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On the Sum of the Angles of Orthogonal Polyhedra, and Guarding Orthogonal Polyhedra

Jorge Urrutia

Abstract

In this paper we review some recent results on orthogonal polyhedra. We give a characterization of the orthogonal polyhedra in $\mathbb{R}^3$ that minimize and maximize the sum of their internal solid angles. To prove our result we generalize the well-known result that any orthogonal polygon with n vertices has $\frac{n+4}{2}$ convex and $\frac{n-4}{2}$ reflex vertices. We prove that an orthogonal polyhedron whose 1-skeleton is a connected cubic graph has $\frac{n+8}{2}$ convex and $\frac{n-8}{2}$ reflex vertices. A general bound for orthogonal polyhedra of arbitrary genus allowing for disconnected 1-skeletons is also given. These results are then used to obtain bounds on the number of edge lights used to guard orthogonal polyhedra in $\mathbb{R}^3$.

1 Introduction

In this paper we deal with *orthogonal polyhedra*. A *polyhedron* in $\mathbb{R}^3$ is a compact set bounded by a piecewise linear manifold. A *face* of a

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polyhedron is a maximal planar subset of its boundary whose interior is connected and non-empty. A polyhedron is *orthogonal* if all of its faces are parallel to the xy-, xz- or yz-planes. Faces of a polyhedron can be polygons with holes, and if the polyhedron is orthogonal, its faces and its holes are orthogonal polygons. A *vertex* of a polyhedron is a vertex of any of its faces. An *edge* is a minimal positive-length straight line segment shared by two faces and joining two vertices of the polyhedron.

In the plane, to measure the size of an angle α at a vertex of a polygon P, we consider a small enough circle C centered at v and not containing any other vertices of the polygon; the size of α is the length of the portion of the circle that lies inside the polygon divided by the radius of C. It has been well-known since antiquity that the sum of the angles of a triangle is π. Since a simple polygon of n vertices can be partitioned into exactly $n-2$ triangles using diagonals joining pairs of vertices of the polygon, the sum of the internal angles of a polygon is always $(n-2)\pi$.

The natural generalization of the size of an angle to dimension three is the following: Let $\mathcal{P}$ be a polyhedron in $\mathbb{R}^3$ and let v be a vertex of $\mathcal{P}$. The *solid angle* of $\mathcal{P}$ at v is defined as follows: Consider a small enough sphere $\mathcal{C}$ centred at v. The size of the angle of $\mathcal{P}$ at v is the *area* of the portion of the boundary of $\mathcal{C}$ that lies within $\mathcal{P}$ divided by the square of the radius of $\mathcal{C}$. Since the area of a unit sphere is 4π, it follows that the maximum size of the solid angles at vertices of a polyhedron is at most 4π; see Figure 1.

It is not hard to see that the sum of the solid angles of a tetrahedron is at most 2π. An easy way to see this is as follows: Consider a very small tetrahedron $\mathcal{T}$ contained in a unit sphere $\mathcal{S}$ and containing the center of $\mathcal{C}$. If we think of an angle at a vertex v of $\mathcal{T}$ as a floodlight f illuminating the cone generated by v and its opposite triangular face in $\mathcal{T}$, it is easy to see that for each point p on $\mathcal{S}$ illuminated by f, there is another point q that is not illuminated; that is, the point opposite to p with respect to to v; see Figure 2.

Some problems arise when we consider solid angles of a polyhedron in

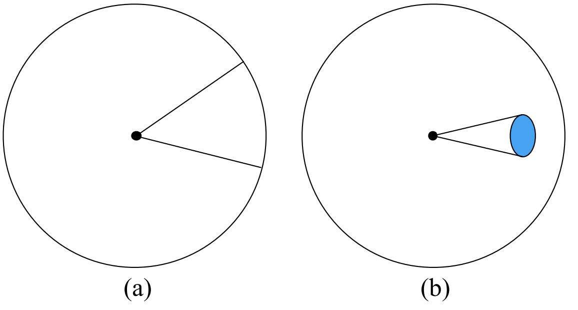


Figure 1: (a) An angle in the plane (b) A solid angle in R^3.

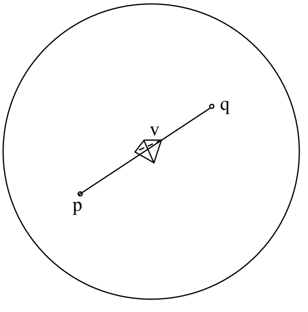
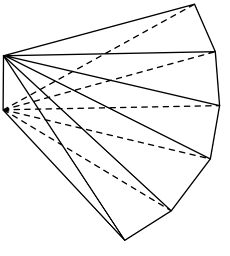


Figure 2: The point p is illuminated from v, while q is not.

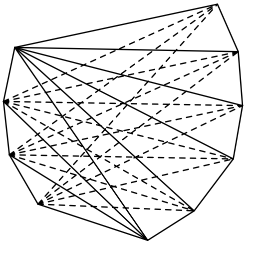
R^3. To start, it is not true that the sum of the solid angles of a tetrahedron is constant. In fact for any $0 < \alpha < 2\pi$ there is tetrahedron such that the sum of its solid angles is α [9].

One proof of this follows from the well-known result that there are convex polyhedra that can be decomposed into a linear and a quadratic number of tetrahedra. Well-known examples of this are the so-called neighborly polytopes [10]. Another example of convex polytopes that can be decomposed into a quadratic number of tetrahedra are what we call (n, m)-seashells.

The vertices of a $(1 \times r)$-seashell are the n vertices of a convex chain with $r+1$ vertices contained in a plane P in $\mathbb{R}^3$, plus the vertices of a segment not parallel to P, facing the convex chain, and far enough from it. In Figure 3(a) we show a $(1,5)$-seashell. Clearly a $(1,r)$-seashell has a tetrahedralization with r tetrahedra. An $(n{\times}m)$-seashell contains $n{+}m{+}2$ vertices, n on a convex chain and m on a second convex chain contained in two orthogonal planes, facing each other, and far enough apart. In Figure 3(b) we show a $(3,5)$-seashell. Clearly an (n,m)-seashell can be decomposed into $n \times m$ tetrahedra; see Figure 3(b). But the sum of the solid angles of these tetrahedra is the sum of the solid angles of the (n,m)-seashell, which is at most $2(n+m+2) \times \pi$ (the size of the solid angle at each vertex of a convex polytope is at most 2π), and thus for at least one of them the sum of its solid angles is at most $\frac{2(n+1)(m-1)\times\pi}{n\times m}$, which approaches zero as n and m grow.



(a) A (1,5)-sea shell



(b) A (3,5)-sea shell

Figure 3: Two (n,m)-seashells examples.

It is not hard to construct polyhedra with an arbitrarily large number of vertices such that the sum of the solid angles at their vertices is as small as possible. This is not, however, the case for orthogonal polyhedra; the size of the solid angle at each of their vertices is at least $\pi/2$ and at most $7\pi/2$, as each vertex of an orthogonal polyhedron covers one, three, four, five, or seven octants; see Figure 6.

Thus a natural question that arises is the following: Can we characterize orthogonal polyhedra with n vertices that minimize or maximize the sum of their solid angles? We prove that the sum of the solid angles of an orthogonal polyhedron is at least $(n-4+4g)\pi$ and at most $(3n-24-4g)\pi$, where g is the genus of the polyhedron. Our bounds are tight. To prove our main result, we will need to generalize to $\mathbb{R}^3$ the following well-known result for orthogonal polygons in the plane:

Theorem 1. *[12] Any orthogonal polygon in the plane $\mathbb{R}^2$ has $\frac{(n+4)}{2}$ convex and $\frac{(n-4)}{2}$ reflex vertices.*

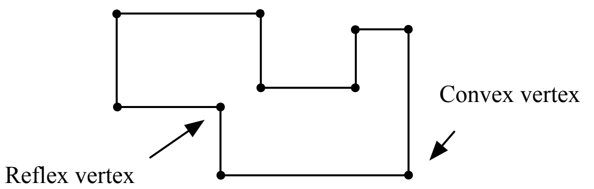


Figure 4: Reflex and convex vertices of an orthogonal polygon.

We define a vertex of a polyhedron $\mathcal{P}$ to be *convex on the faces*, *convex* for short, if it is a convex vertex in all of the faces of $\mathcal{P}$ to which it belongs, otherwise we call it a *reflex vertex*. We prove that the 1-skeleton of an orthogonal polyhedron of genus g with n vertices, k of which have degree greater than or equal to 4, has $(n+8-8g+3k)/2$ convex, and $(n-8+8g-3k)/2$ reflex vertices.

We then use this result to address a variant of the Art Gallery Problem in orthogonal polyhedra. Given an edge f of a polyhedron $\mathcal{P}$, a point $q \in \mathcal{P}$ is guarded by f if there is a point $p \in f$ such that the line segment joining p to q is contained in $\mathcal{P}$. We prove that if the orthogonal polyhedron has k_4 vertices of degree 4, k_6 vertices of degree 6, e edges, genus g and h_m holes on its faces, then we can guard it using at most $\frac{(11e-k_4-3k_6-12g-24h_m+12)}{72}$ $\frac{\pi}{2}$-edge guards (i.e., guards that have a visibility angle of $\frac{\pi}{2}$ towards the interior of the polyhedron), slightly refining the bound given in [4] for open edge guards. Finally, we show a family of orthogonal polyhedra that needs $\frac{4}{45}e$ $\frac{\pi}{2}$-edge guards to guard it.

Most of the research on art gallery problems has been focused on polygons on the plane. For example, it is well-known that every simple polygon with n vertices can always be guarded with at most $\lfloor \frac{n}{3} \rfloor$ vertex guards; see Chvátal [7]. For orthogonal polygons $\lfloor \frac{n}{4} \rfloor$ vertex guards are always sufficient; see Khan, Klawe and Kleitman [11]. Estivill-Castro and Urrutia [8] proved that every orthogonal polygon can be guarded with at most $\frac{3(n-1)}{8}$ *orthogonal floodlights*; that is, vertex guards that have an angle of vision of $\frac{\pi}{2}$. Later in [1] it was proved that $\frac{(3n+4(h-1))}{8}$ orthogonal floodlights are always sufficient to guard an orthogonal polygon with n vertices and h holes.

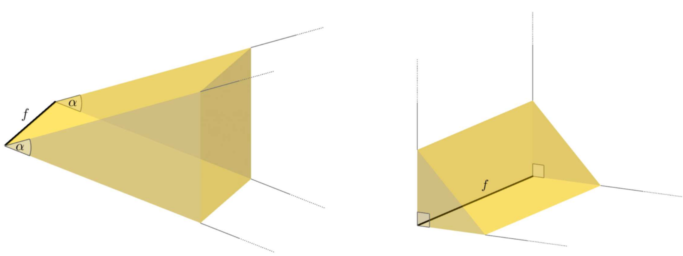
The problem of determining bounds on the number of edge guards required to guard a polyhedron was introduced by Urrutia [14]. He conjectured that any polyhedra with e edges in $\mathbb{R}^3$ can always be guarded with $\frac{e}{6} \pm c$ edges, and for orthogonal polyhedra he conjectured that $\frac{e}{12} \pm c$ edges always suffice. These conjectures remain open. Benbernou *et al.* [4] proved that every orthogonal polyhedron with n vertices of genus g can always be guarded by $\frac{11e}{72} - \frac{g}{6} - 1$ *open edge guards* (i.e., excluding their endpoints). Cano *et al.* [6] proved that any polyhedron can always be guarded by $\frac{27e}{32}$ edge guards, and if the faces of the polyhedron are all triangles the bound improves to $\frac{29e}{36}$.

2 Some definitions and preliminary results

Given a point p on the plane, an α-guard (also called an α-floodlight) is a guard that surveils an area within an angular cone of size α whose apex is at p; see [5]. We now extend the definition of α-guards to $\mathbb{R}^3$. A *wedge* in $\mathbb{R}^3$ is the intersection, or the union, of two semispaces whose supporting planes intersect. The line of intersection of the supporting planes is called the *axis* of the wedge. A wedge is called *small* if it is the intersection of two semispaces, and *big* if it is the union of two semispaces. Note that if a wedge $\mathcal{W}$ is small, then the intersection of $\mathcal{W}$ with a plane orthogonal to the axis of $\mathcal{W}$ determines an angular region $\mathcal{A}$ of size less than or equal to

π. If $\mathcal{W}$ is a big wedge, then the size of the $\mathcal{A}$ is greater than π. The wedge $\mathcal{W}$ will be called an α-wedge if the size of $\mathcal{A}$ is α. An *orthogonal wedge* in $\mathbb{R}^3$ is the intersection or the union of two semispaces whose supporting planes are orthogonal. If an orthogonal wedge is small, it is a $\frac{\pi}{2}$-wedge; if it is big it is a $\frac{3\pi}{2}$-wedge.

Let f be an edge of a polyhedron $\mathcal{P}$. We call f an α-edge guard of $\mathcal{P}$ if f guards all of the points of $\mathcal{P}$ visible from f and is contained in an α-wedge whose axis contains f; see Figure 5a. If $\alpha = \frac{\pi}{2}$ we call f an *orthogonal edge guard.* We note that α-guards and orthogonal edge guards are natural generalizations of α-floodlights [5], and $\frac{\pi}{2}$-floodlights on the plane [8]. We assume that an α-edge guard f can be rotated around f until it reaches a final orientation. In the rest of this paper we will assume that orthogonal-edge guards are always placed in such a way that their supporting planes are parallel to the xy-, xz- or yz-planes of $\mathbb{R}^3$. In our language, the open edge guards used by Benbernou *et al.* are open $\frac{3\pi}{2}$-edge guards in which the endpoints of the edges are not included. In our paper we will prove a similar result to that proved by Benbernou *et al.* but using $\frac{\pi}{2}$-edge guards. In fact we will use ortho-$\frac{\pi}{2}$-edge guards f, where f is an edge of a polyhedron. These guards surveil only points p within a $\frac{\pi}{2}$-wedge with the additional restriction that the shortest line segment joining p to f is a line segment orthogonal to f; see Figure 5b. For the sake of simplicity, we will refer to these edge-guards as $\frac{\pi}{2}$-edge guards.



(a) α-wedge (b) $\pi/2$-wedge

Figure 5: An illustration of an α-wedge and a $\pi/2$-wedge.

3 The number of convex and reflex vertices of an orthogonal polyhedron

In this section, we will assume that the 1-skeleton of the orthogonal polyhedra considered here is a cubic connected graph. We now prove:

Theorem 2. *Let $\mathcal{P}$ be an orthogonal polyhedron in $\mathbb{R}^3$ homeomorphic to the sphere, with $n = 2k$ vertices, and such that its 1-skeleton is a* 3-regular *graph. Then $\mathcal{P}$ has $\frac{(n+8)}{2}$ convex vertices and $\frac{(n-8)}{2}$ reflex vertices.*

Proof. Since each vertex has degree 3, the number of edges e of $\mathcal{P}$ is $3k$. By Euler's formula, which asserts that for any planar graph with e edges, f faces and v vertices $f - e + v = 2$, it follows that the number of faces f is $k + 2$.

Note that each face f_i of $\mathcal{P}$ is an orthogonal polygon. If f_i has n_i vertices, by Theorem 1 it has $(n_i - 4)/2$ reflex vertices. Observe that any vertex of $\mathcal{P}$ appears in three faces of $\mathcal{P}$, and it is a reflex vertex of at most one of these faces. Then the number of reflex vertices of $\mathcal{P}$ satisfies the following equation:

$$r = \sum_{i=1}^{k+2} \frac{n_i - 4}{2}. \tag{1}$$

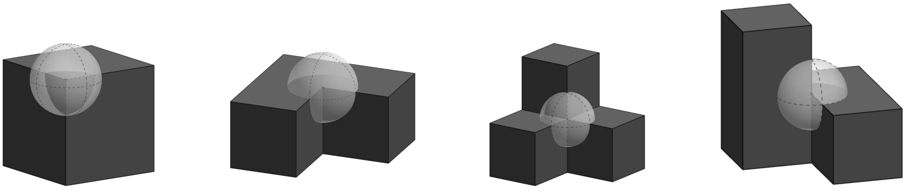
Multiplying Equation (1) by 2, we have

$$2r = \sum_{i=1}^{k+2} n_i - \sum_{i=1}^{k+2} 4.$$

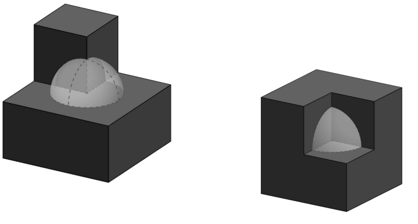
As each vertex belongs to three faces of $\mathcal{P}$, it is counted three times when we calculate the first sum; thus we have:

$$2r = 6k - 4(k + 2)$$
$$r = k - 4.$$

Since $n = 2k$, it follows that $r = \frac{(n-8)}{2}$, and since $n = c + r$, $c = \frac{(n+8)}{2}$. $\square$



(a) 1-octant vertex (b) 3-octant vertex (c) 4-octant vertex (d) 4-octant vertex



(e) 5-octant vertex (f) 7-octant vertex

Figure 6: Vertex classification for orthogonal polyhedra.

3.1 Minimizing the angle sum for orthogonal polyhedra whose 1-skeleton is a cubic connected graph.

Observe that the convex vertices of $\mathcal{P}$ are 1- or 7-octant vertices; see Figure 6(a) and (f), while reflex vertices are 3- or 5-octant vertices; see Figure 6(b) and (e).

Then by Theorem 2 if we can construct orthogonal polyhedra with $n = 2k$ vertices without 5- or 7-octant vertices, they will minimize their sum of solid angles. These polyhedra can be constructed as follows:

Take an orthogonal polygon Q contained in the $x-y$ plane in R^3, and a copy Q' of Q obtained by translating Q vertically by one unit; see Figure 7. Then the convex hull of $Q \cup Q'$ is an orthogonal polyhedron having only 1- and 3-octant vertices.

Since the size of the solid angle of an k-octant vertex is $k \times \frac{\pi}{2}$, the sum

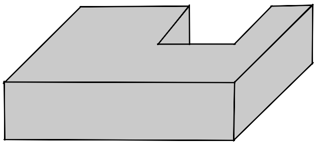


Figure 7: An orthogonal polyhedron with 16 vertices such that the sum of its angles is the smallest possible.

of the solid angles of these polyhedra is:

$$\frac{\pi}{2}\left[\frac{n+8}{2}+\frac{3(n-8)}{2}\right],$$

which equals $\pi(n-4)$.

Thus we have proved:

Theorem 3. *The sum of the solid angles of an orthogonal polyhedron whose* 1*-skeleton is a connected cubic graph is at least* $\pi(n-4)$*. The bound is tight.*

The following result, given without proof, generalizes the previous result to orthogonal polyhedra of arbitrary genus for which the vertices of their 1-skeleton may contain vertices with degree 3, 4 and 6. Given an orthogonal polyhedron $\mathcal{P}$, let k_i be the number of vertices of degree i in the 1-skeleton of $\mathcal{P}$. Vertices of degree 3 correspond to 1-, 3-, and 7-octant vertices. Vertices with degree 4 correspond to some of the 4-octant vertices, and vertices of degree 6 also correspond to some 4-octant vertices. See Figure 6.

Theorem 4. *Let* $\mathcal{P}$ *be an orthogonal polyhedron in* $\mathbb{R}^3$ *with* $n = k_3+k_4+k_6$ *vertices and arbitrary genus* g*. Then* $\mathcal{P}$ *has* $(n+3(k_4+k_6)-8g+8)/2$ *convex vertices and* $(n-3(k_4+k_6)+8g-8)/2$ *reflex vertices.*

This result is proved using the Euler-Poincaire formula, which states that for any polyhedron of genus g with f faces, e edges, v vertices and a total of h holes on its faces, the following identity holds [13]:

$$v-e-h+f=2-2g.$$

The details of the proof can be found in [2].

4 Minimizing the sum of the solid angles of orthogonal polyhedra

To solve the problem of characterizing orthogonal polyhedra with genus g that minimize the sum of their solid angles, we will use the Gauss-Bonnet formula, which asserts that for a polyhedron $\mathcal{P}$ of genus g, the sum of the deficiencies at its vertices equals $2\pi(2-g)$. Let V_i be the number of i-octant vertices, $i = 1, 3, 4, 5, 7$. Thus

$$V_1 + V_3 + V_4 + V_5 + V_7 = n. \tag{2}$$

Note that the sum of the solid angles of an orthogonal polyhedron is

$$S = \frac{\pi}{2}V_1 + \frac{3\pi}{2}V_3 + 2\pi V_4 + \frac{5\pi}{2}V_5 + \frac{7\pi}{2}V_7. \tag{3}$$

Since the defect of 1-octant and 7-octant vertices is $\pi/2$, the deficiency of 3-octant and 5-octant vertices is $-\pi/2$ and the deficiency of 4-octant vertices is $-\pi$. Applying the Gauss-Bonnet theorem, where g is the genus of the polyhedron, we get

$$\frac{\pi}{2}(V_1 + V_7) - \frac{\pi}{2}(V_3 + 2V_4 + V_5) = 4\pi - 4\pi g. \tag{4}$$

Multiplying (2) by π and subtracting (4) we obtain

$$\frac{\pi}{2}V_1 + \frac{3\pi}{2}V_3 + 2\pi V_4 + \frac{3\pi}{2}V_5 + \frac{\pi}{2}V_7 = n\pi - 4\pi + 4\pi g. \tag{5}$$

Adding $\pi V_5 + 3\pi V_7$ to both sides of (5) yields

$$\frac{\pi}{2}V_1 + \frac{3\pi}{2}V_3 + 2\pi V_4 + \frac{5\pi}{2}V_5 + \frac{7\pi}{2}V_7 = \pi n - 4\pi + 4\pi g + \pi V_5 + 3\pi V_7. \tag{6}$$

The left side of (6) corresponds to the angle sum

$$S = \pi(n - 4 + 4g + V_5 + 3V_7). \tag{7}$$

Thus (7) is minimized when V_5 and V_7 are both equal to zero. The next result follows.

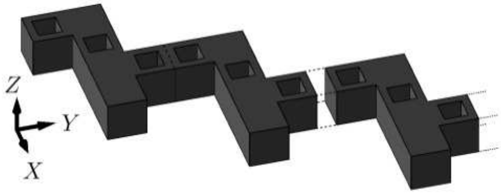


Figure 8: Family of polyhedra that minimize their solid angle sum.

Theorem 5. *The sum of the solid angles of an orthogonal polyhedron with n vertices and genus g is at least $(n - 4 + 4g)\pi$. This bound is sharp and it is achieved by polyhedra having only 1-octant, 3-octant, and possibly 4-octant vertices.*

In Figure 8 we show polyhedra of arbitrary genus all of whose vertices have degree 3.

The problem of characterizing orthogonal polyhedra that maximize their solid angle sum can be easily obtained from Theorem 5. This is done by taking an orthogonal polyhedron such that the sum of its angles is minimized, and turning it inside-out by enclosing it in a cube, see Figure 9.

Theorem 6. *The sum of the solid angles of orthogonal polyhedra with n vertices and genus g is at most $(3n - 24 - 4g)\pi$. The bound is sharp.*

5 Guarding orthogonal polyhedra

In this section we will use the results of the previous section to obtain bounds on the number of edge guards needed to guard an orthogonal polyhedron with e edges. J. Urrutia [14] conjectured in 1996 that every orthogonal polyhedron with e edges can can always be guarded using at most $\frac{e}{12} \pm c$ edge guards; see Figure 10.

He proved that $\frac{e}{6}$ edge guards always suffice. This is proved as follows: Let $\mathcal{P}$ be an orthogonal polyhedron. An edge of $\mathcal{P}$ is called an x-edge if it is parallel to the x-axis; y- and z-edges are defined in a similar way. Let

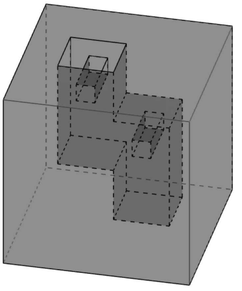


Figure 9: Turning an orthogonal polyhedron inside-out by enclosing it in a cube.

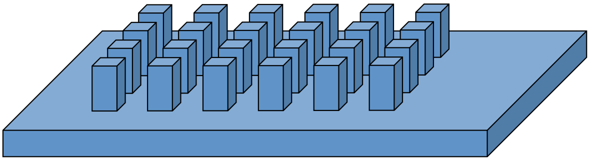


Figure 10: Family of orthogonal polyhedra that requires $\frac{e}{12}-1$ edge guards to guard it.

E_x, E_y and E_z be the sets containing, respectively, all of the x-, the y-, or the z-edges of $\mathcal{P}$.

It is easy to see that the sets of edges E_x, E_y, and E_z guard $\mathcal{P}$. Consider E_x and a point p in the interior of $\mathcal{P}$. Take the plane through p orthogonal to the x-axis. The intersection of this plane with $\mathcal{P}$ is an orthogonal polygon R_p, see Figure 11. It is clear that p is guarded by at least one vertex v of Q (in fact by at least four of them), which implies that p is guarded by the x-edge of $\mathcal{P}$ intersecting R_p at v. Since one of E_x, E_y, or E_z contains at most $\frac{e}{3}$ edges, it follows that $\mathcal{P}$ can be guarded using at most $\frac{e}{3}$ of its edges.

One might try to use the well-known result that that any orthogonal polygon can be guarded with at most a quarter of its vertices [11], but this

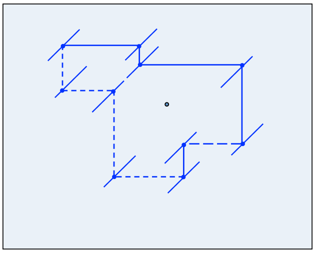


Figure 11: The intersection of a plane orthogonal to the x-axis and an orthogonal polyhedron.

is not possible; the reason is the following: Suppose that for two points p and q in the interior of $\mathcal{P}$, the polygons R_p and R_q intersect an x-edge f of $\mathcal{P}$. This edge determines a vertex in each of the two polygons; call these vertices v_p and v_q respectively. But it could happen that v_p is in the set with at most a quarter of the vertices of R_p that guard R_p, but v_q is not in the set of at most a quarter of the vertices of R_q that guard R_q!

Thus what we need is a consistent way to choose x-edges of P such that if v_p is chosen in R_p, then v_q is also chosen in R_q. To achieve this we need to review some results on guarding orthogonal polygons.

Consider an orthogonal polygon P. We split its edges into four classes; *top*-, *bottom*-, *left*-, and *right*-edges as follows: A *top*-edge e of P is one such that if we draw a small enough vertical segment ℓ through the mid point of e, the portion of ℓ below e belongs to P; *bottom*-, *left*-, and *right*-edges are defined in a similar way; see Figure 12.

A vertex of P is called a *top-left* vertex if it is a vertex of a *top*-edge and a *left*-edge of P. In a similar way we define *top-right*, *bottom-left*, and *bottom-right* vertices. It is straightforward to see that the union of the top-left and the bottom-right vertices of P are exactly half of the vertices of P, and that the remaining points are the union of the top-right and the bottom-left vertices. Furthermore, each of these sets of vertices guard P.

Consider now an orthogonal polyhedron $\mathcal{P}$. We call a face h of $\mathcal{P}$ a *top*-face if for a small enough vertical segment ℓ that intersects h at a point

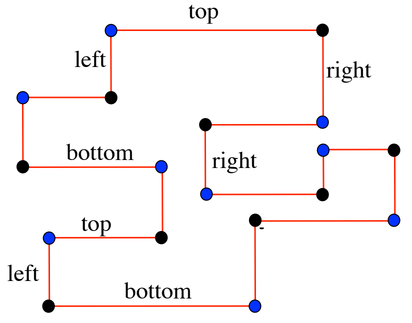


Figure 12: The top-left and bottom-right vertices are colored blue, and the top-right and the bottom-left vertices are colored black.

in the interior of h, the sector of ℓ below h belongs to $\mathcal{P}$. In a similar way we define *bottom*, *right*, *left*, *front*, and *back*-faces of $\mathcal{P}$. Consider now the faces of $\mathcal{P}$ not perpendicular to the x-axis; these are its top, bottom, right, and left faces. An x-edge of $\mathcal{P}$ is called a *top-left* edge if it belongs to a top and a left face of $\mathcal{P}$. In a similar way we also define *top-right*, *bottom-left* and *bottom-right* edges.

It now follows easily that the set containing all of the *top-left* and the *bottom-right* edges of $\mathcal{P}$ guard it, as does the set containing all of its *top-right* and the *bottom-left* edges. Thus we have:

Theorem 7. *Any orthogonal polyhedron can be guarded using at most one sixth of its edges.*

Note that the proof of Theorem 7 holds for any orthogonal polyhedron regardless of its genus or its 1-skeleton being connected or not.

5.1 Improving the bound of Theorem 7

We now show how to use the results of the previous section to obtain a slight improvement on the $\frac{e}{6}$ bound we just proved. To do this we will need the following result, proved in [8]. Let p be a point in the plane, and let $C_1(p)$, $C_2(p)$, $C_3(p)$, and $C_4(p)$ be the translations of the four quadrants of the plane when the origin is translated to p. The *top-left* guarding rule to guard an orthogonal polygon P is the following:

Top-left guarding rule: At each top-left vertex p of P place an orthogonal guard that guards $C_4(p)$.

Theorem 8. *[8] The floodlights used by the top-left guarding rule guard P.*

In a similar way we can define the *top-right*, *bottom-right*, and *bottom-left* guarding rules, each of which guards P; see Figure 13.

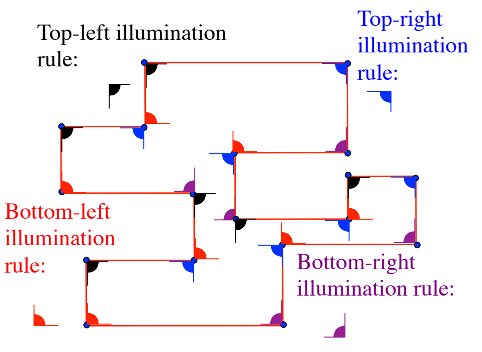


Figure 13: The top-left and the bottom right vertices are colored blue, and the top-right and the bottom-left vertices are colored black.

Using the four guarding rules defined above, we place two orthogonal edge guards at each reflex vertex of P, and we place one at each convex vertex of P. It now follows that these four guarding rules for orthogonal polygons induce four guarding rules on the x-edges of an orthogonal polyhedron $\mathcal{P}$. We would place two orthogonal edge guards on each reflex x-edge of $\mathcal{P}$, and one for each convex x-edge.

Applying a similar procedure to the y- and the z-edges of P we obtain a set of twelve guarding rules, each of which guards $\mathcal{P}$. Thus we have:

Lemma 1. *Let $\mathcal{P}$ be an orthogonal polyhedron with e edges, c of which* *are convex, and r of which are reflex. Then we can guard $\mathcal{P}$ with $\frac{c+2r}{12}$ orthogonal edge guards.*

To improve on the bounds of Theorem 7 we need to guarantee that we have a large bound on the number of convex edges of $\mathcal{P}$, for otherwise the existence (which does not happen) of an orthogonal polyhedron $\mathcal{P}$ in which most of its edges were reflex edges would yield close to $\frac{e}{6}$ edge guards in Lemma 1. The bounds we need are those obtained in Theorem 4.

Theorem 9. *Let $\mathcal{P}$ be an orthogonal polyhedron with n vertices, k_4 of degree 4 and k_6 of degree 6; e edges, genus g and h_m holes in the faces of $\mathcal{P}$. Then $(11e - k_4 - 3k_6 - 12g - 24h_m + 12)/72$ is the number of $\pi/2$-edge guards that are always sufficient to guard the interior of $\mathcal{P}$.*

Proof. First we look at the types of vertices of the polyhedron $\mathcal{P}$, and describe the number of convex and reflex edges that each kind of vertex is incident to.

Each 1-octant vertex is incident to three convex edges. Each 3-octant vertex is incident to two convex edges and one reflex edge. Each 4-octant vertex with degree four is incident to two convex edges and two reflex edges. Each 4-octant vertex with degree six is incident to three convex edges and three reflex edges. Each 5-octant vertex is incident to one convex edge and two reflex edges. Finally, each 7-octant vertex is incident to three reflex edges.

By Theorem 4, $\mathcal{P}$ has $c = (n + 3(k_4 + k_6) - 8g + 8)/2$ convex vertices and $r = (n - 3(k_4 + k_6) + 8g - 8)/2$ reflex vertices. Recall that according to our definition, 4-octant vertices, whether they have degree four or six, are convex. Then $\mathcal{P}$ has $c' = (n + k_4 + k_6 - 8g + 8)/2$ convex vertices of degree three, k_4 4-octant vertices of degree four, k_6 4-octant vertices of degree six, and $r = (n - 3(k_4 + k_6) + 8g - 8)/2$ reflex vertices.

In the worst case every convex vertex is adjacent to three reflex edges, every 4-octant vertex of degree four is adjacent to two reflex edges and two convex edges, every 4-octant vertex of degree six is adjacent to three reflex edges and three convex edges, and every reflex vertex is incident to two reflex edges and one convex edge.

Place $\pi/2$-edge guards on all of the x-edges using the top-left, top-right, bottom-left and bottom-right rules. In a similar way, place $\pi/2$-edge guards in all of the y- and z-edges of $\mathcal{P}$. This uses in total $(6c + 6k_4 + 9k_6 + 5r)/2$ $\pi/2$-edge guards. Suppose that $\mathcal{P}$ has fewer than or the same number of x-edges as y- or z-edges. Then choose among the top-left, top-right, bottom-left and bottom-right guarding rules the one that places the fewest $\pi/2$-edge guards.

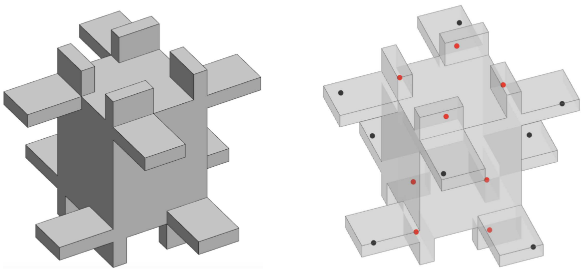
It follows that $(6c' + 6k_4 + 9k_6 + 5r)/24$ $\pi/2$-edge guards are always sufficient to guard $\mathcal{P}$. Substituting c' and r in the equation above, we have a total of $(11n + 3k_4 + 9k_6 - 8g + 8)/48$ $\pi/2$-edge guards.

As $\mathcal{P}$ has h_m holes on its faces, and for each of them we save four edge guards, we conclude that the total number of $\pi/2$-edge guards in $\mathcal{P}$ is $(11n+3k_4+9k_6-8g-16h_m+8)/48$. If we substitute $n = (2e-k_4-3k_6)/3$ into the number of $\pi/2$-edge guards, then we obtain that $(11e - k_4 - 3k_6 - 12g - 24h_m + 12)/72$ $\pi/2$-edge guards are always sufficient to guard the interior of $\mathcal{P}$. □

We finish this paper by obtaining lower bounds on the number of orthogonal edge guards needed to guard orthogonal polyhedra. Consider the polyhedron $\mathcal{P}_1$ illustrated in Figure 14a. It is formed by a cube each of whose vertices is replaced by an L-shaped polyhedron. Each L shape is formed by 21 edges and 14 vertices. These L-shapes together with 12 edges of the original cube result in a polyhedron with a total of 180 edges.

Let $\mathcal{A}$ be the set of red and black points in the interior of $\mathcal{P}_1$ shown in Figure 14. It can be shown that no $\frac{\pi}{2}$-edge guard of $\mathcal{P}$ guards more than one point of $\mathcal{A}$, and that it can be guarded with 16 $\frac{\pi}{2}$-edge guards.

By pasting together k copies of $\mathcal{P}_1$ as shown in Figure 15, we get a family of polygons with $180k$ edges that requires $16k$ $\frac{\pi}{2}$-edge guard to guard them. Thus we have:



(a) (b)

Figure 14: The orthogonal polyhedron shown in *(a)* requires 16 edge guards. Any orthogonal edge guard can see at most one of the colored points in *(b)*.

Theorem 10. *There exist orthogonal polyhedra with e edges such that the number of $\pi/2$-edge guards necessary to guard them is at least $\frac{4}{45}e$.*

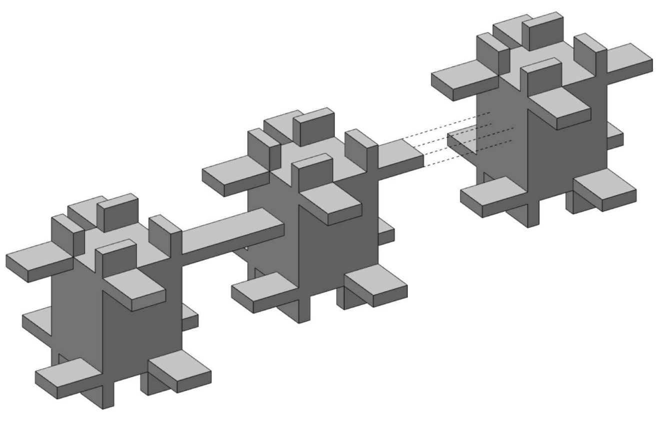


Figure 15: Family of orthogonal polyhedra that need $\frac{4}{45}e$ guards to surveil the polyhedron.

Note that in Urrutia's original conjecture for guarding orthogonal polyhedra with e edges using edge guards, he conjectures that $\frac{e}{12}$ edges are always sufficient to guard them. Theorem 10 tells us that if we use $\pi/2$-edge guards, $\frac{e}{12}$ guards are not sufficient.

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Jorge Urrutia
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Ciudad de México, México
urrutia@matem.unam.mx