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On nested and 2-nested graphs: two subclasses of graphs between threshold and split graphs.

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Abstract

A $(0,1)$-matrix has the Consecutive Ones Property (C1P) for the rows if there is a permutation of its columns such that the ones in each row appear consecutively. We say a $(0,1)$-matrix is nested if it has the consecutive ones property for the rows (C1P) and every two rows are either disjoint or nested. We say a $(0,1)$-matrix is 2-nested if it has the C1P and admits a partition of its rows into two sets such that the submatrix induced by each of these sets is nested. We say a split graph G with split partition (K,S) is nested (resp. 2-nested) if the matrix $A(S,K)$ which indicates the adjacency between vertices in S and K is nested (resp. 2-nested). In this work, we characterize nested and 2-nested matrices by minimal forbidden

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submatrices. This characterization leads to a minimal forbidden induced subgraph characterization of these graph classes, which are superclasses of threshold graphs and subclasses of split and circle graphs.

1 Introduction

Let $A = (a_{ij})$ be a $n \times m$ $(0,1)$-matrix. We denote $a_{i.}$ and $a_{.j}$ the ith row and the jth column of matrix A. Let $l_i = \min\{j\colon a_{ij} = 1\}$ and $r_i = \max\{j\colon a_{ij} = 1\}$ for each $i \in \{1, \dots, n\}$. Two rows $a_{i.}$ *and* $a_{k.}$ *are disjoint* if there is no j such that $a_{ij} = a_{kj} = 1$. We say that $a_{i.}$ is *contained* in $a_{k.}$ if for each j such that $a_{ij} = 1$ also $a_{kj} = 1$. We say that $a_{i.}$ *and* $a_{k.}$ *are nested* if $a_{i.}$ is contained in $a_{k.}$ or $a_{k.}$ is contained in $a_{i.}$. Finally, we say that $a_{i.}$ *and* $a_{k.}$ *start* (resp. *end*) *in the same column* if $l_i = l_k$ (resp. $r_i = r_k$), and we say $a_{i.}$ *and* $a_{k.}$ *start (end) in different columns* otherwise. We say a $(0,1)$-matrix A has the *consecutive ones property for the rows* (for short, C1P) if there is permutation of the columns of A such that the 1's in each row appear consecutively. Tucker characterized all the minimal forbidden submatrices for the C1P, later known as *Tucker matrices*. For the complete list of Tucker matrices, see [8], where a graphic representation of them can be found in Figure 3.

We say a $(0,1)$-matrix is *nested* if it has the consecutive ones property for the rows (C1P) and every two rows are either disjoint or nested. We say a $(0,1)$-matrix is *2-nested* if it has the C1P for the rows and there is a partition S_1, S_2 of the rows such that each submatrix obtained is nested.

All graphs in this work are simple. The pair (K, S) is a *split partition* of a graph G if $\{K, S\}$ is a partition of the vertex set of G and the vertices of K (resp. S) are pairwise adjacent (resp. nonadjacent), and we denote it $G = (K, S)$. A graph G is a *split graph* if it admits some split partition. Let G be a split graph with split partition (K, S), $n = |S|$, and $m = |K|$. Let $s_1, \dots, s_n$ and $v_1, \dots, v_m$ be linear orderings of S and K, respectively. Let $A = A(S, K)$ be the $n \times m$ matrix defined by $A(i, j) = 1$ if s_i is

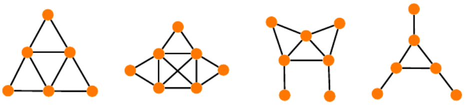

Figure 1: Some useful graphs, from left to right: tent, 4-tent, co-4-tent and net.

adjacent to v_j and $A(i, j) = 0$, otherwise.

A split graph $G = (K, S)$ is *nested* (resp. *2-nested*) if there is a linear ordering Π of K, such that the associated matrix $A(S, K)$ is nested (resp. 2-nested) and if its columns are ordered as in Π then the ones in each row occur in consecutive columns.

Circle graphs [3] are intersection graphs of chords in a circle. These graphs were characterized by Bouchet [2] in 1994 by forbidden induced subgraphs under local complementation, and by Geelen and Oum [5] in terms of pivoting. These graphs can be recognized in $\mathcal{O}((n+m)\alpha(n+m))$-time, where α is the inverse of the Ackermann function [6]. The characterization of the entire class of circle graphs by forbidden induced subgraphs of the graph itself is still an open problem. However, some partial characterizations are known [1]. It follows from the definition that nested and 2-nested graphs are common subclasses of circle graphs. Furthermore, nested and 2-nested graphs are also a superclass of threshold graphs (see Golumbic [7] for more details on these definitions).

The problem of characterizing 2-nested graphs by minimal forbidden induced subgraphs arises as a natural subproblem in our ongoing efforts to obtain the same kind of characterization of those split graphs that are circle graphs. We started by considering a split graph H such that H is minimally non-circle. Since comparability graphs are a subclass of circle graphs, in particular H is not a comparability graph. Notice that permutation graphs are those comparability graphs for which their complement is also a comparability graph. It is easy to prove that permutation graphs are precisely those circle graphs having a circle model with an equator. See Gallai [4] for the complete list of minimal forbidden subgraphs of comparability graphs. Using the list of minimal forbidden subgraphs of comparability graphs and the fact that H is also a split graph, we conclude that H contains either a tent, a 4-tent, a co-4-tent or a net as a subgraph (see Figure 1). We first considered the case in which H contains an induced tent as a subgraph, thus reaching a problem when trying to give a circle model for H. Once analyzed the compatibilities between the vertices in the complete and independent partitions of such a graph, it arises that there is exactly one subclass –which we denoted α– of independent vertices for which both endpoints of each vertex could be entirely drawn in two distinct areas of the circle model, when for every other vertex there is a unique possible placement. Hence, for the subgraph induced by taking the tent graph union the subclass α to admit a circle model, the subclass α must be partitioned into two disjoint subsets such that, for each subset, every pair of vertices are either disjoint or nested, thus leading to the definition of 2-nested graphs.

2 Nested matrices

We begin by giving the following characterization of nested matrices.

Theorem 1. *A $(0,1)$-matrix is nested if and only if it contains no G_0 as a submatrix (see Figure 2).*

$$G_0 = \begin{pmatrix} 110 \\ 011 \end{pmatrix}$$

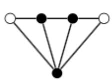

Figure 2: The G_0 matrix and the gem graph

Proof. Since no Tucker matrix has the C1P and the rows of G_0 are neither disjoint nor nested, no nested matrix contains a Tucker matrix or G_0 as submatrices. Conversely, as each Tucker matrix contains G_0 as a submatrix, every matrix containing no G_0 as a submatrix is a nested matrix. □

Corollary 2. *A split graph is nested if and only if it contains no induced gem.*

3 2-nested matrices

We define the following matrices, since they play an important role in the sequel.

$$F_0 = \begin{pmatrix} 11100 \\ 01110 \\ 00111 \end{pmatrix} \qquad F_1(k) = \begin{pmatrix} 011...111 \\ 111...110 \\ 000...011 \\ 000...110 \\ ..... \\ ..... \\ ..... \\ 110...000 \end{pmatrix} \qquad F_2(k) = \begin{pmatrix} 0111...10 \\ 1100...00 \\ 0110...00 \\ ..... \\ ..... \\ ..... \\ 0000...11 \end{pmatrix}$$

Figure 3: F_0, $F_1(k) \in \{0,1\}^{k \times k-1}$, and $F_2(k) \in \{0,1\}^{k \times k}$, for any odd $k \geq 5$.

Theorem 3. *A* $(0,1)$*-matrix* A *is 2-nested if and only if there is a linear ordering* Π *of the columns such that the matrix* A *with its columns ordered according to* Π *does not contain any Tucker matrix, or* F_0*,* $F_1(k)$*,* $F_2(k)$ *for every odd* $k \geq 5$ *as a configuration.*

We define the auxiliary graph $H(A) = (V, E)$ where the vertex set $V = \{w_1, \ldots, w_n\}$ has one vertex for each row in A, and two vertices w_i and w_k in V are adjacent if and only if the rows $a_{i.}$ and $a_{k.}$ are neither disjoint nor nested. With a minor misuse of notation, w_i will refer to both the vertex w_i in $H(A)$ and the row $a_{i.}$ of A. In particular, the definitions given in the introduction apply to the vertices in $H(A)$; i.e., we say two vertices w_i and w_k in $H(A)$ are *nested* (resp. *disjoint*) if the corresponding rows $a_{i.}$ and $a_{k.}$ are nested (resp. disjoint). And two vertices w_i and w_k in $H(A)$ *start* (resp. *end*) *in the same column* if the corresponding rows $a_{i.}$ and $a_{k.}$ start (resp. end) in the same column. It follows from the definition of 2-nested matrices that A is a 2-nested matrix if and only if there is a bicoloring of the auxiliary graph $H(A)$ or, equivalently, if $H(A)$ is bipartite (i.e., $H(A)$ does not contain cycles of odd length).

Proof. Since A admits a C1P, then A contains no Tucker matrices. Moreover, if A contains F_0, $F_1(k)$ or $F_2(k)$ for some odd $k \geq 5$, since the corresponding subgraphs in $H(A)$ of every such matrix induces an odd cycle, then it follows that $H(A)$ does not admit a proper 2-coloring and this results in a contradiction. Therefore, A does not contain any F_0, $F_1(k)$ or $F_2(k)$ for any odd $k \geq 5$ as a configuration.

Conversely, let Π be a linear ordering of the columns such that the matrix A does not contain any $F_0, F_1(k), F_2(k)$ for any odd $k \geq 5$ or Tucker matrices as configurations. Due to Tucker's Theorem, since there are no Tucker submatrices in A, the matrix A has the C1P.

For a proof by contradiction, suppose that the auxiliary graph $H(A)$ is not bipartite. Hence there is an induced odd cycle C in $H(A)$.

Suppose first that $H(A)$ has an induced odd cycle $C = w_1, w_2, w_3, w_1$ of length 3, and suppose without loss of generality that the first rows of A are those corresponding to the cycle C. Since w_1 and w_2 are adjacent, both begin and end in different columns. The same holds for w_2 and w_3, and w_1 and w_3. We assume without loss of generality that the vertices start in the order of the cycle, in other words, that $l_1 < l_2 < l_3$.

Since w_1 starts first, it is clear that $a_{2l_1} = a_{3l_1} = 0$, thus the column $a_{.l_1}$ of A is the same as the first column of the matrix F_0.

Since A has the C1P and w_1 and w_2 are adjacent, then $a_{1l_2} = 1$. As stated before, w_2 starts before w_3 and thus $a_{3l_2} = 0$. Hence, column $a_{.l_2}$ is equal to the second column of F_0.

The third column of F_0 is $a_{.l_3}$, for w_3 is adjacent to w_1 and w_2, hence it is straightforward that $a_{1l_3} = a_{2l_3} = a_{3l_3} = 1$.

To find the next column of F_0, let us look at column $a_{.(r_1+1)}$. Notice that $r_1 + 1 > l_3$. Since w_1 is adjacent to w_2 and w_3, and w_2 and w_3 both start after w_1, then necessarily $a_{2(r_1+1)} = a_{3(r_1+1)} = 1$, and thus $a_{.(r_1+1)}$ is equal to the fourth column of F_0.

Finally, we look at the column $a_{.(r_2+1)}$. Notice that $r_2+1 > r_1+1$. Since A has the C1P, $a_{1(r_2+1)} = 0$ and $r_2 + 1 > r_1 + 1$, then $a_{1(r_2+1)} = 0$ and $a_{3(r_2+1)} = 1$, which is equal to last column of F_0. Therefore we reached a contradiction that came from assuming that there is a cycle of length 3 in $H(A)$.

Suppose now that $H(A)$ has an induced odd cycle $C = w_1, \dots, w_k, w_1$ of length $k \geq 5$. We assume without loss of generality that the first k rows of A are those in C and that A is ordered according to the C1P.

Remark 1. *Let w_i, w_j be vertices in $H(A)$. If w_i and w_j are adjacent and w_i starts before w_j, then $a_{ir_i} = a_{jr_i} = 1$ and $a_{i(r_i+1)} = 0$, $a_{j(r_i+1)} = 1$.*

Remark 2. If $l_{i-1} > l_i$ and $l_{i+1} > l_i$ for some $i = 3, \dots, k-1$, then for all $j \geq i+1$, w_j is nested in w_{i-1}. The same holds if $l_{i-1} < l_i$ and $l_{i+1} < l_i$. Since $l_{i-1} > l_i$ and $l_{i+1} > l_i$, then w_{i-1} and w_{i+1} are not disjoint, thus necessarily w_{i+1} is nested in w_{i-1}. It follows from this argument that this holds for $j \geq i+1$.

Notice that w_2 and w_k are nonadjacent, hence they are either disjoint or nested. Using this fact and Remark 1, we split the proof into two cases.

Case 1. *w_2 and w_k are nested*

We may assume without loss of generality that w_k is nested in w_2, for if not, we can rearrange the cycle backwards as w_1, w_k, $w_{k-1}, \dots, w_2, w_1$. Moreover, we will assume without loss of generality that both w_2 and w_k start before w_1. First, we need the following Claim.

Claim 1. *If w_2 and w_k are nested, then w_i is nested in w_2, for $i = 4, \dots, k-1$.*

Suppose first that w_1 and w_3 are disjoint, and for a proof by contradiction suppose that w_2 and w_4 are disjoint. In this case, $l_4 < l_3 < r_4 < l_2 < r_3 < r_2$. The contradiction is clear if $k = 5$. If instead $k > 5$ and w_5 starts before w_4, then $r_i < l_3$ for all $i > 5$, which contradicts the assumption that w_k is nested in w_2. Hence, necessarily w_5 is nested in w_3 and w_5 and w_2 are disjoint. This implies that $l_3 < l_5 < r_4 < r_5 < l_2$ and once more, $r_i < l_2$ for all $i > 5$, which contradicts the fact that w_k is nested in w_2.

Suppose now that w_3 is nested in w_1. For a proof by contradiction, suppose that w_4 is not nested in w_2. Thus, w_2 and w_4 are disjoint since they are nonadjacent vertices in $H(A)$. Notice that, if w_3 is nested in w_1, then $l_2 < l_3$ and $r_2 < r_3$. Furthermore, since w_4 is adjacent to w_3 and nonadjacent to w_2, then $l_3 < r_2 < l_4 < r_3 < r_4$. This holds for every odd $k \geq 5$.

If $k = 5$, since w_5 is nested in w_2, then $r_5 < r_2 < l_4$, which results in a contradiction for w_4 and w_5 are adjacent.

Suppose that $k > 5$. If w_2 and w_i are disjoint for all $i = 5, \ldots, k-1$, then w_{k-1} and w_k are nonadjacent for w_k is nested in w_2, which results in a contradiction. Conversely, if w_i and w_2 are not disjoint for some $i > 3$, then they are adjacent, which also results in a contradiction that came from assuming that w_2 and w_4 are disjoint. Therefore, since w_4 is nested in w_2, w_2 and w_i are nonadjacent and w_i is adjacent to w_{i+1} for all $i > 4$, then necessarily w_i is nested in w_2, which finishes the proof of the Claim.

Claim 2. *Suppose that w_2 and w_k are nested. Then, if w_3 is nested in w_1, then $l_i > l_{i+1}$ for all $i = 3, \ldots, k-1$. If instead w_1 and w_3 are disjoint, then $l_i < l_{i+1}$ for all $i = 3, \ldots, k-1$.*

Recall that, by the previous Claim, since w_i is nested in w_2 for all $i = 4, \ldots, k$, in particular w_4 is nested in w_2. Moreover, since w_3 and w_4 are adjacent, notice that, if w_3 is nested in w_1, then $l_3 > l_4$, and if w_1 and w_3 are disjoint, then $l_3 < l_4$.

It follows from Remark 2 that, if $l_5 > l_4$, then w_i is nested in w_3 for all $i = 5, \ldots, k$, which contradicts the fact that w_1 and w_{k-1} are adjacent. The proof of the first statement follows from applying this argument successively.

The second statement is proven analogously by applying Remark 2 if $l_5 < l_4$, and afterwards successively for all $i > 4$.

If w_1 and w_3 are disjoint, then we obtain $F_2(k)$ first, by putting the first row as the last row, and considering the submatrix given by columns $j_1 = l_1 - 1$, $j_2 = l_3$, …, $j_i = l_{i+1}$, …, $j_k = r_1 + 1$ (using the new ordering of the rows). If instead w_3 is nested in w_1, then we obtain $F_1(k)$ by taking the submatrix given by the columns $j_1 = l_1 - 1$, $j_2 = r_k$, …, $j_i = l_{k-i+2}$,

..., $j_{k-1} = r_3$.

Case 2. *w_2 and w_k are disjoint*

We assume without loss of generality that $l_2 < l_1$ and $l_k > l_1$.

Claim 3. *If w_2 and w_k are disjoint, then $l_i < l_{i+1}$ for all $i = 2, \dots, k-1$.*

Notice first that, in this case, w_i is nested in w_1, for all $i = 3, \dots, k-1$. If not, then using Remark 2, we notice that it is not possible for the vertices $w_1, \dots, w_k$ to induce a cycle. This implies, in particular, that w_3 is nested in w_1 and thus $l_2 < l_3$. Furthermore, using this and the same remark, we conclude that $l_i < l_{i+1}$ for all $i = 2, \dots, k-1$, therefore proving Claim 3.

In this case, we obtain $F_2(k)$ by considering the submatrix given by the columns $j_1 = l_1 - 1$, $j_2 = l_3$, ..., $j_i = l_{i+1}$, ..., $j_k = r_1 + 1$. □

4 Conclusions

Nested and 2-nested graphs are a particular case of those split graphs that are also circle. When comparing the structural characterization given in this work with known partial characterizations of circle graphs by minimal forbidden induced subgraphs, it is oddly interesting that this result gives several families of forbidden subgraphs which are not only different to those already known but are also infinite. As a consequence of this fact, a structural charaterization by minimal forbidden induced subgraphs for the entire class of circle graphs may be even harden than expected.

A possible sequel of this work could be characterizing those self-complementary graphs that are also circle graphs.

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