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Timber Game with Caterpillars

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Abstract

Timber is a two player game played on a directed graph, with a domino on each arc. The direction of the arc indicates the direction in which the domino can be initially toppled. If one domino is toppled, it topples the dominoes in the direction it was toppled and creates a chain reaction. The goal of this game is to topple all the dominoes and the player who topples the last dominoes wins. A P-position is a configuration where the second player can always force a win. We contribute to the open problem of determining the number of P-positions showing structural properties to establish whether a configuration D of a caterpillar is a P-position, expanding the existing results for paths. We prove that a caterpillar 1 has no P-position, and we generalize the characterization of paths to double brooms and to caterpillars with even number of adjacent legs to each vertex of the spine, as well as to caterpillars with an odd number of legs adjacent to each vertex of the spine.

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1 Introduction

A *combinatorial game* is a finite two player game with perfect information[1] and the aim of its study is usually to know a winning strategy. Because these games are of pure strategy, they have aroused the interest of many researchers and there exists a rich and recent literature on Game Theory [ANW07, BCG01, Ren13, Sie13, PDG14]. We studied the combinatorial game called *Timber*.

Timber is a two player game introduced by Nowakowski et al. in [NRL$^+$13]. It is an *Impartial game*, because the players are not distinguished, i.e., they both have the same allowed set of moves. Let $G = (V(G), E(G))$ be a graph. A *configuration* $D = (V(D), \vec{E}(D))$ is an orientation of G. Timber is played on a directed graph D, with a domino on each arc. The orientation of the arc represents the available movement of the domino piece. Each player chooses a domino on some arc (x, y) and topples it in the direction of vertex y, removing from D all vertices of the connected component of G minus the edge xy containing y. See Figure 1.

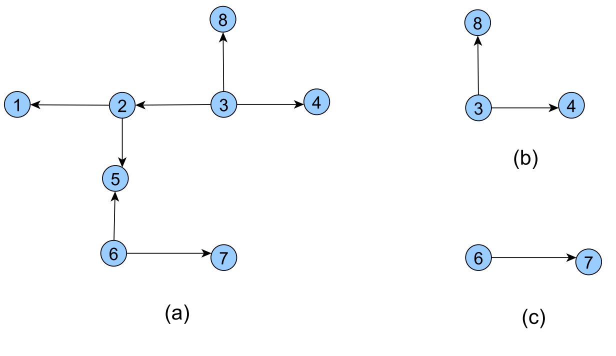

Figure 1: (a) Digraph D; What remains after toppling: (b) $(3, 2)$, (c) $(6, 5)$.

A *P-position* is a configuration D in which the second player wins,

[1]Perfect information is a situation in which all the relevant information is public to both players, and the set of available moves is also public.

independent of what the first player plays. The *number of P-positions of a graph G* is the number of configurations of the graph G that are a P-position. The number of P-positions of a path is known [NRL$^+$13]. Our goal is to contribute to the open problem of determining the number of P-positions of a tree by studying the case where G is a caterpillar. A N-position is a configuration in which the first player wins [NRL$^+$13]. Notice that the number of configurations in which the first player wins is $2^{|E(G)|}$ minus the number of P-positions of G.

In Section 2, we briefly present the results shown in [NRL$^+$13] that give us the number of P-positions of paths and reduce the problem of deciding whether a tree is a P-position to the same problem in a smaller tree.

Using these results, in Section 3, we study special cases of caterpillars determining their number of P-positions. Finally, we conclude our work exposing open problems in Section 4.

2 Known results for Paths and Trees

In [NRL$^+$13], the authors presented the number of P-positions of paths P_s with s vertices and size $(s-1)$ (the number is 0 when s is even and the $\frac{s-1}{2}$ Catalan number when s is odd, i.e., $(\frac{(s-1)!}{(\frac{s-1}{2})!(\frac{s+1}{2})!})$, and the following three lemmas about a configuration $\vec{T}$ of a tree.

Lemma 2.1 [NRL$^+$13] says that if $\vec{T}$ has a leaf with outdegree 1, then $\vec{T}$ is not a P-position. Since we intend to study the number of P-positions of a digraph, thus we do not analyze the cases where there is a leaf with outdegree 1, because it is not a P-position (the first player wins toppling this leaf).

Figure 2 and Figure 3 illustrate the Lemmas 2.2 and 2.3, respectively. Lemma 2.2 [NRL$^+$13] shows that, for each $\vec{T}$ such that there exists a source vertex x with $N^+(x) = \{y, z\}$ and $N^-(x) = \emptyset$, we can remove the vertex x and contract the vertices y and z without changing the outcome of $\vec{T}$ of being or not a P-position. Lemma 2.3 [NRL$^+$13] shows that in $\vec{T}$

if $\vec{A}$ and $\vec{B}$ are two oriented paths starting at w, such that $|\vec{A}| = a$ and $|\vec{B}| = b$, then we can replace $\vec{A}$ and $\vec{B}$ by just one oriented path $\vec{Q}$ starting at w, such that $|\vec{Q}| = a \oplus b$, using the operator XOR[2], without changing the outcome of $\vec{T}$ of being or not a P-position.

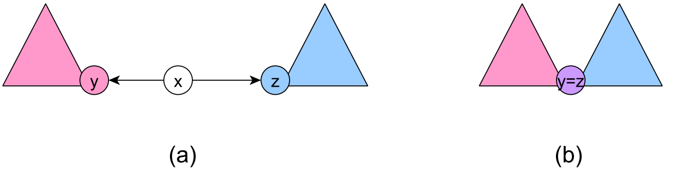

Figure 2: (a) $\vec{T}_1$; (b) $\vec{T}_2$; $\vec{T}_1$ and $\vec{T}_2$ have the same outcome.

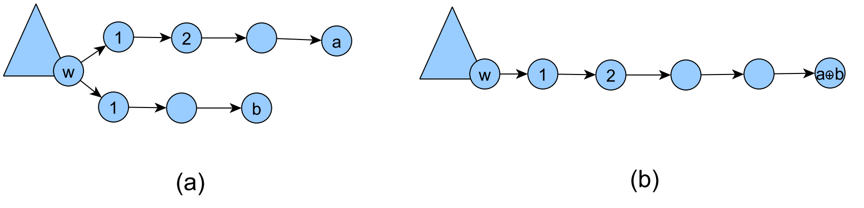

Figure 3: (a) $\vec{T}_1$; (b) $\vec{T}_2$; $\vec{T}_1$ and $\vec{T}_2$ have the same outcome.

These three lemmas determine an algorithm to decide if a an oriented tree is a P-position (with complexity $O(n^2)$), presented in [NRL+13], in which Lemma 2.1 solves the trivial cases and Lemmas 2.2 and 2.3 reduce a large oriented tree into a smaller one. Observe that the Lemmas 2.2 and 2.3 do not address the number of P-positions of a tree. We use these lemmas to reduce a particular tree, the caterpillar, in order to contribute to the open problem of determining the number of P-positions.

[2]The operator XOR (represented by $\oplus$) returns a bit 1 when the number of operands equals to 1 is odd. For example, $11 \oplus 5 = 14$.

3 Timber in Caterpillars

A *caterpillar* $cat(k_1, k_2, \ldots, k_s)$ is a tree which is obtained from a central path $v_1, v_2, v_3, ..., v_s$ (called *spine*) by joining k_i new leaf vertices to v_i (called *legs*), for each $i = 1, \ldots, s$. Thus, the number of vertices is $n = s + \sum_{i=1}^{s} k_i$. See in Figure 4, an example of caterpillar. Using this definition, a *caterpillar 1* is a $cat(1, \ldots, 1)$, i.e., $k_i = 1$, for all $i = 1, \ldots, s$; and a *double broom* is a $cat(k_1, 0, \ldots, 0, k_s)$, i.e., $k_i = 0$ for $i = 2, \ldots, s - 1$. Caterpillars are used in Chemical to represent the structure of the benzenoid hydrocarbon molecule and for this reason are also known as *tree benzenoid* or *Gutman trees*, a researcher who developed the work in this area.

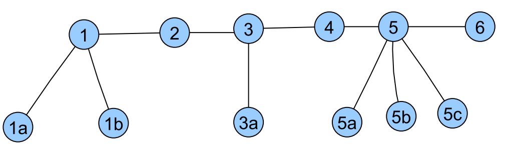

Figure 4: $cat(2, 0, 1, 0, 3, 0)$.

Next, we present our main results:

Theorem 3.1. Let T be a caterpillar and D be a configuration of T. If T has a leaf whose outdegree in D is 1, then D is not a P-position.

Proof. Let u be the out-neighbour of v. The first player wins by toppling the domino on the arc (v, u). (It is a particular case of Lemma 2.1.) ■

Theorem 3.2. Every configuration of a *caterpillar 1* is not P-position.

Proof. (by induction in $|V(D)|$) Base of induction: The shortest caterpillar 1 is $cat(1)$ that has 2 vertices. It is already known that the path with 2 vertices has no P-positions.

Suppose a caterpillar 1 with $2s$ vertices has no P-positions.

Let us add one vertex to the spine of D. By the definition of a *caterpillar 1*, it is impossible to add one vertex in the spine without adding another leaf. So we will add two vertices: $s+1$ and $s+1'$. We have the following options to the orientation of the new arcs in D:

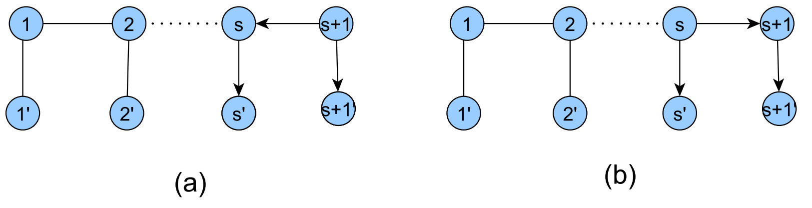

Figure 5: Caterpillar 1 after adding two vertices. (a) Case 1 (b) Case 2

In Case 1, applying Lemma 2.2, we remove the vertex v_{s+1} and contract the vertices v_s and $v_{s+1'}$ without changing the outcome. Therefore, we obtain exactly the case of the induction hypothesis and can conclude that the caterpillar has no P-positions.

In Case 2, we apply Lemmas 2.2 and 2.3 from the right to the left until we obtain just a path oriented to the right. The length of this oriented path is equal to $2R+1$, where R is the number of arcs (v_i, v_{i+1}) to the right, for $i \in \{1, ..., s\}$. Hence, at the end of the process, we get a path $(u_1, ..., u_{2R+1})$. This final path has an odd number of arcs. Thus, the digraph in Case 2 has no P-position.

Therefore, there is no P-position independent of the configuration of the caterpillar 1.

■

Theorem 3.3. Let G be a caterpillar $cat(k_1, ..., k_s)$. The number of P-positions of G is equal to the number of P-positions of a caterpillar $cat(l_1, ..., l_s)$, such that if k_i is even, then $l_i = 0$, and if k_i is odd, then $l_i = 1$, for $i = 1, ..., s$.

Proof. For $i = 1, ..., s$:

If k_i is even, then we can suppose $k_i = 2t$, for $t \in Z_+$, and the vertex v_i has the leaves $\{u_1, u_2, ..., u_{2t}\}$. We apply Lemma 2.3 to pairs of paths v_i, u_1 and v_i, u_2, ..., v_i, u_{2t-1} and v_i, u_{2t}. Thus after this process, vertex v_i does not have any adjacent leaf, and we say that in the new caterpillar $l_i = 0$.

If k_i is odd, then we can suppose $k_i = 2t+1$, for $t \in Z_+$, and the vertex v_i has the leaves $\{u_1, u_2, ..., u_{2t}, u_{2t+1}\}$. We apply Lemma 2.3 to pairs of paths v_i, u_1 and v_i, u_2, ..., v_i, u_{2t-1} and v_i, u_{2t}. Thus after this process, vertex v_i has only u_{2t+1} as an adjacent leaf, and we say that in the new caterpillar $l_i = 1$. ■

So the number of P-positions of a $cat(k_1, ..., k_s)$ is equal to the number of P-positions of a $cat(k_1 mod 2, ..., k_s mod 2)$ and the study of the caterpillars can be reduced to the study of the binary caterpillars in which $k_i \in \{0, 1\}$.

Corollary 3.4. The number of P-positions of $cat(k_1, k_2, \ldots, k_s)$ satisfies:

(i) if each k_i is even, for $i = 1, \ldots, s$, then the number of P-positions is equal to the number of P-positions of a path with s vertices.

(ii) if each k_i is odd, for $i = 1, \ldots, s$, then the number of P-positions is zero.

Proof.

(i) By Theorem 3.3, if k_i is even, $i = 1, \ldots, s$, we can replace k_i for 0 without changing the outcome. Therefore, this caterpillar has the same outcome of $cat(0, 0, \ldots, 0)$, that is a P_s.

(ii) By Theorem 3.3, if k_i is odd, $i = 1, \ldots, s$, we can replace k_i for 1 without changing the outcome. Therefore, this caterpillar has the same outcome of *caterpillar 1*, and by Theorem 3.2 this caterpillar has no P-position. ■

Corollary 3.5. Let $cat(k_1, 0, \ldots, 0, k_s)$ be a double broom. The number of P-positions of a double broom is equal to the number of P-positions of a path with $s + k_1 mod 2 + k_s mod 2$ vertices.

Proof. By Theorem 3.3, we can replave k_1 and k_s for 0 or 1 depending on the parity of them. Thus, we can make the following analysis of the number the vertices after this process:

k_1	k_s	vertices in the path
even	even	0+s+0=s
even	odd	0+s+1=s+1
odd	even	1+s+0=s+1
odd	odd	1+s+1=s+2

Therefore, the double broom′s case is reduced to the case of the path with $s, s+1$ or $s+2$ vertices, that is the same of the path with $s+k_1 mod 2+k_s mod 2$ vertices.

∎

4 Conclusion

In this paper, we studied Timber game restricted to caterpillars, and give four results that allow us to know the number of P-positions for certain configurations of caterpillars.

In future works, we intend to find a solution to the number of P-positions for any caterpillar, to study the minimum number of steps to the first player win the game in caterpillars and, finally, to get a more efficient algorithm to reduce the problem of deciding whether a configuration of a tree is a P-position.

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