Matemática Contemporânea, Vol. 46, 194–203
http://doi.org/10.21711/231766362020/rmc4620

Meyniel's Conjecture on graphs with few P4's

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Abstract

The Meyniel's conjecture has been one of the most interesting topics of study regarding the cops and robber game on graphs. It states that given any connected graph G with n vertices, $c(G) \leq C\sqrt{n}$, for a constant C large enough, where $c(G)$ is the cop-number, the minimum number of cops to guarantee that the robber cannot escape. We use the primeval and modular decompositions of the $(q, q-4)$-graphs and the P_4-tidy graphs to find polynomial algorithms to calculate the cop number $c(G)$ of a graph in these classes. Futhermore, we prove that the Meyniel's conjecture is true for P_4-tidy and for $(q, q-4)$-graphs with at least q vertices.

Let G be a finite graph. Two players, $\mathcal{C}$ and $\mathcal{R}$, face each other on a match in which the cops, controlled by $\mathcal{C}$ move between adjacent vertices of G in order to capture the robber, controlled by $\mathcal{R}$. The game has perfect information; that is, each player can see the position of the pieces controlled by his adversary.

On turn 0, player $\mathcal{C}$ chooses in which vertices he will place his cops. After that, $\mathcal{R}$, aware of the initial position of the cops, places his robber. Players then, alternately, starting with $\mathcal{C}$, move each of their pieces to an

2000 AMS Subject Classification: 68R10, 05C75 and 05C85
Key Words and Phrases: Cops and Robber, cop number, Modular Decomposition.

adjacent vertex of its current position or make it stay on the same vertex for that turn. The game ends, with the victory of player $\mathcal{C}$, if in a finite number of turns one of the cops capture the robber(i.e. move to the same vertex of the robber). If the robber can avoid capture indefinitely, then $\mathcal{R}$ wins.

The *Game of Cops and Robber*, described above, was introduced by Quilliot [13] and independently by Nowakowski and Winkler [11]. We say that a graph G is k-cop-win if there is a strategy for $\mathcal{C}$ to always win, using k cops, no matter the strategy adopted by $\mathcal{R}$. The cop number of a graph G, denoted by $c(G)$, is the minimum k such that G is k-cop-win. Aigner and Fromme proved that for any plannar graph G, $c(G) \leq 3$ [1].

Hahn and MacGillivray presented an algorithm that decides if a graph G is k-cop-win, for fixed integer k, in polynomial time [8]. However, Kratochvíl et al. proved that, when k is not fixed, decide if $c(G) \leq k$ is NP-hard [6]. Recently, Kinnersley proved that the problem is indeed ExpTime-complete [9].

The most important open problem related to the game of cops and robber is to limit, for any connected graph $G = (V, E)$, the value of $c(G)$ by a function of $|V|$. Let n be a positive integer, $c(n)$ is the minimum k such that for any connected graph G with n vertices, $c(G) \leq k$. In 1985, Meyniel conjectured that $c(n) = O(\sqrt{n})$. Lu and Peng proved that $c(n) = O\left(\frac{n}{2^{1-o(n)}\sqrt{\log_2 n}}\right)$ [10].

On this article we show polynomial algorithms to calculate the cop number for P_4-tidy graphs and $(q, q-4)$-graphs. Futhermore, we prove that the Meyniel's conjecture is true for P_4-tidy and for $(q, q-4)$-graphs with at least q vertices.

1 Graphs with few P4's

A *spider* graph is a graph whose vertex set can be partitioned on sets R, C, S where $C = \{c_1, \ldots, c_k\}$ is a clique and $S = \{s_1, \ldots, s_k\}$ is an independent set with $k > 1$ such that: (i) Every vertex in R is adjacent to every vertex in C and non-adjacent to every vertex in S; (ii) s_i is adjacent to c_j if and only if $i = j$ (thin spider) or s_i is adjacent to c_j if and only if $i \neq j$ (thick spider). We refer to R, C and S, respectively, as the *head*, *body* and *legs* of the spider. In the case where $R = \emptyset$ the spider is said to be *headless*. Some examples of spider graphs are presented in [5]. A graph is a *quasi-spider* if it can be obtained from a spider by replacing at most one vertex in $C \cup S$ by a K_2 or a $\overline{K_2}$ maintaining adjacencies. A graph G is *P_4-tidy* if for every induced P_4 H in G, there is at most one vertex outside H that induces at least two P_4's with the vertices of H [7]. The P_4-tidy graphs contains the P_4-lite graphs, P_4-extendible and extended P_4-sparse graphs.

Theorem 1 ([7])**.** *A graph G is P_4-tidy if and only if exactly one of the following holds: (a) G is the disjoint union or the join of two P_4-tidy graphs; (b) G is a quasi-spider, with partition (R, C, S), such that R induces a P_4-tidy graph or is empty; (c) G is isomorphic to a C_5, P_5, $\overline{P_5}$ or K_1.*

Theorem 1 imply a tree decomposition of the P_4-tidy graphs, where the leaves are headless quasi-spiders, C_5, P_5, $\overline{P_5}$ or K_1 and the internal nodes are the results of operations(applied on the sons of the node) of disjoint union(a), join(b), or an operation that adds all the edges between the vertices of a subgraph to the body of a headless quasi-spider, creating a complete quasi-spider(c). Such decomposition can be computed in linear time $O(m + n)$ [7], where m is the number of edges and n is the number of vertices of the input graph.

A graph $G = (V, E)$ is said to be a *$(q, q-4)$-graph* if no subset of V with at most q vertices induces more than $q - 4$ distinct P_4's. The $(q, q-4)$-graphs also have a nice tree decomposition similar to the one presented above.

A graph $G = (V, E)$ is *p-connected* if, for every bipartition (A, B) of V there is an induced P_4 with vertices from both A and B. A *p-component* of a graph G is a maximal p-connected subgraph of G. A graph $G = (V, E)$

is *separable* if there is a particular bipartition (H_1, H_2) of its vertex set such that for every induced P_4 $wxyz$, $x, y \in H_1$ and $w, z \in H_2$.

Theorem 2 ([3])**.** *Let $q \geq 4$ be fixed and G a $(q, q-4)$-graph. Then one of the following holds (a) G is the disjoint union of two $(q, q-4)$-graphs; (b) G is the join of two $(q, q-4)$-graphs; (c) G is spider (R, C, S) such that $G[R]$ is a $(q, q-4)$-graph; (d) G contains separable p-component H, with $|H| < q$ with separation (H_1, H_2), such that every vertex in $G - H$ is adjacent to every vertex in H_1 and not adjacent to any vertex of H_2; (e) G has less than q vertices.*

The decomposition sugested by Theorem 2 is the *Primeval Decomposition* and can be obtained in linear time $O(m + n)$ [2]. Given graphs G and H, $G \cup H$ denotes the disjoint union of G and H, and $G \vee H$ denotes the join of G and H.

2 The cop number of graphs with few P_4's

The cop number of a disconnected graph is the sum of the cop number of its connected components. Thus, we have the following straightforward lemma.

Lemma 1. *Let $G = G_1 \cup G_2$, then $c(G) = c(G_1) + c(G_2)$.*

Next we show the exact value for the cop number of the join of two graphs.

Lemma 2. *Let $G = G_1 \vee G_2$, then $c(G) = \min\{2, c(G_1), c(G_2)\}$.*

Proof. Clearly $c(G) \leq 2$ since $\mathcal{C}$ can initially place a cop on a vertex of G_1 and another one on a vertex of G_2 and then any vertex of G would be adjacent to at least one cop. Therefore, no matter what vertex the player $\mathcal{R}$ chooses as the initial position of the robber, he will be captured on the first round.

Suppose $c(G_1) > 1$ and $c(G_2) > 1$. Assume w.l.g. that $\mathcal{C}$ chooses to initially place his single cop in a vertex of G_1. Since $c(G_1) > 1$, $\mathcal{R}$ can use a winning strategy to initially place the robber on a vertex of G_1 and avoid capture as long as the cop remain in G_1. Afterwards if $\mathcal{C}$ chooses to move his cop to a vertex of G_2, since all vertices of G_1 are adjacent to all vertices in G_2 and $c(G_2) > 1$, the robber can use a winning strategy to move to a vertex of G_2, as if the game had just started, and avoid capture as long as the cop remains in G_2. Therefore, if $c(G) = 1$ then $c(G_1) = 1$ or $c(G_2) = 1$.

Suppose w.l.g. that $c(G_1) = 1$, then $\mathcal{C}$ can use a winning strategy in G_1 to initially place his cop in a vertex of G_1. By doing so, he forbids $\mathcal{R}$ from placing the robber on any vertex of G_2, otherwise it would be captured on the next turn. Therefore, the game will be always restricted to G_1. Since $c(G_1) = 1$, $\mathcal{C}$ has a winning strategy to capture the robber in a finite number of moves. Consequently, $c(G) = \min\{2, c(G_1), c(G_2)\}$. □

Lemma 3. *Let G be a spider with partition (R, C, S), then $c(G) = 1$.*

Proof. If G is a thin spider, then $\mathcal{C}$ can place his cop initially in $c_1 \in C$. The only vertices of G that are not adjacent to c_1 are the vertices s_i with $2 \leq i \leq |C|$. We can assume w.l.g. that $\mathcal{R}$ chooses to initially place the robber on s_2. Then on the next turn $\mathcal{C}$ would move his cop to c_2 and $\mathcal{R}$ would keep his robber on s_2, since the only vertex he could move is now occupied by a cop.

If G is a thick spider, then $\mathcal{C}$ can use a similar strategy. He places one cop on some vertex of the clique and afterwards moves the cop towards the robber. Assuring capture on at most 2 moves. □

In the next lemma, we assume from Lemma 3 that G is not a spider.

Lemma 4. *Let G be a quasi-spider with partition (R, C, S). If G is a thin quasi-spider, then:*

$$c(G) = \begin{cases} 2, & \text{if the duplicate vertices } v, v' \in C \text{ and induce a } \overline{K_2}; \\ 1, & \text{otherwise.} \end{cases}$$

If G is a thick quasi-spider, then $c(G) = 1$.

Proof. Assume that G is a thin quasi-spider such that the duplicate vertex $c_i \in C$ and induces a $\overline{K_2}$ with his copy c'_i. We show a strategy for the robber to avoid capture indefinitely when playing against just one cop in G. The strategy for the robber depends only on the position of the cop. If the cop is in c_i, $\mathcal{R}$ will place the robber in c'_i and vice-versa. If the cop is in s_i, $\mathcal{R}$ must place the robber on a vertex of C distinct from c_i and c'_i. If the cop is in any other vertex of G, $\mathcal{R}$ must place the robber on s_i. Since the player $\mathcal{R}$ can choose the initial position of the robber accordingly to the strategy and keep the cop at a distance of at least 2 from the robber, then $c(G) > 1$.

However, if there are two cops, the robber is captured in at most two moves. It suffices to $\mathcal{C}$ place the cops initially on c_i and c'_i, hence if $\mathcal{R}$ place the robber on s_i or any other vertex of $C \cup R \setminus \{c_i, c'_i\}$ he will be captured on the next turn. Therefore, $\mathcal{R}$ must place the robber on some vertex $s_j \in S \setminus \{s_i\}$. If he does so, one of the cops can be moved to c_j, since $N[s_j] \subseteq N[c_j]$, the robber has no place to flee and will be captured on the next move of the cops.

Suppose that G is a thin quasi-spider such that the duplicate vertex belongs to S or induces a K_2 with its copy. We can show that only one cop can guarantee the capture of the robber with at most two moves. Player $\mathcal{C}$ must choose to initially place the only cop on a vertex $c_i \in C$. Hence, if $\mathcal{R}$ initially places the robber on s_i (or possibly s'_i) or in any vertex from $C \cup R$ he will be captured on the next turn. Therefore, $\mathcal{R}$ must place the robber on a vertex s_j (possibly s'_j) from S such that $i \neq j$. It suffices now for the cop to be moved to c_j to assure the capture of the robber on the next turn.

Since a thick spider with $|C| = 2$ is also a thin spider, we can assume that a thick quasi-spider has at least 3 vertices on C, 4 if one of them is duplicate. One cop is enough to capture a robber on this case. Initially $\mathcal{C}$ places a cop on a non duplicate vertex of $c_j \in C$. Observe that, by doing so, the cop is adjacent to all vertices of the graph, except s_j (and possibly s'_j). Consequently, $\mathcal{R}$ must place the robber on s_j (or its copy). Now it suffices for $\mathcal{C}$ to move the cop to another non duplicate vertex of C (that is possible because C has at least 3 vertices). On the robber next move he can refrain from moving or move to a vertex of C, either way he will end on a vertex adjacent to the vertex of the cop and will be captured on the next turn. □

With the results presented so far we obtain a linear algorithm to calculate $c(G)$ for any P_4-tidy graph G.

Theorem 3. *Given a P_4-tidy graph G we can determine $c(G)$ in time $O(n+m)$. Furthermore, $c(G) \leq 2$.*

Proof. Directly from Theorem 1 and Lemmas 1, 2, 3, 4. □

Observe that, since $c(G) \leq 2$, this implies that Meyniel's conjecture is true for P_4-tidy graphs. A *module* on a graph G is a set of vertices M such that for each vertex $v \in V(G) \setminus M$, v is adjacent to all vertices in M or v is not adjacent to any vertex in M. Jamison and Olariu investigated the process of turning each module of a separable p-component H with separation (H_1, H_2) in a single vertex, keeping the adjacencies. They proved that the graph obtained in such process, which they called *characteristic p-component* of H, is a split graph. The vertices of the clique represent the modules in H_1 and the vertices of the independent set represent the modules in H_2. Furthermore, the characteristic p-component of a graph H can be obtained in polynomial time on the size of H [12].

Theorem 4. *Let q be a fixed integer and G be a $(q, q-4)$-graph there is a polynomial algorithm to calculate $c(G)$. Furthermore, for all connected $(q, q-4)$-graph G with at least q vertices, $c(G) \leq 2$.*

Proof. The union, join and spider operations are solved in Lemmas 1, 2 and 3. Suppose that G has a separable p-component $H = (H_1, H_2)$ with less than q vertices. We can use the characteristic p-component of H to show that $c(G) \leq 2$. $\mathcal{C}$ begins the game by placing two cops c_1 and c_2 in vertices from distinct modules of H_1. Those 2 cops are adjacent to all vertices in $G' = G \setminus H$, all the vertices of H_1, since the modules form a clique in the characteristic p-component, and some vertex of H_2. Therefore, $\mathcal{R}$ has no other option but place the robber on some vertex of H_2 not adjacent to the vertices occupied by the cops.

On the next turn, $\mathcal{C}$ moves the cop c_1 to a vertex of H_1 adjacent to the vertex occupied by the robber. Observe that, since the modules of H_2 are independent set on the characteristic p-component of H, then the robber can only move to a vertex of H_1 or a vertex of H_2 from the same module where he is. Either way the robber will be captured on the next turn. This assure that $c(G) \leq 2$. The algorithm from Hahn and MacGillivray [8] gives us a polynomial solution to decide if $c(G) = 1$. For the case when G has less than q vertices, we can also use the algorithm from Hahn and MacGillivray [8] for j cops, with $1 \leq j < q$. Since q is a fixed value, such procedure also takes polynomial time. □

Corolary 1. *The Meyniel's Conjuecture holds for both P_4-tidy graphs and $(q, q-4)$-graphs with at least q vertices.*

Conclusion: In this paper, we present two new classes of graphs for which Meyniel's conjecture is true and polynomial algorithms to calculate the cop number of a graph in these classes. Since any graph is a $(q, q-4)$-graph for some $q > 0$, this could lead to another way to prove the Meyniel's conjecture. Unfortunately, the decomposition of $(q, q-4)$-graphs do not give us any information about the structural properties of these graphs when they have less than q vertices. It is also worth noticing that these results imply a polynomial algorithm to calculate, for any integer k, the k-capture time [4] of P_4-tidy graphs and $(q, q-4)$-graphs.

References

[1] M. Aigner and M. Fromme, *A game of cops and robbers*, Discrete Applied Mathematics **8** (1984), 1–12.

[2] L.Babel, T.Kloks, J.Kratochvíl, D.Kratsch, H.Müller, S.Olariu, *Efficient algorithms for graphs with few P4?s*, Disc. Math. **235**(2001), 29–51.

[3] L. Babel and S. Olariu, *On the structure of graphs with few P 4 s*, Discrete Appl. Math. **84** (1998), 1–13.

[4] A. Bonato, P. Golovach, G. Hahn and J. Kratochvíl, *The capture time of a graph*, Discrete Math. **309**(18) (2009), 5588–5595.

[5] R. R. Del-Vecchio and A. A. Jones, *Laplacian integrality in P4-sparse and P4-extendible graphs*, Applied Mathematics and Computation **330** (2018), 307–315.

[6] F. V. Fomin, P. Golovach, J. Kratochvíl and N. Nisse NISSE, *Pursuing fast robber in graph*, Journal Theoretical Computer Science **411** (2010), 1167–1181.

[7] V. Giakoumakis, H. Roussel and H. Thuillier, *On P 4 -tidy graphs*, Discrete Mathematics and Theoretical Computer Science **1** (1997), 17–41.

[8] G. Hahn and G. MacGillivray, *A note on k-cop, l-robber games on graphs* Discrete Mathematics **306** (2006), 2492–2497.

[9] W. B. Kinnersley, *Cops and Robbers is EXPTIME-complete*, J. Comb. Theory Ser. B **111** (2015), 201–220.

[10] L. Lu and X. Peng, *On Meyniel?s conjecture of the cop number*, Journal of Graph Theory **71** (2012), 192–205.

[11] R. J. Nowakowski and P. Winkler, *Vertex-to-vertex pursuit in a graph*, Discrete Mathematics **43** (1983), 235–239.

[12] B. Jamison and S. Olariu, *P-components and the homogeneous decomposition of graphs*, SIAM Journal of Discrete Mathematics, **8** (1995), 448–463.

[13] A. Quilliot, *Jeux et pointes fixes sur les graphes*, Thèse de 3ème cycle, Université de Paris **VI** (1978), 131–145.

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