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SOME INVARIANTS OF CONVEX MANIFOLDS

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Abstract

A smooth submanifold in the affine space is called convex if it lies on the boundary of its convex hull. We obtain some topological restrictions on the position of the convex manifold in the ambient space. The homologies of the singular support hyperplanes set of a convex manifold and Euler number of the tangent elements set of these hyperplanes are calculated.

Introduction

Consider a C^∞-smooth closed connected k-dimensional submanifold in n-dimensional affine space $\mathbb{R}^n$. A tangent hyperplane to this manifold is called nonsingular support if it has only one common point with the manifold and this point is a nondegenerate critical point for every function obtained by the restriction on the manifold of any nonzero linear function in $\mathbb{R}^n$ which is equal to 0 on the hyperplane.

In the first section of this paper we consider strictly convex manifolds. A manifold is called strictly convex if for every its point there is a nonsingular support hyperplane passing through this point. The homologies of the singular support hyperplanes set of a strictly convex manifold and Euler number of the tangent elements set of such hyperplanes are calculated in the nontrivial case when this manifold does not lie in any hyperplane of $\mathbb{R}^n$ and its codimension $n-k$ is greater than 1.

In the second section we consider convex manifolds. A manifold is called convex if it lies on the boundary of its convex hull. Any convex generic manifold is strictly convex if $k = 1$ or $n \leq 7$.

In the third section we apply the previous results to obtain numerical relations between different classes of singular support hyperplanes of a convex generic manifold. These relations imply (see sections 4 and 5) some restrictions on the position of the convex manifold in the ambient space.

1. Strictly convex manifolds

Let M be a smooth closed connected submanifold in $\mathbb{R}^n$. A tangent hyperplane to the manifold M is called a *support* hyperplane of M if the manifold M lies entirely in one of the two closed semispaces defined by this hyperplane.

Definition 1.1 A support hyperplane π of the manifold M is called *nonsingular* if:

1. the hyperplane π is tangent to M at only one point P;
2. the point P is a nondegenerate critical point of any composed function

$$M \longrightarrow \mathbb{R}^n \longrightarrow R$$

   where the first arrow is the embedding and the second one is a nonzero linear function in $\mathbb{R}^n$ which is equal to 0 on π.

Other support hyperplanes are called *singular*.

Definition 1.2 The manifold M is called *strictly convex* if for every point of M there is a nonsingular support hyperplane passing through this point.

For example, any manifold which lies on a smooth hypersurface in $\mathbb{R}^n$ having the positive-definite second quadratic form is strictly convex.

Definition 1.3 The manifold M is called *volumetric* if it does not lie in any hyperplane of $\mathbb{R}^n$.

The set A of all support hyperplanes of the volumetric manifold M is homeomorphic to the sphere S^{n-1}. The set $\sum$ of singular support hyperplanes of M is a compact subset of A.

Theorem 1.4 *Let M be a smooth closed connected volumetric strictly convex k-dimensional submanifold in $\mathbb{R}^n$ where $k < n-1$. Then the set Σ of singular support hyperplanes of M has the following homologies and Euler number:*

$$H_0(\Sigma, \mathbf{Z}) = \begin{cases} \mathbf{Z} & , \text{ if } k < n-2 , \\ \mathbf{Z} \oplus \mathbf{Z} & , \text{ if } k = n-2; \end{cases}$$

$$H_m(\Sigma, \mathbf{Z}) \cong \begin{cases} H^{n-m-2}(M, \mathbf{Z}) & , \text{ if } 0 < m < n-2 , \\ 0 & , \text{ if } m \geq n-2; \end{cases}$$

$$\chi(\Sigma) = \chi(S^{n-1}) + (-1)^{n-k}\chi(M). \tag{1}$$

The proof will be published in other paper.

Corollary 1.5 *If $k = n-2$ then the set $\sum$ has two connected components and*

$$H_m(\Sigma, \mathbf{Z}) \cong H_m(M, \mathbf{Z}), \text{ if } 0 < m < n-2 .$$

The set of all hyperplanes in $\mathbb{R}^n$ which are tangent to a k-dimensional manifold M at some fixed point is diffeomorphic to the projective space $\mathbb{R}P^{n-k-1}$.

Definition 1.6 The pair (π, P) where π is a hyperplane in $\mathbb{R}^n$ which is tangent to M at some point P is called a *tangent element* to the manifold M.

The set L of all tangent elements to M is a smooth $(n-1)$-dimensional manifold. The mapping

$$L \longrightarrow M, \quad (\pi, P) \longmapsto P$$

defines a smooth fibration of L over M with a fibre $\mathbb{R}P^{n-k-1}$.

Theorem 1.7 *Let M be a smooth closed connected volumetric strictly convex k-dimensional submanifold in $\mathbb{R}^n$ where $k < n-1$. Denote by Ω the set of tangent elements of singular support hyperplanes of M. Then the mapping.*

$$\Omega \longrightarrow M, \quad (\pi, P) \longmapsto P$$

defines a C^0-fibration of Ω over M with a fibre S^{n-k-2}. In particular, the Euler number of the set Ω is calculated by the formula:

$$\chi(\Omega) = \chi(M)\chi(S^{n-k}). \tag{2}$$

The proof will be published in other paper.

Corollary 1.8 *If $k = n - 2$ then the set Ω consists of two connected components which are homeomorphic to M.*

From the formulas (1) and (2) we have

Corollary 1.9 *If $k < n - 1$ then*

$$\chi(\Sigma)\chi(S^{n-k}) - \chi(\Omega) = \chi(S^{n-1})\chi(S^{n-k})$$

for every smooth closed connected volumetric strictly convex k-dimensional submanifold in $\mathbb{R}^n$.

2. Convex manifolds

The intersection of all semispaces which contain the manifold M is called its *convex hull.* The boundary V of the convex hull of the volumetric manifold M is homeomorphic to a sphere S^{n-1}.

Remark 2.1 This homeomorphism is defined as follows (see [1]): Let O be some inside point of the convex hull of the manifold M and S^{n-1} be some sphere of a small radius with a centre at the point O. Consider the projection

$$\rho : \mathbb{R}^n \setminus O \longrightarrow S^{n-1}$$

of the space $\mathbb{R}^n$ without the point O on the sphere S^{n-1} by means of rays going out the point O. Then the restriction of ρ on the set V defines the homeomorphism

$$V \longrightarrow S^{n-1}.$$

Definition 2.2 The manifold M is called *convex* if it lies on the boundary V of its convex hull.

If the manifold M is convex then for every its point there is a support hyperplane passing through this point.

Remark 2.3 Support hyperplanes do not pass through inside points of the convex hull of the manifold M. Therefore the mapping

$$M \longrightarrow \mathbb{R}^n \setminus O \longrightarrow S^{n-1}.$$

where the first arrow is the embedding and the second one is the projection ρ is a smooth embedding for any volumetric convex manifold M.

In particular, every smooth closed connected volumetric convex submanifold of the codimension 2 in $\mathbb{R}^n$ 1) is oriented; 2) has the even Euler number; 3) divides the boundary of its convex hull into two connected components. For example, such submanifold in $\mathbb{R}^3$ is a closed unknoted curve; such submanifold in $\mathbb{R}^4$ is a sphere of the arbitrary genus g.

A strictly convex manifold is obviously convex. The reverse is not, generally speaking, true.

Equip the space of all smooth embeddings of the manifold M into $\mathbb{R}^n$ by the fine Whitney C^∞-topology. The submanifolds corresponding to the embeddings from some open dense set in this space are called *generic*. Generic manifolds are volumetric (this fact follows from the Thom transversality theorem).

Theorem 2.4 *Any smooth closed connected convex k-dimensional generic submanifold in $\mathbb{R}^n$ where $k = 1$ or $n \leq 7$ is strictly convex.*

The proof will be published in other paper.

3. Relations between singular support hyperplanes

Let us consider a sequence of odd natural numbers

$$\mu_1 \leq \mu_2 \leq \ldots \leq \mu_m.$$

Definition 3.1 A support hyperplane π of the k-dimensional manifold M is called an $A_{\mu_1} \ldots A_{\mu_m}$ *-plane* if

1. the hyperplane π is tangent to M exactly at m points $A_{\mu_1}, \ldots, A_{\mu_m}$ which are the vertices of a $(m-1)$-dimensional simplex;

2. a germ at the point A_{μ_i} of any composed function

$$M \longrightarrow \mathbb{R}^n \longrightarrow R$$

where the first arrow is the embedding and the second one is a nonzero linear function in $\mathbb{R}^n$ which is equal to 0 on π and is nonnegative on M is defined by the formula

$$t_1^{\mu_i+1} + t_2^2 + \ldots + t_k^2$$

in suitable local coordinates $t_1, \ldots, t_k$ on the manifold M.

The number

$$d = \mu_1 + \ldots + \mu_m$$

is called the *degree* of a support $A_{\mu_1} \ldots A_{\mu_m}$ -plane.

Remark 3.2 A support hyperplane of the manifold M is nonsingular if and only if it is an A_1-plane. The support $A_{\mu_1} \ldots A_{\mu_m}$ -planes of the degree $d > 1$ are singular.

Let M be a generic k-dimensional submanifold in $\mathbb{R}^n$ where $k = 1$ or $n \leq 7$. Then the set A of all support hyperplanes of M consists of the support $A_{\mu_1} \ldots A_{\mu_m}$-planes of the degree $d \leq n$ (see [7]).

The set $A(\mu_1, \ldots, \mu_m)$ of all support $A_{\mu_1} \ldots A_{\mu_m}$-planes of M is a smooth submanifold of the codimension $d = \mu_1 + \ldots + \mu_m$ in the n-dimensional manifold of all hyperplanes in $\mathbb{R}^n$. The dividing of the set A into connected components *(strata)* of the manifolds $A(\mu_1, \ldots, \mu_m)$ for all possible sequences $\mu_1, \ldots, \mu_m$ is a minimal Whitney C^∞-stratification.

Denote by $\sum$ the set of singular support hyperplanes of the manifold M and by Ω the set of the tangent elements of these hyperplanes. Then the above stratification of the set A induces the finite Whitney C^∞-stratification of the sets $\sum$ and Ω (see [2]).

Let $\chi(\mu_1, \ldots, \mu_m)$ be the Euler number of the manifold $A(\mu_1, \ldots, \mu_m)$. Then the Euler numbers of the sets $\sum$ and Ω are given by the formulas

$$\begin{aligned} \chi(\Sigma) &= \sum_{d=2}^{n} (-1)^{n-d} \sum_{(\mu_1,\ldots,\mu_m) \in N(d)} \chi(\mu_1,\ldots,\mu_m) , \\ \chi(\Omega) &= \sum_{d=2}^{n} (-1)^{n-d} \sum_{(\mu_1,\ldots,\mu_m) \in N(d)} m\chi(\mu_1,\ldots,\mu_m) \end{aligned} \qquad (3)$$

where $N(d)$ is the set of all sequences of odd natural numbers $\mu_1 \leq \ldots \leq \mu_m$ such that $\mu_1 + \ldots + \mu_m = d$.

Theorem 3.3 *Let M be a smooth closed connected convex k-dimensional generic submanifold in $\mathbb{R}^n$ where $k < n-1$ and $k = 1$ or $n \leq 7$. Then*

$$\sum_{d=2}^{n} (-1)^{n-d} \sum_{(\mu_1,\ldots,\mu_m) \in N(d)} \chi(\mu_1,\ldots,\mu_m) = \chi(S^{n-1}) + (-1)^{n-k}\chi(M) ,$$

$$\sum_{d=2}^{n} (-1)^{n-d} \sum_{(\mu_1,\ldots,\mu_m) \in N(d)} m\chi(\mu_1,\ldots,\mu_m) = \chi(M)\chi(S^{n-k}) . \qquad (4)$$

This statement follows from the theorems 1.4,1.7,2.4 and the formulas (3).

Corollary 3.4 *If $k < n-1$ and $k = 1$ or $n \leq 7$ then*

$$\sum_{d=2}^{n} (-1)^{n-d} \sum_{(\mu_1,\ldots,\mu_m) \in N(d)} \left[\chi(S^{n-k}) - m\right] \chi(\mu_1,\ldots,\mu_m) = \chi(S^{n-1})\chi(S^{n-k})$$

for every smooth closed connected convex k-dimensional generic submanifold in $\mathbb{R}^n$. In particular, if n and k are odd numbers then

$$\sum_{d=3}^{n} (-1)^{d} \sum_{(\mu_1,\ldots,\mu_m) \in N(d)} (m-2)\chi(\mu_1,\ldots,\mu_m) = 4. \qquad (5)$$

Remark 3.5 Let M be a smooth closed connected convex $(n-2)$-dimensional generic submanifold in $\mathbb{R}^n$ where $3 \leq n \leq 7$. Then (see corollary 1.5) the set $\sum$ of singular support hyperplanes of M has two connected components $\sum_i, i = 1,2$ and

$$\chi(\Sigma_i) = \sum_{d=2}^{n} (-1)^{n-d} \sum_{(\mu_1,\ldots,\mu_m) \in N(d)} \chi_i(\mu_1,\ldots,\mu_m) \qquad (6)$$

where $\chi_i(\mu_1,\dots,\mu_m)$ is the Euler number of the manifold $\Sigma_i \cap A(\mu_1,\dots,\mu_m)$.

On the other hand, the set Ω of singular support hyperplanes tangent elements of M has two connected components which are homeomorphic to M (see corollary 1.8). Therefore

$$\sum_{d=2}^{n}(-1)^{n-d}\sum_{(\mu_1,\dots,\mu_m)\in N(d)} m\chi_i(\mu_1,\dots,\mu_m) = \chi(M) \tag{7}$$

and the formulas (6) and (7) imply the equality

$$\chi(\Sigma_i) = (1/2)\left[\chi(M) + \sum_{d=3}^{n}(-1)^{n-d}\sum_{(\mu_1,\dots,\mu_m)\in N(d)}(2-m)\chi_i(\mu_1,\dots,\mu_m)\right] \tag{8}$$

for every $i = 1, 2$.

4. The case $n = 3$

Let M be a smooth closed connected convex generic curve in $\mathbb{R}^3$. Then the relation (5) has the form

$$C - T = 4 \tag{9}$$

where

$C = \chi(3)$ is the number of the *osculating support planes* of the curve M;

$T = \chi(1, 1, 1)$ is the number of the support planes of the curve M which are tangent to M at three distinct points *(tritangent support planes)*.

The relation (9) was obtained by Romero-Fuster [5] who used it in the proof of the 4-vertex theorem for generic convex curves in Euclidean space $\mathbb{R}^3$. Namely, if the osculating plane at some point of the curve is the support one then this point is a *vertex* (zero-torsion point) of the curve. Therefore the curve M has at least C vertices where $C = T + 4 \geq 4$.

The 4-vertex theorem is true for nongeneric curves too.

Theorem 4.1 *Every C^3-smooth closed connected convex curve in $\mathbb{R}^3$ with nowhere vanishing curvature has at least four vertices.*

For the proof see [6].

Remark 4.2 The set of singular support planes of the curve M has two connected components Σ_1 and Σ_2 (see corollary 1.5). Let $C_i = \chi_i(3)$ and $T_i = \chi_i(1,1,1)$ be the numbers of the support osculating planes and the support tritangent planes, respectively, which belong to the i-th component Σ_i. Then for every $i = 1,2$ the theorem 1.4 and the formula (8) imply the equality

$$C_i - T_i = 2.$$

5. The case $n = 4$

Let M be a smooth closed connected convex one-dimensional or two-dimensional generic submanifold in $\mathbb{R}^4$. Then the formulas (4) imply the equalities

$$E - 2C + S = \chi(M) + Q$$

and

$$T - C = 2Q \tag{10}$$

where

$E = \chi(1,1)$ is the Euler number of the set $A(1,1)$ of support hyperplanes which are tangent to the manifold M at two distinct points and are nonsingular for the germs of M at these points;

$T = \chi(1,1,1)$ is the number of connected simply-connected components of the set of support hyperplanes which are tangent to the manifold M at three distinct points;

$C = \chi(3)$ is the number of connected simply-connected components of the set of support hyperplanes being tangent to the manifold M at only one point and having the third degree of a tangency with a germ of some curve on the manifold;

$Q = \chi(1,1,1,1)$ is the number of support hyperplanes which are tangent to the manifold M at four distinct points;

$S = \chi(1,3)$ is the number of support hyperplanes being tangent to the manifold M at two distinct points and having at one of these points the third degree of a tangency with a germ of some curve on the manifold.

The relation (10) is true for any smooth closed generic submanifold in $\mathbb{R}^4$ (for three-dimensional submanifolds this was proved in [4]). In fact it is a corollary of the two equalities

$$2Q + S = T + C, \qquad S = 2C \tag{11}$$

which follow from the incidence rules on the graph formed by 0-dimensional and 1-dimensional strata of the Maxwell set for a generic 3-parameter family of functions (see [3]).

Theorem 5.1 *Let M be a smooth closed connected convex one-dimensional or two-dimensional generic submanifold in $\mathbb{R}^4$. Then*

$$E = \chi(M) + Q, \quad T = C + 2Q, \quad S = 2C .$$

On the other hand, every connected component of the set $A(1,1)$ is diffeomorphic to the open 2-dimensional disk with holes and handles (see theorem1.4). Therefore

$$E = D - B - 2R$$

where D is the number of such disks and B is the total number of holes in them and R is the total number of handles.

Corollary 5.2 *For every smooth closed connected convex one-dimensional or two-dimensional generic submanifold in $\mathbb{R}^4$*

$$D = \chi(M) + Q + B + 2R. \tag{12}$$

For example, if dim $M = 1$ then $\chi(M) = 0, R = 0$ and the relation (12) has the form $D = Q + B$. If dim $M = 2$ then M is a sphere of the genus $g, \chi(M) = 2 - 2g$ and the relation (12) has the form

$$D = 2 - 2g + Q + B + 2R.$$

Remark 5.3 If M is a sphere of the genus g then $D \geq 2$ since the set $\sum$ of singular support hyperplanes of M has two connected components $\sum_i, i = 1, 2$

(see corollary 1.5). The relations (11) are true for every connected component Σ_i. Therefore

$$T_i - C_i = 2Q_i \tag{13}$$

where $T_i = \chi_i(1,1,1), C_i = \chi_i(3)$ and $Q_i = \chi_i(1,1,1,1)$.

From the formulas (8), (13) and the theorem 1.4 we have

$$\chi(\Sigma_i) = \chi(M)/2 = 1 - g,$$

$$H_1(\Sigma_i, \mathbf{Z}) = \mathbf{Z} \oplus \ldots \oplus \mathbf{Z}(g\ times)$$

for every $i = 1, 2$. In particular, if $g < 2$ then $R = 0$ and $D = 2 - 2g + Q + B$.

6. About 4-vertex theorem

The purpose of this work consists in generalizing the 4-vertex theorem (see [5 - 6]) for higher dimensional spaces. We can consider the formula (5) as a first possible step in this direction.

The *vertex* of a generic submanifold of codimension higher than 1 in the odd-dimensional space $\mathbb{R}^n$ is a point at which this manifold has a support A_n-plane. The formula (5) relates the number $\chi(n)$ of vertices with other topological characteristics of the position of the convex manifold in the ambient space.

Observe that we can obtain other corollaries from the theorems 1.4, 1.7, 2.4 and the formulas (3). For instance,

Corollary 6.1 *If n and k are odd numbers, $k < n - 1$ and $k = 1$ or $n \leq 7$ then*

$$\sum_{d=2}^{n} (-1)^d \sum_{(\mu_1, \ldots, \mu_m) \in N(d)} [m - (n+1)/2]\, \chi(\mu_1, \ldots, \mu_m) = n + 1$$

for every smooth closed connected convex k-dimensional generic submanifold in $\mathbb{R}^n$.

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