Matemática Contemporânea, Vol. 45, 68–76
http://doi.org/10.21711/231766362017/rmc458

On Clique Corona Graphs - A Short Survey

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Abstract

Let $\mathcal{H} = \{H_v : v \in V(G)\}$ be a family of complete graphs indexed by the vertex set of a graph G. The *clique corona graph* $G \circ \mathcal{H}$ of G and $\mathcal{H}$ is the disjoint union of G and $H_v, v \in V(G)$, with additional edges joining each vertex $v \in V(G)$ to all the vertices of H_v. In this paper, we survey interconnections between clique coronas, well-covered graphs and independence polynomials.

1 Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of G spanned by X. By $G - W$ we mean the subgraph $G[V - W]$, for $W \subset V(G)$. We denote by $G - F$ the subgraph of G obtained by deleting the edges of F, for $F \subset E(G)$, and we write shortly $G - e$, whenever $F = \{e\}$. A vertex v is a *leaf* if $|N(v)| = 1$. We let $C_n, K_n, P_n, K_{p,q}$ denote respectively, the cycle on $n \geq 3$ vertices, the complete graph on $n \geq 1$ vertices, the path

2000 AMS Subject Classification: 05C76, 05C31 and 05C69.
Key Words and Phrases: independent set, independence polynomial, unimodal polynomial, well-covered graph, corona of graphs.

on $n \geq 1$ vertices, and the complete bipartite graph with the bipartition of size (p, q). An *independent* set in G is a set of pairwise non-adjacent vertices. An independent set of the largest size is a *maximum independent set*. Its cardinality is denoted by $\alpha(G)$.

The *disjoint union* of the graphs G_1, G_2 is the graph $G_1 \cup G_2$ having the disjoint unions $V(G_1) \cup V(G_2)$ and $E(G_1) \cup E(G_2)$ as a vertex set and an edge set, respectively. In particular, qG denotes the disjoint union of $q > 1$ copies of the graph G. If G_1, G_2 are vertex disjoint graphs, then their *join* (or *Zykov sum*) is the graph $G = G_1 + G_2$ with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{v_i v_j : v_i \in V(G_1), v_j \in V(G_2)\}$. Let $\mathcal{H} = \{H_v : v \in V(G)\}$ be a family of graphs indexed by the vertex set of a graph G. The corona $G \circ \mathcal{H}$ of G and $\mathcal{H}$ is the disjoint union of G and $H_v, v \in V(G)$, with additional edges joining each vertex $v \in V(G)$ to all the vertices of H_v. If $H_v = H$ for every $v \in V(G)$, then we write $G \circ H$ instead of $G \circ \mathcal{H}$ [10]. This graph compound operation attracts a lot of attention from people dealing with vertex coloring [30], Szeged index and Zagreb indices [35], Merrifield-Simmons index [36], adjacency spectrum and Laplacian spectrum [2, 32] etc.

If all H_v are complete graphs, then $G \circ \mathcal{H}$ is a *clique corona graph*. In this paper we discuss the clique coronas that are essentially involved in the context of well-covered graphs and independence polynomials.

2 Clique corona and well-covered graphs

A graph is *well-covered* if all its maximal independent sets are of the same size [29]. If, in addition, G has no isolated vertices and $|V(G)| = 2\alpha(G)$, then G is a *very well-covered graph* [7].

Recall that the *girth* of a graph G is the length of a shortest cycle contained in G, and it is defined as plus infinity for every forest. Using the clique corona operation one can build well-covered graphs of any girth as follows.

Theorem 2.1. *[34] The corona $G \circ \mathcal{H}$ of G and $\mathcal{H} = \{H_v : v \in V(G)\}$*

is well-covered if and only if each $H_v \in \mathcal{H}$ is a complete graph on at least one vertex, i.e., $G \circ \mathcal{H}$ is a clique corona graph.

For instance, all the graphs in Figure 1 are of the form $G \circ \mathcal{H}$, but only G_1 is not well-covered.

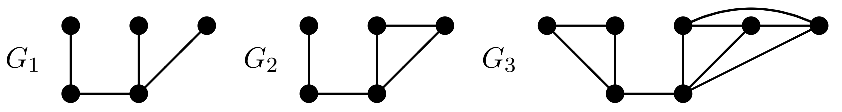

Figure 1: $G_1 = P_2 \circ \{K_1, 2K_1\}$, $G_2 = P_2 \circ \{K_1, K_2\}$, $G_3 = P_2 \circ \{K_2, K_3\}$.

It is worth mentioning that there are well-covered graphs different from clique corona graphs; e.g., C_3, C_4, C_5 and C_7. However, starting from some value, the girth becomes essential in characterizing well-covered graphs.

Theorem 2.2. *[8] Let G be a connected graph of girth ≥ 6, which is isomorphic to neither C_7 nor K_1. Then G is well-covered if and only if $G = H \circ K_1$ for some graph H.*

Notice that C_4 is the unique very well-covered cycle.

Theorem 2.3. *[20] Let G be a connected graph of girth ≥ 5. Then G is very well-covered if and only if $G = H \circ K_1$ for some graph H.*

As a consequence of Theorems 2.2 and 2.3, a tree on at least two vertices is (very) well-covered if and only if it can be represented as the clique corona $H \circ K_1$ for some tree H.

A graph G belongs to the class $\mathbf{W}_n, n \geq 1$, if every n disjoint independent sets in G are contained in n disjoint maximum independent sets [33]. Clearly, $\mathbf{W}_n \neq \emptyset$, because $K_n \in \mathbf{W}_n$, for all n. For instance, K_2 belongs to class $\mathbf{W}_2$, while P_4 is very well-covered, but not in the class $\mathbf{W}_2$. A number of ways to build graphs in the class $\mathbf{W}_n$ are presented in [33].

Theorem 2.4. (i) *[14] Let L be a connected graph without 4-cycles and of order at least two. Then $L \in \mathbf{W}_2$ if and only if L is isomorphic to K_2, C_5 or $L = G \circ K_2$, for some graph G.*

(ii) *[25] Let $L = G \circ \mathcal{H}$, where $\mathcal{H} = \{H_v : v \in V(G)\}$ and G has no isolated vertices. Then $L \in \mathbf{W}_2$ if and only if L is a clique corona graph such that every $H_v \in \mathcal{H}$ is a nontrivial complete graph.*

3 Independence polynomials of clique coronas

Let s_k be the number of independent sets of size k in a graph G. The polynomial $I(G;x) = \sum\limits_{k=0}^{\alpha(G)} s_k x^k$ is called the *independence polynomial* of G [13]. For a survey on independence polynomials of graphs see [18]. Computing the independence polynomial is **NP**-hard, since the evaluating of $\alpha(G)$ is **NP**-hard [11]. Moreover, it is intractable to calculate the value of the independence polynomial at any non-zero number [4]. More recent research concerning the independence polynomial computational complexity may be found in [15].

Theorem 3.1. (i) *[13] $I(G_1 \cup G_2; x) = I(G_1; x) \cdot I(G_2; x)$;*

(ii) *[13] $I(G_1 + G_2; x) = I(G_1; x) + I(G_2; x) - 1$;*

(iii) *[12] $I(G \circ H; x) = (I(H;x))^n \cdot I(G; \frac{x}{I(H;x)})$, where $n = |V(G)|$.*

Using Theorem 3.1, one can efficiently compute the independence polynomial for various recursive families of graphs, for example, see [16, 21, 23, 37].

A polynomial $P(x) = \sum\limits_{k=0}^{q} a_k x^k$ with real coefficients is called:

(i) unimodal if there exists an index $k \in \{0, 1, ..., q\}$ (called the *mode*) such that $a_0 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_q$;

(ii) log-concave if $a_i^2 \geq a_{i-1} \cdot a_{i+1}$ for all $i \in \{1, ..., q-1\}$.

(iii) palindromic (or *self-reciprocal*) if $a_i = a_{q-i}, i \in \{0, ..., \lfloor q/2 \rfloor\}$.

It is known that if a polynomial P with positive coefficients has only real roots, then it is log-concave, and if P is log-concave, then it is unimodal as well.

For instance, the independence polynomial:

- $I(K_{127}+3K_7;x)=1+148x+\mathbf{147}x^2+343x^3$ is non-unimodal and non-palindromic;
- $I(K_{43}+3K_7;x)=\mathbf{1}+64x+147x^2+\mathbf{343}x^3$ is unimodal, non-log-concave and non-palindromic;
- $I(K_{18}+3K_3+4K_1;x)=1+\mathbf{31}x+33x^2+\mathbf{31}x^3+x^4$ is unimodal, log-concave and palindromic;
- $I(K_{52}+3K_4+4K_1;x)=1+68x+\mathbf{54}x^2+68x^3+x^4$ is non-unimodal and palindromic.

There are infinite families of graphs, in general [1], and well-covered graphs, in particular [19, 28], whose independence polynomials are not unimodal. It was proved in [1] that for each permutation σ of $\{1,2,...,\alpha\}$ there is a graph G with $\alpha(G)=\alpha$ such that $I(G;x)$ satisfies $s_{\sigma(1)} < s_{\sigma(2)} < \cdots < s_{\sigma(\alpha)}$ [1].

Conjecture 3.2. *[1] $I(T;x)$ is unimodal for every tree T.*

The following results give partial support to Conjecture 3.2. It was shown in [17] that the inequality $\alpha(G) \leq 4$ ensures the unimodality of $I(G\circ K_1;x)$. Further it was improved up to $\alpha(G) \leq 8$ [5].

Conjecture 3.3. *$I(G\circ K_1;x)$ is unimodal for every graph G.*

An observation due to [17] claims that $I(K_{1,q}\circ K_1;x)$ is log-concave. Further, it was shown that the same is true for $I(K_{2,q}\circ K_1;x)$, and conjectured that log-concavity holds for each $I(K_{p,q}\circ K_1;x)$ [5].

Theorem 3.4. *[38] If $G \in \{K_{p,q} : p,q \geq 1\}$, then $I(G\circ K_1;x)$ is log-concave.*

There are some more cases of clique coronas with log-concave independence polynomials.

Theorem 3.5. *[17] If $\alpha(G) \leq 3$ or $G = P_n$, then $I(G\circ K_1;x)$ is log-concave.*

A graph G with $|V(G)| \geq 2$ with exactly two vertices of the same degree is known as *antiregular*.

Theorem 3.6. *[23] If G is an antiregular graph, then $I(G;x)$ is log-concave, and has at most two real roots.*

Theorem 3.7. *[24] If $H = K_r - e, r \geq 2$, then the polynomial $I(G \circ H;x)$ is unimodal and palindromic for every graph G. Moreover, the mode of $I(G \circ H;x)$ is unique and equal to the order of G.*

Unlike the matching polynomial, the independence polynomial may have non-real roots. Clearly, the real roots of independence polynomials are negative.

Theorem 3.8. *[24] If G has a non-empty edge set and all the roots of $I(G \circ H;x)$ are real, then the same is true for both $I(G;x)$ and $I(H;x)$.*

The converse of Theorem 3.8 is not necessarily true.

Theorem 3.9. *[22] Let G be a connected well-covered graph of girth ≥ 6, which is not isomorphic to C_7, K_1, K_2. Then the real roots of its independence polynomial are in $[-1, -1/n)$, where $n = 2\alpha(G)$.*

It is known that a root of smallest modulus of $I(G;x)$ is real [9].

Theorem 3.10. *[6] If G is a claw-free graph (i.e., G has no subgraph isomorphic to $K_{1,3}$), then all the roots of $I(G;x)$ are real.*

For clique corona graphs, we have the following.

Theorem 3.11. *[27] For every graph G of order n that has at least one edge, there exists a bijection between the set of roots of $I(G \circ K_p;x)$ different from $-1/p$ and the set of roots of $I(G;x)$, respecting the multiplicities of the roots. Moreover, the rational roots in $I(G \circ K_p;x)$ but $-1/p$ correspond to the rational roots in $I(G;x)$, and the same happens with the real roots.*

The case $p = 1$ in Theorem 3.11 was established in [22]. The roots of the independence polynomial of well-covered graphs were first investigated in [3].

Combining Theorems 3.11, 3.8, and 3.10 we get the following.

Corollary 3.12. (i) *If G is claw-free, then $I(G \circ K_p; x)$ has only real roots.*

(ii) *$I(G \circ K_p; x)$ has only real roots if and only if the same is true for $I(G; x)$.*

4 Conclusions

This paper is not meant to be a comprehensive survey, but rather a bird's-eye view on some advances and developments in the rich area of research concerning clique corona graphs. Nevertheless, even during this short journey around clique coronas, one may feel their mathematical beauty and depth. Surprisingly, it turns out that recursive clique corona graphs $((G \circ K_{p_1}) \circ K_{p_2} \cdots) \circ K_{p_r}$ compete to be adequate tools in modelling so-called small-world networks [26, 31].

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