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SUBMERSIONS, MAPS OF CONSTANT RANK, SUBMERSIONS WITH FOLDS, AND IMMERSIONS

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1. Introduction

The aim of these notes is to give global characterizations of maps of constant rank and submersions with definite folds by means of their Stein factorizations. We also stablish a relation between the problem of existence of these maps and the problem of extensions of immersions. A theorem on the existence of submersions with definite folds is proved.

Let M and N be connected differentiable manifolds of dimensions m and n. Assume that M is closed and $m \neq n$. Let $f : M \longrightarrow N$ be differentiable. The Stein factorization of f is given by the commutative diagram

$$\begin{array}{ccc} M & & \\ q\downarrow & \overset{f}{\searrow} & \\ W & \overset{g}{\longrightarrow} & N, \end{array}$$

where W is the quotient space of M by the equivalence relation that identifies to a point each connected component of each fiber of f, q is the quotient projection map onto W and g is the induced map from W to N.

As an example, let N be $\mathbb{R}$ and let $f : M \longrightarrow \mathbb{R}$ be a Morse function. Then W is the Reeb graph of f.

Let f be a submersion. Then the local form of f as a projection is known. Its Stein factorization provides a global characterization of f as the composite of the projection of a bundle with connected fiber with a covering map.

When f is a map of constant rank k, $0 < k < m, n$, its local form is also known. f is locally the composite of a projection with an embedding. In this case, the Stein factorization of f also gives a global version of this local characterization of f.

Let now f be a submersion with definite folds. Then f is a quotient of a submersion $h : E \longrightarrow N$, where E is an m-manifold with boundary ∂E and the restriction $h\mid_{\partial E}$ has rank $n-1$. There are several recent results in the study of submersions with folds [1], [3], [4], [6]. We also prove a result in this direction.

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2. Maps of constant rank

Let $f : M \longrightarrow N$ be a submersion. Then $m > n$, f is onto N and N must also be closed. In this case g is a local homeomorphism; thus, W is a closed n-manifold with the differentiable structure induced by g. It follows that g is an immersion. As $\dim N = \dim W$ and g is onto then g is a covering map. Also q is a submersion. It follows that $M \xrightarrow{q} W$ is a differentiable bundle with connected fiber. Conversely, if $M \xrightarrow{q} W$ is a bundle with connected fiber and $W \xrightarrow{g} N$ is a differentiable covering map then $f = g \circ q$ is a submersion.

We have many non-trivial examples. When N is simply-connected g is a diffeomorphism. Take $N = S^4$. Then there is an S^7-bundle over S^4 that is not orthogonal [7]. If $N = S^1$ then W is diffeomorphic to S^1 and $g : W \longrightarrow S^1$ is an r-fold covering, where $r =$ number of connected components of a fiber of f. In this case the bundles over S^1 are determined by $\pi_0(\text{Diff } F)$, where F is the typical fiber of the bundle $M \xrightarrow{q} W$. We have for example the non-orthogonal S^6-bundle over S^1 obtained from $S^6 \times [0,1]$ under identification of $S^6 \times \{0\}$ with $S^6 \times \{1\}$ by a diffeomorphism not isotopic to an orthogonal one [2].

Now let $f : M \longrightarrow N$ be of constant rank k, $0 < k < m, n$. In this case, $f(M)$ is locally given by submanifolds of dimension k of N. Thus $g : W \longrightarrow N$ is locally given by topological embeddings with submanifolds as images. We can give a differentiable structure to W so that it becomes a smooth manifold and that g becomes an immersion. Since W is compact, it becomes a closed k-dimensional manifold. Here also $M \xrightarrow{q} W$ is a differentiable bundle with connected fiber. Thus we have proved the following proposition.

Proposition *Let $f : M \longrightarrow N$ have constant rank k, $0 < k < m, n$. Then W is a closed k-manifold, $M \xrightarrow{q} W$ is a bundle with connected fiber and $g : W \longrightarrow N$ is an immersion, where $f = g \circ q$ is the Stein factorization of f. The converse is also true.*

3. Submersions with folds

Let $f : M \longrightarrow N$ be a submersion with definite folds ($m > n$). This means that all the singularities of f are definite fold points. At those points f is locally given by $y_i = x_i$, $i = 1, \ldots, n-1$, $y_n = x_n^2 + \cdots + x_m^2$. Let S_0 denote the singular set of f. This is an $(n-1)$-submanifold of M. In this case, $g\,|_{q(M-S_0)}: q(M-S_0) \longrightarrow N$ is a local homeomorphism. It follows from this and from the normal forms of f at fold points that W is an n-manifold with boundary ∂W diffeomorphic to S_0 and that $g : W \longrightarrow N$ is an immersion. We also have that $q\,|_{M-S_0}$ is a proper submersion. Thus $q\,|_{M-S_0}: M - S_0 \longrightarrow W - \partial W$ is a bundle map. This bundle extends to a bundle $E \xrightarrow{p} W$ where E is a manifold with boudary ∂E. That follows from the behavior of the bundle on a collar neighbourhood of ∂W. We also have a projection map $j : E \longrightarrow M$ such that $p = q \circ j$. Now it follows from the normal forms of f at points of S_0 that the fiber of those bundles is diffeomorphic to S^{m-n}. Let $C = c(\partial W \times [0,1])$ be a collar neighbourhood of ∂W in W, where $c : \partial W \times [0,1] \longrightarrow W$ is a diffeomorphism into W. Then, if C is sufficiently small, $p_1 \circ c^{-1} \circ q : q^{-1}(C) \longrightarrow \partial W$ is a D^{m-n+1}-bundle, where $p_1 : \partial W \times [0,1] \longrightarrow \partial W$ is the projection to the first factor. The structure group of this bundle can be reduced to the orthogonal group (see [6]). This implies that $\partial E \xrightarrow{p|_{\partial E}} \partial W$ is an orthogonal bundle.

There are many recent results on the study of manifolds which admit submersions with folds into $\mathbb{R}^n$. We present here a contribution.

Let $f : M^{n+1} \longrightarrow \mathbb{R}^n$ be a submersion with folds such that M is closed and $n \geq 6$. Assume that S_0 is the disjoint union of two simply connected components S_1 and S_2. Also assume that a closed tubular neighbourhood N of

S_1 with boundary ∂N is such that $\pi_*(M - S_0,\ \partial N) = 0$.

Theorem. *Under the above conditions M is a differentiable S^2-bundle over S_1.*

Proof: The boundary ∂W of W is the union of two components $\partial_1 W = q(S_1)$ and $\partial_2 W = q(S_2)$. For $i = 1,2$ let $c_i : \partial_i W \times [0,\varepsilon) \longrightarrow W$ be a diffeomorphism into W such that $C_i = c_i(\partial_i W \times [0, \varepsilon/2])$ is a closed collar neighbourhood of $\partial_i W$ in W. Then $N_i = q^{-1}(C_i)$ is a tubular neighbourhood of S_i in M. It follows that $\pi_*(M - S_0,\ \partial N_1) = 0$. We may assume that $C_1 \cap C_2 = \emptyset$. There is a strong deformation retraction from $M - S_0$ onto the closure E_0 of $M - (N_1 \cup N_2)$. Thus we have $\pi_*(E_0,\ \partial N_1) = 0$. Set W_0 = closure of $W - (C_1 \cup C_2)$. Then $E_0 \xrightarrow{q|E_0} W_0$ is a fibration that maps ∂N_1 onto $c_1(\partial_1 W \times \{\varepsilon/2\})$. It follows that $\pi_*(W_0,\ c_1(\partial_1 W \times \{\varepsilon/2\})) = 0$. From the collar neighbourhoods we get a diffeomorphism between W and W_0. This implies $\pi_*(W,\ \partial_1 W) = 0$. Now for $i = 1,2$, $\partial_i W$ is diffeomorphic to S_i. Thus $\partial_1 W$ and $\partial_2 W$ are simply connected. It follows from the condition $\pi_*(W, \partial_1 W) = 0$, together with the fact that $\partial_1 W$ is simply connected, that W is also simply connected. As $n \geq 6$ we have from the h-cobordism theorem that W is diffeomorphic to $\partial_1 W \times [0,1]$. We may now assume that $W = \partial_1 W \times [0,1]$. The composite map $p_1 \circ q : M \longrightarrow \partial_1 W$ is a submersion, where $p_1 : W \longrightarrow \partial_1 W$ is the projection to the first factor. This follows from the normal forms of f on S_0. This means that $M \xrightarrow{p_1 \circ q} \partial_1 W$ is a differentiable bundle. Now for any $x \in \partial_1 W$, $Q_x = (p_1 \circ q)^{-1}(x)$ is a differentiable 2-manifold. If $p_2 : \partial_1 W \times [0,1] \longrightarrow [0,1]$ denotes the projection to the second factor then $p_2 \circ q\ |_{Q_x} : Q_x \longrightarrow [0,1]$ is a Morse function with two critical points. This implies that Q_x is diffeomorphic to S^2. This completes the proof.

The S^2-bundle $M \longrightarrow S_1$ in the theorem must be orthogonal. Indeed its structure group can be reduced to $O(2)$ $(\subset O(3))$. In fact, there is a bundle map from the bundle $E \xrightarrow{p} \partial_1 W \times [0,1]$ to $\partial_1 E \times [0,1] \xrightarrow{(p|_{\partial_1 E}) \times identity} \partial_1 W \times [0,1]$, that covers the identity on $\partial_1 W \times [0,1]$, where $\partial_1 E = p^{-1}(\partial_1 W)$. We view E as E_0, for simplicity of notation. Recall that $p_i \circ q : N_i \longrightarrow \partial_1 W \times \{i-1\}$ are orthogonal

D^2-bundles, for $i = 1, 2$, where $p_i : \partial_1 W \times [0,1] \longrightarrow \partial_1 W \times \{i-1\}$ are the obvious projections. Then there are bundle maps $h : \partial N_1 \longrightarrow \partial_1 E \times \{0\}$ and $h' : \partial N_2 \longrightarrow \partial_1 E \times \{1\}$, that cover the respective identities on the base spaces, such that M is diffeomorphic to $N_1 \cup_h (\partial_1 E \times [0,1]) \cup_{h'} N_2$. The structure group of E is reduced to $O(2)$, since every smooth S^1-bundle is orthogonal. Thus the structure group of $\partial_1 E \times [0,1] \longrightarrow \partial_1 W \times [0,1]$ is also reduced to $O(2)$. Since any diffeomorphism of ∂N_2 on itself, which is a bundle map, is isotopic to an orthogonal bundle map, the structure group of the D^2-bundle $(\partial_1 E \times [0,1]) \cup_{h'} N_2$ can also be reduced to $O(2)$. From the orthogonal structure on this last bundle we get a bundle map $k : \partial N_1 \longrightarrow \partial_1 E \times \{0\}$. Now, $k^{-1} \circ h : \partial N_1 \longrightarrow \partial N_1$ is isotopic to an orthogonal bundle map. It then follows that the structure group of $M \longrightarrow \partial_1 W$ can also be reduced to $O(2)$.

Now $j : E \longrightarrow M$ identifies to a point in S_0 each fiber over a point of ∂W. This fiber is diffeomorphic to S^1. Thus the diffeomorphism of E that corresponds to the map $(y,t) \longrightarrow (y, 1-t)$ on $\partial_1 E \times [0,1]$ induces a homeomorphism of M that interchanges S_1 and S_2. This implies that (M, S_1) and (M, S_2) are homeomorphic.

Let $n = 6$ and $S_1 = S^5$. Then M is an S^2-bundle over S^5. Its structure group is reduced to $O(2)$. Since every principal $O(2)$-bundle over S^5 is trivial, this bundle is also trivial. Thus M must be diffeomorphic to $S^5 \times S^2$. The knots $S^5 \to S_1 \subset M$ and $S^5 \to S_2 \subset M$ are then equivalent in the sense that there is a homeomorphism of M that maps S_1 onto S_2 [5]. Indeed they are the sections of $S^5 \times S^2 \to S^5$ given by south pole and north pole of S^2.

4. Immersions

In all previous cases of maps $f : M \longrightarrow N$ immersions $g : W \longrightarrow N$ are involved. We may ask several questions about extensions. For example, let S_1 and S_2 be $(n-1)$-manifolds and let immersions $g_1 : S_1 \longrightarrow \mathbb{R}^n$ and $g_2 : S_2 \longrightarrow \mathbb{R}^n$ as well as disjoint embeddings $i_1 : S_1 \longrightarrow M^{n+1}$ and $i_2 : S_2 \longrightarrow M^{n+1}$ be given. We may ask if there is a submersion with definite folds f :

$M^{n+1} \longrightarrow \mathbb{R}^n$ such that $S_0 = i_1(S_1) \cup i_2(S_2)$ and $g_1 = f \circ i_1$, $g_2 = f \circ i_2$. The answer is negative in general. If $n = 6$, $S_1 = S_2 = S^5$ and $i_1(S_1)$ satisfies the conditions of the theorem in section 3 then M must be diffeomorphic to $S^5 \times S^2$. On the other hand, there must be an immersion $g: S^5 \times [0,1] \longrightarrow \mathbb{R}^6$ such that $g|_{S^5\times\{0\}} = g_1$ and $g|_{S^5\times\{1\}} = g_2$. This is not always true. As an example, let g_1 and g_2 be embeddings with images given by disjoint spheres with disjoint interiors. Then such an extension $g: S^5 \times [0,1] \longrightarrow \mathbb{R}^6$ does not exist.

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