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Complexity of Comparing the Domination Number to the Independent Domination, Connected Domination, and Paired Domination Numbers

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Abstract

The domination number $\gamma(G)$, the independent domination number $\iota(G)$, the connected domination number $\gamma_c(G)$, and the paired domination number $\gamma_p(G)$ of a graph G (without isolated vertices, if necessary) are related by the simple inequalities $\gamma(G) \leq \iota(G)$, $\gamma(G) \leq \gamma_c(G)$, and $\gamma(G) \leq \gamma_p(G)$. Very little is known about the graphs that satisfy one of these inequalities with equality. I.E. Zverovich and V.E. Zverovich studied classes of graphs defined by requiring equality in one of the first two above inequalities for all induced subgraphs (without isolated vertices, if necessary). In this article we prove hardness results which suggest that the extremal graphs for some of the above inequalities do not have a simple structure.

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1 Introduction

We consider finite, simple, and undirected graphs and use standard terminology and notation. Let D be a set of vertices of some graph G. The set D is a *dominating set of* G if every vertex of G that does not belong to D, has a neighbour in D. The set D is an *independent dominating set of* G if it is a dominating and an independent set of G. The set D is a *connected dominating set of* G if it is dominating and the graph $G[D]$ is connected. Finally, the set D is a *paired dominating set of* G if it is dominating and the graph $G[D]$ has a perfect matching. The *domination number* $\gamma(G)$, the *independent domination number* $\iota(G)$, the *connected domination number* $\gamma_c(G)$, and the *paired domination number* $\gamma_p(G)$ ([HS95], [McC13]) of G are the minimum cardinalities of a dominating, an independence dominating, a connected dominating and a paired dominating set of G, respectively. These definitions immediately imply:

$$\gamma(G) \leq \iota(G) \tag{1}$$
$$\gamma(G) \leq \gamma_c(G) \tag{2}$$
$$\gamma(G) \leq \gamma_p(G) \tag{3}$$

for every graph G where the parameters are well defined.

Very little is known about the extremal graphs for these inequalities. It is usual to work with a less complex class. In that sense, V.E. Zverovich and I.E. Zverovich [ZZ95], I.E. Zverovich [Zve03], and J.D. Alvarado et al. [ADR15], studied classes of graphs defined by requiring equality in (1), (2), or (3), respectively, for all induced subgraphs (where the parameters are well defined). Their results are characterizations of these classes in terms of their minimal forbidden induced subgraphs.

In this work, we prove hardness results which suggest that the extremal graphs for some of the above inequalities do most likely not have a simple description.

2 Hardness Results

In this section we state and prove our results. For a positive integer n, let $[n]$ be the set of positive integers at most n.

Theorem 1. For a given graph G, it is NP-hard to decide whether $\gamma(G) = \iota(G)$.

Proof. We describe a reduction from 3-CNF-SAT. Therefore, let f be a 3-CNF-SAT instance with clauses $C_1, ..., C_m$ over the boolean variables $x_1, ..., x_n$. We construct a graph G whose order is polynomially bounded in terms of n and m such that f is satisfiable if only if $\gamma(G) = \iota(G)$.

For every variable x_i, we create a copy $G(x_i)$ of the graph shown the Figure 1 and denote its vertices as indicated in the figure.

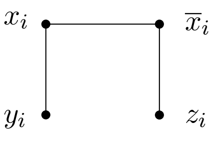

Figure 1: Graph $G(x_i)$ created for the variable x_i.

For every clause C_j with literals x_r, x_s and x_t, we create a vertex C_j and the three edges C_jx_r, C_jx_s and C_jx_t. This completes the construction of G. Clearly the order of G is $4n+m$. Since $\{x_i : i \in [n]\} \cup \{\overline{x}_i : i \in [n]\}$ is a dominating set of G, we have $\gamma(G) \leq 2n$. Since every dominating set of G contains at least two vertices from each $G(x_i)$, because of the pendant vertices, we have $\gamma(G) = 2n$ and $\iota(G) \geq 2n$. Furthermore, if $\iota(G) = 2n$, then G has a minimum independent dominating set D_ι that contains exactly two non-adjacent vertices from each $G(x_i)$. Note that this implies that $D_\iota \cap V(G(x_i)) \in \{\{x_i, z_i\}, \{\overline{x}_i, y_i\}, \{y_i, z_i\}\}$. Therefore, if $\iota(G) = \gamma(G) = 2n$, then the intersection of a minimum independent dominating set with the graph $G(x_i)$ indicates a satisfying truth assignment for f. Conversely, if f is satisfiable, we consider a satisfying truth assignment for f. Now $\{x_i : i \in [n]$ and x_i is set to true$\} \cup \{\overline{x}_i : i \in [n]$ and x_i is set to false$\} \cup \{y_i : i \in [n]$ and x_i is set to false$\} \cup \{z_i : i \in [n]$ and x_i is set to true$\}$ is an independent dominating set of G of order $2n$, which implies $\iota(G) = \gamma(G)$. This completes the proof. ■

For the benefit of the reader, we present in Figure 2 an example of the graph G, associated with a particular instance f, of the proof of Theorem 1. This example also illustrates in a similar way the following proofs.

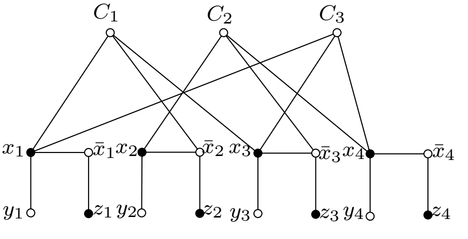

Figure 2: Graph G associated with the instance $f \equiv (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_2 \vee \bar{x}_3 \vee x_4) \wedge (x_1 \vee x_3 \vee x_4)$, where $C_1 \equiv (x_1 \vee \bar{x}_2 \vee x_3), C_2 \equiv (x_2 \vee \bar{x}_3 \vee x_4)$ and $C_3 \equiv (x_1 \vee x_3 \vee x_4)$. The black vertices represent a minimum independent dominating set of G, corresponding to a satisfying truth assignment of f.

Theorem 2. For a given graph G, it is NP-hard to decide whether $\gamma(G) = \gamma_c(G)$.

Proof. We describe a reduction from 3-CNF-SAT. Therefore let f be a 3-CNF-SAT instance with clauses $C_1, ..., C_m$ over the boolean variables $x_1, ..., x_n$. We construct a graph G whose order is polynomially bounded in terms of n and m such that f is satisfiable if only if $\gamma(G) = \gamma_c(G)$.

For every variable x_i, we create a copy $G(x_i)$ of a graph shown in Figure 3 and denote its vertices as indicated in the figure.

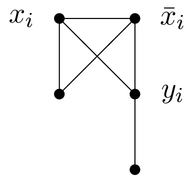

Figure 3: Graph $G(x_i)$ created for the variable x_i.

For every clause C_j with literals x_r, x_s and x_t, we create a copy $G(C_j)$ of K_2 and denote one of its two vertices by u_j. Furthermore, we create the three edges u_jx_r, u_jx_s and u_jx_t. Finally, for each $i \in [n-1]$, we create the edge y_iy_{i+1}. This completes the construction of G. Clearly, the order of G is $5n+2m$. Since $\{x_i : i \in [n]\} \cup \{y_i : i \in [n]\} \cup \{u_j : j \in [m]\}$ is a dominating set of G, we have $\gamma(G) \leq 2n+m$. Since every dominating set of G contains at least two vertices of $G(x_i)$ and at least one vertex of $G(C_j)$, we have $\gamma(G) = 2n+m$ and $\gamma_c(G) \geq 2n+m$. Furthermore, if $\gamma_c(G) = 2n+m$, then G has a minimum connected dominating set D_c that contains exactly two vertices from each $G(x_i)$ and exactly one vertex from each $G(C_j)$. Note that this implies that $D_c \cap V(G(x_i)) \in \{\{x_i, y_i\}, \{\overline{x}_i, y_i\}\}$ and $D_c \cap V(G(C_j)) = \{u_j\}$. Therefore, if $\gamma_c(G) = \gamma(G) = 2n+m$, then the intersection of a minimum connected dominating set with the graphs $G(x_i)$ indicates a satisfying truth assignment for f. Conversely, if f is satisfiable, we consider a satisfying truth assignment for f. Now, $\{x_i : i \in [n]$ and x_i is set to true$\} \cup \{\overline{x}_i : i \in [n]$ and x_i is set to false$\} \cup \{y_i : i \in [n]\} \cup \{u_j : j \in [m]\}$ is a connected dominating set of G of order $2n+m$, which implies $\gamma_c(G) = \gamma(G)$. This completes the proof. ■

Theorem 3. For a given graph G, it is NP-hard to decide whether $\gamma(G) = \gamma_p(G)$.

Proof. We describe a reduction from 3-CNF-SAT. Therefore let f be a 3-CNF-SAT instance with clauses $C_1, ..., C_m$ over the boolean variables $x_1, ..., x_n$. We construct a graph G whose order is polynomially bounded in terms of n and m such that f is satisfiable if only if $\gamma(G) = \gamma_p(G)$.

For every variable x_i, we create a copy $G(x_i)$ of a graph shown in Figure 3 and denote its vertices as indicated in the figure. For every clause C_j with literals x_r, x_s and x_t, we created a copy $G(C_j)$ of the graph shown in Figure 4 and denote its vertices as indicated in the figure.

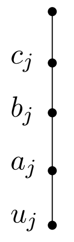

Figure 4: Graph $G(C_j)$ created for the clause C_j.

Furthermore, we create the three edges u_jx_r, u_jx_s and u_jx_t. This completes the construction of G. Clearly, the order of G is $5n+5m$. Since $\{x_i : i \in [n]\} \cup \{y_i : i \in [n]\} \cup \{a_j : j \in [m]\} \cup \{c_j : j \in [m]\}$ is a dominating set of G, we have $\gamma(G) \leq 2n+2m$. Since every dominating set of G contains at least two vertices of $G(x_i)$ and at least two vertices of $G(C_j)$, we have $\gamma(G) = 2n+2m$ and $\gamma_p(G) \geq 2n+2m$. Furthermore, if $\gamma_p(G) = 2n+2m$, then G has a minimum paired dominating set D_p that contains exactly two vertices from each $G(x_i)$ and exactly two vertices from each $G(C_j)$. Note that this implies that $D_p \cap V(G(x_i)) \in \{\{x_i, y_i\}, \{\overline{x}_i, y_i\}\}$ and $D_p \cap V(G(C_j)) = \{b_j, c_j\}$. Therefore, if $\gamma_p(G) = \gamma(G) = 2n+2m$, then the intersection of a minimum paired dominating set with the graphs $G(x_i)$ indicates a satisfying truth assignment for f. Conversely, if f is satisfiable, we consider a satisfying truth assignment for f. Now, $\{x_i : i \in [n] \text{ and } x_i \text{ is set to true}\} \cup \{\overline{x}_i : i \in [n] \text{ and } x_i \text{ is set to false}\} \cup \{y_i : i \in [n]\} \cup \{b_j : j \in [m]\} \cup \{c_j : j \in [m]\}$ is a paired dominating set of G of order $2n+2m$, which implies $\gamma_p(G) = \gamma(G)$. This completes the proof. ∎

3 Conclusion

This work suggests that the extremal graphs for the inequalities $\gamma(G) \leq \iota(G)$, $\gamma(G) \leq \gamma_c(G)$, and $\gamma(G) \leq \gamma_p(G)$ will not have a simple description. It seems interesting to study the complexity of $(\gamma_t, 2\gamma)$-*extremal graphs*, that is, the graphs (without isolated vertices) that satisfy the inequality $\gamma_t(G) \leq 2\gamma(G)$ with equality ([BC79], [CDH80], [HS95], [Hen00]).

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