d\omega)^{k-1}(Y_1, ..., Y_{2k-1}).$$

Here, δ is a volume form on M, and $Y_1, ..., Y_{2k-1}$ are arbitrary vector fields on M. The field of directions generated by v is invariantly related to the Pfaffian equation $\omega = 0$. It is singular at points where v vanishes, i.e. at points α such that $\omega(d\omega)^{k-1}|_\alpha = 0$. These points form a codim 3 (maybe stratified) submanifold (this is a non-trivial fact, see [M]).

4. Vector fields with a manifold of singular points related to Pfaffian systems

A field of directions on a manifold M of dimension n can be given as the intersection of the kernels of $n-1$ differential 1-forms $\omega_1, \dots, \omega_{n-1}$. So, every Pfaffian system $\omega_1 = \dots = \omega_{n-1} = 0$ defines a field of directions on M. This field is generated by a vector field v given by the equation

$$\delta(v, Y_1, \dots, Y_{n-1}) = \omega_1\omega_2\dots\omega_{n-1}(Y_1, \dots, Y_{n-1}),$$

where δ is a fixed volume form on M. The field v vanishes at the points where the 1-forms $\omega_1, \dots, \omega_{n-1}$ are dependent, these points form a codimension 2 submanifold.

5. Stable germs of vector fields with a codim ≥ 2 manifold of singular points

The vector fields defined in Sections 1-4 are not invariantly related to the singularities of differential forms, they depend on the choice of the volume form δ. On the other hand, generated by them fields of directions do not depend on the choice of δ, they are the invariants of the singularities. This means that an interesting question is the classification of fields of directions generated by vector fields with a manifold of singular points, or equivalently, the orbital classification of vanishing on a submanifold vector fields.

Definition. Two germs of vector fields v_1 and v_2 are said to be C^r *orbitally equivalent* ($r \leq \infty$) if there exists a local diffeomorphism Φ of the class C^r and a non-vanishing function germ H of the same smoothness class such that $\Phi_* v_1 = H v_2$. Two germs are called almost C^∞ orbitally equivalent if they are C^r orbitally equivalent for any $r < \infty$.

The basic definition in any classification problem is that of stability. Local stability means that the local structure does not change under a small perturbation of a globally defined object (it is a local property). Defining stability one has to fix a class of all possible perturbations. We are interested in the stability within the class of vector fields vanishing on some submanifold (the submanifold is not fixed, but its codimension is).

Let $\tilde{V}_n^k$ be the space of vector fields on R^n vanishing on some codim k submanifold of R^n, $n \geq 2$. By V_n^k we denote the space of germs at singular points of vector fields of $\tilde{V}_n^k$.

Definition. A germ $v \in V_n^k$ of a vector field $V \in \tilde{V}_n^k$ is called *C^r-stable in* V_n^k if for any neighbourhood U of the source point of the germ there exists a neighbourhood $\tilde{U}$ of V such that for any vector field $V_1 \in \tilde{U} \cap \tilde{V}_n^k$ there exists a point of the neighbourhood U such that the germ of V_1 at this point is C^r orbitally equivalent to the germ v. A germ is called almost C^∞ stable in V_n^k if it is C^r stable in V_n^k for any $r < \infty$.

Note. From now we will use the notion of stability meaning stability in V_n^k, and the notion of equivalence meaning the orbital equivalence.

Theorem 1. *If $k > \frac{n+1}{2}$ then none of germs of V_n^k is almost C^∞ stable.*

Theorem 2. *If $2 \leq k \leq \frac{n+1}{2}$ then a generic germ of V_n^k is almost C^∞ stable and almost C^∞ equivalent to one and only one of the germs*

$$(\lambda_1 + y_1)x_1\frac{\partial}{\partial x_1} + \dots + (\lambda_{k-1} + y_{k-1})x_{k-1}\frac{\partial}{\partial x_{k-1}} + x_k\frac{\partial}{\partial x_k}. \qquad (1)$$

Here $x_1, \dots, x_k$ and $y_1, \dots, y_{n-k}$ are the coordinates on R^n (the coordinates x are transversal to the manifold of singular points $\{x_1 = \dots = x_k = 0\}$, the parameters $\lambda_1, \dots, \lambda_{k-1}$ are the moduli of normal form (1)). It is interesting that the moduli do not prevent stability, we shall explain this in the proof of Theorem 2. The genericness conditions in Theorem 2 are as follows.

Let $v \in V_n^k$ be the germ of a vector field $V \in \tilde{V}_n^k$ at a singular point $0 \in R^n$. Let $\tilde{S}$ be the set of singular points of V, S be the germ of this set at the origin. At every singular point $\alpha \in \tilde{S}$ the spectrum of the V's linearization contains

$n-k$ zero eigenvalues (corresponding to the directions tangent to $\tilde{S}$). Denote by $\lambda_1(\alpha), ..., \lambda_k(\alpha)$ the others k eigenvalues. Let $\lambda_j = \lambda_j(0)$. The first genericness condition is that these eigenvalues are different and form a non-resonant tuple, i.e. for any integer non-negative numbers $m_1, ..., m_k$ such that $m_1 + ... + m_k \geq 2$ and every $j = 1, ..., k$

$$\lambda_j \neq m_1\lambda_1 + ... + m_k\lambda_k. \tag{2}$$

Condition (2) implies that $\lambda_i(0) \neq 0, i = 1, ..., k$. Multiplying v by a nonvanishing function we can make $\lambda_k(\alpha)$ be equal to 1. Then the function germ

$$f_v : S \to R^{k-1}, f_v(\alpha) = (\lambda_1(\alpha), ..., \lambda_{k-1}(\alpha)) \tag{3}$$

is an invariant of the field of directions generated by v. The second genericness condition in Theorem 2 is that the origin is a non-singular point of f_v, i.e.

$$rank f_v'(0) = k - 1. \tag{4}$$

Theorem 3. *A germ of V_n^k is C^∞ stable if the eigenvalues $\lambda_1, ..., \lambda_k$ form a non-resonant tuple and lie on one side of a straight line passing through the origin of the complex plane (i.e. the tuple of the eigenvalues belongs to the Poincaré domain, see [AI]).*

In the proofs of the theorems we follow the notations given after the formulation of Theorem 2.

Proof of Theorem 1: The function f_v maps the $(n-k)$–dimensional manifold S to R^{k-1}. If $k > \frac{n+1}{2}$ then $n-k < k-1$, and f_v is not stable with respect to the R-equivalence (see [AVG], moreover, the image of f_v is a functional modulus). The germ f_v is invariantly related to v (i.e., if v_1 is equivalent to v_2 then f_{v_1} is R-equivalent to f_{v_2}), and Theorem 1 follows.

Sketch of the proof of Theorem 2: Let $S = \{x_1 = ... = x_k = 0\}$, and $y_1, ..., y_k$ be the coordinates on S. Consider v of the form

$$v = \sum_{i,j} a_{i,j}(y)x_i\frac{\partial}{\partial x_j} + \sum_{i,l} b_{i,l}(y)x_i\frac{\partial}{\partial y_l} + o(||x||). \tag{5}$$

It follows from the genericness conditions that a coordinate transformation

$$x \to T(y)x, \; y \to y + B(y)x$$

with a suitable non-degenerated $k \times k$ matrix $T(y)$ and a suitable $(n-k) \times k$ matrix $B(y)$ reduces (5) to the form

$$v = \sum_{i=1}^{k} (\lambda_i + h_i(y)) x_i \frac{\partial}{\partial x_i} + o(||x||), \tag{6}$$

where $h_i(0) = 0$. Multiplying v by a function we can reduce λ_k to 1 and $h_k(y)$ to 0. Condition (4) allows to change the coordinates y ($y \to \Phi(y)$) to reduce $h_i(y)$ to y_i, $i = 1, ..., k-1$. After this

$$v = \sum_{i=1}^{k-1} (\lambda_i + y_i) x_i \frac{\partial}{\partial x_i} + o(||x||). \tag{7}$$

Arguing as in the proof of the Poincare-Dulak theorem (see [AI]) and using condition (2) one can prove that for given arbitrary $p < \infty$ there exists a transformation of the form

$$x \to x + \sum_{|\alpha|=2}^{p} \varphi_\alpha(y) x^\alpha,$$

$$y \to y + \sum_{|\alpha|=2}^{p} \psi_\alpha(y) x^\alpha, \tag{8}$$

bringing (7) to the form

$$v = \sum_{i=1}^{k-1} (\lambda_i + y_i) x_i \frac{\partial}{\partial x_i} + o(||x||^p). \tag{9}$$

The condition $\lambda_i \neq 0$, $i = 1, ..., k$ (following from (2)) implies the hyperbolicity of v on S and a possibility to "kill" the terms $o(||x||^p)$ by a change of the coordinates of the class C^r, where r depends on p and $r \to \infty$ as $p \to \infty$ (we use results by G.Belitskii, see [Z1,Ch.2,Section 8, B1,B2], see also [H]). Therefore (7) is almost C^∞ eqivalent to (1).

It remains to prove that germ (1) is almost C^∞ stable. Let V be a vector field of $\tilde{V}_n^k$ with the germ v at the origin. Take a small perturbation $V_1 \in \tilde{V}_n^k$

of V. Let $\tilde{S}_1$ be the manifold of singular points of V_1. It follows from (4) that there exists a point $\alpha \in \tilde{S}_1$ close to the origin $0 \in \tilde{S}$ such that $f_{V_1}(\alpha) = f_{V_2}(0)$. Then, as we have proved the germ of V_1 at α is reducible to (1), the stability follows.

Sketch of the proof of Theorem 3. We follow the proof of Theorem 2, but instead of transformations (8) we use transformations of the form

$$x \to x + \sum_{|\alpha|=2}^{\infty} \varphi_\alpha(y)x^\alpha,$$

$$y \to y + \sum_{|\alpha|=2}^{\infty} \psi_\alpha B(y)x^\alpha. \tag{10}$$

Using the condition that the tuple $(\lambda_1, ..., \lambda_k)$ lies in the Poincare domain, and arguing as in the proof of the Poincare-Dulac theorem one can prove that a suitable transformation of form (10) with functions $\varphi_\alpha(y), \psi_\alpha(y)$ defined in a common neighbourhood of the origin brings germ (7) to the form

$$v = \sum_{i=1}^{k-1}(\lambda_i + y_i)x_i\frac{\partial}{\partial x_i} + \tau(x, y), \tag{11}$$

where τ is a flat on S germ, i.e., all the coefficients of the vector field τ vanish on S along with all their derivatives. Now, the hyperbolicity of v on S implies the reducibility of (11) to (1) by a C^∞ transformation (we use results by G.Belitskii, see [Z1,Ch.2,Section 8, B1,B2], see also [H]).

6. Stable germs of vector fields with a hypersurface of singular points

The orbital classification of vector fields of V_n^1 reduces to the classification of pairs consisting of a vector field and a hypersurface (every $v \in V_n^1$ has the form fv_1, where v_1 is a new vector field, and f is a function germ). The degeneration $v_1(0) = 0, f(0) = 0$ has codimension $n + 1$, so it is not typical. For typical singularities $v_1(0) \neq 0$, and we may assume that $v_1 = \frac{\partial}{\partial x}$, where

$x \in R$, $y \in R^{n-1}$ is a suitable C^∞ coordinate system. Now the classification problem reduces to the classification of functions $f : R^n \to R$ with respect to transformations preserving the field of directions generated by $\frac{\partial}{\partial x}$. Such transformations have the form

$$x \to \Phi(x, y), y \to \Psi(x, y).$$

Consider at first transformations of the form

$$x \to \Phi(x, y).$$

We may consider y as a tuple of parameters, then the problem is to find a versal unfolding of a function f in one variable x. This is a well-known problem (see [AVG]), and the answer is

$$f = x^p + a_1(y) + a_2(y)x + ... + a_{p-1}(y)x^{p-2}$$

for the degeneration of codimension p

$$(f(0) = \frac{\partial f}{\partial x}(0) = ... = \frac{\partial^{p-1} f}{\partial x^{p-1}}(0) = 0). \tag{12}$$

For typical singularities $p \leq n$, therefore a typical singularity has a normal form

$$v = (x^p + a_1(y) + a_2(y)x + ... + a_{p-1}(y)x^{p-2})\frac{\partial}{\partial x}. \tag{13}$$

Degeneration (12) has codimension p, and the degeneration

$$(12) + (\text{ } rank \text{ } \frac{\partial(a_1, ..., a_{p-1})}{\partial(y_1, ..., y_{n-1}} < p - 1)$$

has codimension $p + (n - p + 1) = n + 1$, therefore for typical singularities with degeneration (12)

$$rank \text{ } \frac{\partial(a_1, ..., a_{p-1})}{\partial(y_1, ..., y_{n-1})} = p - 1.$$

This condition allows us to reduce (13) to

$$v = (x^p + y_1 + y_2 x + ... + y_{p-1}x^{p-2})\frac{\partial}{\partial x} \tag{14}$$

by a transformation $y \to \Psi(y)$.

We have proved the following

Theorem 4. *A generic vector field* $V \in \tilde{V}_n^1$ *is locally equivalent to a normal form* (14). *Any stable germ* $v \in V_n^1$ *is* C^∞ *equivalent to one and only one of germs* (14).

7. Comments and open questions

1) Classification of singularities of closed differential 2-forms on M^{2k} see in [M,R,AG].

2) Classification of singularities of differential 1-forms and Pfaffian equations see in [M,Z1,Z2,Z3].

3) The fact that the vector fields defined in Sections 1-3 have a manifold of singular points is not their only specific. It is easy to see that the sum of the eigenvalues of these vector fields at every singular point is 0. If a singular point was isolated then this would not mean that a vector field with this property is divergence-free. The vector fields considered in Sections 1-3 are divergence-free (see [R,Z1]). Classification and stable normal forms of vector fields vanishing on a codim 2 or 3 submanifolds and having the zero sum of the eigenvalues at every singular point (see in [M,R] (codim = 2) and in [Z1] (codim = 2,3)).

4) Is it true that the fields of directions defined in Sections 1-4 are the only invariants of the corresponding singularities? For degenerations of small codimension it is true (see [Z1]).

5) We do not know how to describe all the fields of directions which can be obtained by the constructions of Sections 1-4 (they have the mentioned specific, but they might have an extra specific).

6) It seems to be true that if $2 \le k \le \frac{n+1}{2}$ then a germ $v \in V_n^k$ is almost C^∞ stable if and only if conditions (2) and (4) hold true (if some of the eigenvalues are equal then normal form differs slightly from (1)), but this has not been proved.

7) If not all of the eigenvalues $\lambda_1, ..., \lambda_k$ are real, then normal form (1) holds in

special complex coordinates.

8) What are the normal forms of resonant germs of V_n^k, $2 \leq k \leq \frac{n+1}{2}$?

9) It seems to be true that one can change "if" by "if and only if" in the formulation of Theorem 3.

10) Degeneration (12) with $p = 2$ (resp. p=3) corresponds to the fold (resp. cusp) singularity of mappings $(R^n, 0) \to (R^n, 0)$. More deep degenerations ($p \geq 4$) correspond to the generalized Whitney singularities, see [AVG]. A generic germ of V_n^1 reduces to (13) with $p = 1$, i.e., to the normal form $x\frac{\partial}{\partial x}$, the genericness condition is the transversality of the hypersurface of singular points and the field v_1.

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Technion,
Department of Mathematics,
32000 Haifa, Israel