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# Size multipartite Ramsey numbers for bipartite graphs

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**Abstract**

Size multipartite Ramsey numbers were initially investigated when the sought monochromatic graph is complete, balanced and multipartite, extending the celebrated bipartite Ramsey numbers. Nowadays, this generalization has been studied for several classes of graphs. In this note we obtain near-optimal bounds and a few exact classes on the size multipartite Ramsey numbers when the required monochromatic graph is a bipartite graph $K_{2,n}$. In particular, an exact class for the four-cycle $C_4$ is derived by using Dirichlet's Theorem on primes numbers.

## 1 Introduction

Many variants and generalizations of the classical Ramsey numbers have been widely investigated. In this note we deal with the following extension introduced by Burger et al. [2, 3]. Let $K_{c\times s}$ denote the complete multipartite graph having $c$ classes with $s$ vertices per each class. Given a positive

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integer $c \geq 2$ and graphs $G_1, \ldots, G_k$, the *size multipartite Ramsey number* $m_c(G_1, \ldots, G_k)$ denotes the smallest positive integer $s$ (if it exists) such that any $k$-coloring of the edges of $K_{c \times s}$ contains a monochromatic copy of $G_i$ in color $i$ for some $i$, $1 \leq i \leq k$. As usual, the case where $G_i = G$ for $1 \leq i \leq k$ is simplified by $m_c(G; k)$.

Particularly interesting, $m_2(G_1, \ldots, G_k)$ corresponds to the well-known bipartite Ramsey number. We focus now on $m_c(K_{n,n_1}, \ldots, K_{n,n_k})$. For $n = 1$, these numbers were evaluated in [7] (when $c = 2$) and [9] (for an arbitrary $c \geq 2$). The case $n \geq 2$ is much more difficult even for $c = 2$. Indeed, bounds and few exact classes were determined in [1, 7], but the exact value for a general case remains a hard open problem. It is worth mentioning that the exact values of $m_2(C_4; k)$ are known only for $2 \leq k \leq 4$.

As the goal of this note, we estimate $m_c(K_{2,n_1}, \ldots, K_{2,n_k})$ when $c \geq 2$. For this purpose, we obtain an upper bound based on density arguments from [10] and a few lower bounds by using classic results linked to strongly regular graphs, finite fields, and number theory. In some cases these bounds are sharp or near-optimal. In particular, the following exact classes is established.

**Theorem 1.** *For each $u \geq 4$, there are infinitely many primes $k$ such that*

$$m_{uk}(C_4; k) = \lfloor k/u \rfloor + 1.$$

## 2 A few tools

We briefly describe the main tools used in this note.

**Theorem 2.** *(Dirichlet [6]) For relatively prime numbers $t$ and $u$, there are infinitely many prime numbers $p$ such that $p \equiv t$ (mod $u$).*

The celebrated Dirichlet's Theorem on primes in arithmetic progressions plays a central role in the proof of Theorem 1. Concepts and tools from graph theory and Ramsey theory are also applied here. Indeed, a closely related variant of the size multipartite Ramsey number is described as follows. Given a positive integer $s$ and graphs $G_1, \ldots, G_k$, the *set multipartite Ramsey number* $M_s(G_1, \ldots, G_k)$ denotes the smallest positive integer $c$ such that any $k$-coloring of the edges of $K_{c\times s}$ contains a monochromatic copy of $G_i$ in color $i$ for some $i$, $1 \le i \le k$. Both versions of the multipartite Ramsey numbers are linked by the relation below.

**Proposition 3.** *For integers $s \ge 1$, $c \ge 2$ and graphs $G_1, \ldots, G_k$, the equivalence holds: $m_c(G_1, \ldots, G_k) \le s$ if and only if $M_s(G_1, \ldots, G_k) \le c$.*

*Proof:* The proof is a natural extension of a particular case in [3]. ■

The upper bound below is proved by density arguments.

**Proposition 4.** *([10]) Let $s, k, n_1, \ldots, n_k$ be positive integers with $k \ge 2$. If a positive integer $c$ satisfies*

$$\binom{\frac{(c-1)s}{k}}{2} > \sum_{i=1}^{k}(n_i - 1)\binom{cs}{2}, \tag{1}$$

*then $M_s(K_{2,n_1}, \ldots, K_{2,n_k}) \le c$.*

Thus a combination of Propositions 3 and 4 yields the lemma below.

**Lemma 5.** *Let $c, k, n_1, \ldots, n_k$ be positive integers with $c \ge 2, k \ge 2$. If a positive integer $s$ satisfies (1), then $m_c(K_{2,n_1}, \ldots, K_{2,n_k}) \le s$.*

On the other hand, the following relationship is very useful to obtain lower bounds. The proof is similar to that in [10].

**Proposition 6.** *Suppose that $m_c(G_1, \ldots, G_k)$ exists. The following connection holds*

$$\left\lfloor \frac{r(G_1, \ldots, G_k) - 1}{c} \right\rfloor + 1 \le m_c(G_1, \ldots, G_k),$$

*where $r(G_1, \ldots, G_k) = M_1(G_1, \ldots, G_k)$ denotes the classic Ramsey number.*

## 3 Contribution: bounds and a few exact classes

The upper bounds $m_2(K_{2,n}, K_{2,n}) \leq 4n-3$ ([1]), $m_2(K_{2,2}; k) \leq k^2 + k - 1$ ([7]), and $m_2(K_{2,n}; k) \leq (n-1)k^2 + k - 1$ ([4]) are well-known. A corresponding version to an arbitrary $c \geq 2$ follows.

**Proposition 7.** *Given positive integers $c, k, n_1, \ldots, n_k$ with $c, k, n_1 \geq 2$, let $S = \sum_{i=1}^{k} n_i$. The upper bound holds*

$$m_c(K_{2,n_1}, \ldots, K_{2,n_k}) \leq \left\lceil \frac{ck(S-k) + (c-1)k}{(c-1)^2} \right\rceil.$$

*Proof:* The result is an application of Lemma 5. Indeed, note that the inequality (1) is equivalent to the inequality in the variable $s$

$$(c-1)^2 s^2 - \big(ck(S-k) + (c-1)k\big)s + k(S-k) > 0. \tag{2}$$

Thus the result follows if we show that $s_0 := \left\lceil \big(ck(S-k)+(c-1)k\big)/(c-1)^2 \right\rceil$ satisfies the inequality above.

For this purpose, write (2) in the form $\alpha s^2 + \beta s + \gamma > 0$ and let $\Delta := \beta^2 - 4\alpha\gamma$. If $\Delta < 0$, then any real number $s$ satisfies (2), in particular, $s_0$, hence the result is valid in this case. Otherwise, note that $0 \leq \Delta = \beta^2 - 4\alpha\gamma < \beta^2$ because $\alpha, \gamma > 0$ (here we use $n_1 \geq 2$). Since $\sqrt{\Delta} < |\beta| = -\beta$, we conclude that

$$\frac{-\beta + \sqrt{\Delta}}{2\alpha} < \frac{-\beta + (-\beta)}{2\alpha} \leq \left\lceil \frac{-\beta}{\alpha} \right\rceil = \left\lceil \frac{ck(S-k) + (c-1)k}{(c-1)^2} \right\rceil = s_0.$$

Thus $s_0$ satisfies (2) and Lemma 5 concludes the proof. ∎

**Theorem 8.** *(Exoo, Harborth, and Mengersen [5]) For each $n \geq 2$, $r(K_{2,n}, K_{2,n}) = 4n-2$ if and only if there exists a strongly regular graph with parameters $(4n-3, 2n-2, n-2, n-1)$.*

By exploring the literature on $r(G_1, \ldots, G_k)$, Proposition 6 might produce near-optimal lower bounds on $m_c(G_1, \ldots, G_k)$. For instance, Theorem 8 and Propositions 6 and 7 yield the next result.

**Proposition 9.** *Suppose that there is a strongly regular graph with parameters* $(4n-3, 2n-2, n-2, n-1)$. *Then*

$$\left\lfloor \frac{4n-3}{c} \right\rfloor + 1 \leq m_c(K_{2,n}, K_{2,n}) \leq \left\lceil \frac{4c(n-1)}{(c-1)^2} + \frac{2}{c-1} \right\rceil.$$

The classical construction of Paley graph assures that such graph with parameters $(4n-3, 2n-2, n-2, n-1)$ exists whenever $4n-3$ is a prime power, see [5] for instance. Furthermore, for $c \geq 3$, note that

$$\frac{4c(n-1)}{(c-1)^2} + \frac{2}{c-1} \leq \frac{4c(n-1)}{c(c-2)} + \frac{2}{c-2} = \frac{4n-2}{c-2}.$$

The remarks above and Proposition 9 yield the next result.

**Corollary 10.** *Given positive integers* $c, r$ *with* $c \geq 3$ *and a prime* $p$ *such that* $p^r \equiv 1 \pmod 4$, *let* $n = (p^r + 3)/4$. *Then*

$$\lfloor p^r/c \rfloor + 1 \leq m_c(K_{2,n}, K_{2,n}) \leq \lceil (p^r + 1)/(c-2) \rceil. \tag{3}$$

A closer look reveals that if $c = p^t$ for some $t$ such that $r/2 < t < r$, both lower and upper bounds in (3) are sharp, more specifically.

**Corollary 11.** *Let* $r, t$ *be positive integers such that* $r/2 < t < r$. *Given a prime* $p$ *such that* $p^r \equiv 1 \pmod 4$, *let* $n = (p^r + 3)/4$ *and* $c = p^t$. *Then*

$$m_c(K_{2,n}, K_{2,n}) = p^{r-t} + 1.$$

*Proof:* Apply Corollary 10 and note that $(p^r + 1)/(p^t - 2) \leq p^{r-t} + 1$. Indeed,

$$(p^{r-t} + 1)(p^t - 2) = p^r + \left(p^{r-t}(p^{2t-r} - 2)\right) - 2 \geq p^r + p - 2 \geq p^r + 1.$$

■

We now explore bounds on the multicolored case. Motivated by Corollary 11, a question arises: can we find an exact class for $k \geq 3$? In order to answer this question, we recall a result whose proof is based on finite fields.

**Theorem 12.** *(Lazebnik and Mubayi [8]) Given a prime $p$, suppose that $k = p^\alpha$ and $n = p^\beta + 1$ with $\alpha \geq 1$ and $\beta \geq 0$. Then*

$$(n-1)k^2 + 1 \leq r(K_{2,n}; k).$$

**Theorem 13.** *Let $n, u \geq n+2, k \geq 2$ be positive integers such that $k \equiv 0 \pmod{u}$ or $k \equiv 1 \pmod{u}$. Suppose $k = p^\alpha$ and $n = p^\beta + 1$ with $\alpha \geq 1$ and $\beta \geq 0$ for some prime $p$. Then*

$$m_{uk}(K_{2,n}; k) = \lfloor (n-1)k/u \rfloor + 1.$$

*Proof:* Let $s = \lfloor (n-1)k/u \rfloor + 1$. Proposition 6 and Theorem 12 produce the desired lower bound $m_{uk}(K_{2,n}; k) \geq s$. The proof of the upper bound $m_{uk}(K_{2,n}; k) \leq s$ requires accurate estimates. Indeed, Proposition 7 yields

$$m_{uk}(K_{2,n}; k) \leq \left\lceil \frac{(n-1)uk^3 + uk^2 - k}{u^2k^2 - 2uk + 1} \right\rceil \leq \left\lceil \frac{(n-1)k^2 + k}{uk-2} \right\rceil.$$

Since $(n-1)k^2 + k = (uk-2)\left( \frac{(n-1)k}{u} + \frac{2(n-1)+u}{u^2} \right) + \frac{4(n-1)+2u}{u^2}$, we can write

$$m_{uk}(K_{2,n}; k) \leq \left\lceil \frac{(n-1)k}{u} + \frac{2(n-1)+u}{u^2} + \frac{4(n-1)+2u}{u^2(uk-2)} \right\rceil. \tag{4}$$

By hypothesis, there is $q \in \mathbb{N}$ such that $k = qu + r$, where $r = 0$ or $r = 1$. Thus $(n-1)k/u = (n-1)q + r(n-1)/u \leq (n-1)q + (n-1)/u$. Let $\varepsilon = (n-1)u + 2(n-1) + u + (4(n-1)+2u)/(uk-2)$. After a simple algebraic manipulation, (4) and the facts above yield

$$m_{uk}(K_{2,n}; k) \leq \lceil (n-1)q + \varepsilon/u^2 \rceil. \tag{5}$$

It remains to estimate $\varepsilon$. By hypothesis, $n-1 \leq u-3$ holds, and consequently

$$\varepsilon \leq (u-3)u + 2(u-3) + u + \frac{4(u-3)+2u}{uk-2} \leq u^2 - 6 + \frac{6u-12}{2u-2} < u^2 - 3. \tag{6}$$

Finally, the inequalities (5) and (6) imply

$$m_{uk}(K_{2,n}; k) \leq (n-1)q + 1 = (n-1)\lfloor k/u \rfloor + 1 \leq \lfloor (n-1)k/u \rfloor + 1 = s.$$

■

As an immediate consequence, the exact class below is derived.

**Corollary 14.** *Consider integers $\beta \geq 0$, $\alpha \geq \gamma \geq 1$. If $p$ is a prime such that $p^\gamma \geq p^\beta + 3$, then*

$$m_{p^{\alpha+\gamma}}(K_{2,p^\beta+1}; p^\alpha) = p^{\alpha+\beta-\gamma} + 1.$$

*Proof:* Apply Theorem 13 with $k = p^\alpha$, $n = p^\beta + 1$ and $u = p^\gamma$. ■

For example, select $p = 3$, $\beta = 2$ and $\gamma = 3$. The result above yields $m_{3^{\alpha+3}}(K_{2,10}; 3^\alpha) = 3^{\alpha-1}+1$ for any $\alpha \geq 3$. In particular, $m_{729}(K_{2,10}; 27) = 10$.

Moreover, the case $\beta = 0$ in Theorem 13 produces $n = p^\beta + 1 = 2$. Since the graphs $K_{2,2}$ and $C_4$ are isomorphic, the following statement follows.

**Corollary 15.** *For an integer $u \geq 4$ and a prime power $k$ such that $k \equiv 0 \ (mod\ u)$ or $k \equiv 1 \ (mod\ u)$,*

$$m_{uk}(C_4; k) = \lfloor k/u \rfloor + 1.$$

In contrast with Theorem 13, a closer look reveals that $k$ in Corollary 15 does not need to be a power of a pre-determined prime, but a power of any prime. In particular, Theorem 2 states that there are infinitely many prime numbers $k$ such that $k \equiv 1 \ (mod\ u)$. This fact combined with Corollary 15 imply Theorem 1.

Particularly interesting, Theorem 1 is essentially existential, due to the fact that the sequence of prime numbers in Theorem 2 is proved by existential approach.

# 4 Final Remarks

In this work we investigated a few bounds on $m_c(K_{2,n_1}, \ldots, K_{2,n_k})$ by using density arguments and known tools from graph theory, finite fields, and number theory. For future research, it would be interesting to explore bounds on $m_c(K_{n,n_1},\ldots,K_{n,n_k})$ for $n \geq 3$ as well as theirs connections with algebraic and combinatorial structures.

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