Matemática Contemporânea, Vol. 43, 223–246
http://doi.org/10.21711/231766362012/rmc439

On the study of Existence of solutions for a class of equations with critical Sobolev exponent on compact Riemannian Manifold

Carlos Rodrigues da Silva

Abstract

We study the existence of solutions for a class of non-linear differential equation with critical Sobolev's exponent on the compact riemannian manifold (M^n, g), $n > 6$. We show that the equation (1)

$$\Delta u + a(x)u = f(x)u^{2^*-1} + h(x)u^q, \tag{1}$$

where $0 < q < 1$, has solution $u > 0$, if $a, f, h \in C^\infty$ satisfies some growth condition. The equation (1) were studied by [5] in the euclidean case (for $0 < q < 2^* - 1$) and by [7] in the Riemannian context (for $1 < q < 2^* - 1$).

1 Introduction

The study of the theory of nonlinear differential equations on Riemannian manifolds, has began in 1960 with the so-called Yamabe problem. At

2000 AMS Subject Classification: 53C21; 35J60.
Key Words and Phrases: critical Sobolev exponent, compact riemannian manifold, non-linear differential equation
*Partially Supported by CAPES

a time when little was known about the methods of attacking a non-linear equation, the Yamabe problem came to light of a geometric idea and from time sealed a merger of the areas of geometry and differential equations. Let (M, g) be a compact riemannian manifold of dimension n, $n \geq 3$. Given $\widetilde{g} = u^{4/(n-2)} g$ some conformal metrical to the metric g, is well known that the scalar curvatures R and $\widetilde{R}$ of the metrics g and $\widetilde{g}$, respectively, satisfy the law of transformation

$$\Delta u + \frac{n-2}{4(n-1)} R u = \frac{n-2}{4(n-1)} \widetilde{R} u^{2^*-1}$$

where Δ denote the Laplacian operator associated to g.

In 1960, Yamabe [16] announced that for every compact Riemannian manifold (M, g) there exist a metric $\widetilde{g}$ conformal to g for which $\widetilde{R}$ is constant. In another words, this mean that for every compact riemannian manifold (M, g) there exist $u \in C^\infty(M)$, $u > 0$ on M and $\lambda \in \mathbb{R}$ such that

$$\Delta u + \frac{n-2}{4(n-1)} R u = \lambda u^{2^*-1}. \qquad (Y)$$

In 1968, Trüdinger [15] found an error in the work of Yamabe, which generated a race to solve what became known as the Yamabe problem, today it is completely positively resolved, that is, the assertion of Yamabe is true.

The main step towards the resolution of the Yamabe problem was given in 1976 by Aubin in his classic article [1]. In [1] Aubin showed that the statement was true since the manifold satisfy a condition on an invariant (called Yamabe invariant). Then he used tests functions, locally defined, to show that non locally conformal flat manifolds, of dimension $n > 6$, satisfying this condition. Finally, the problem was completely solved by R. Schoen [13].

Several disturbances were considered in the Yamabe Problem, all of analytical characters, both in the sense of equation (with the addition of other factors) and in the sense of the operator (the Laplacian changed for the p-Laplacian), and all (at least those listed in this study) using the idea of estimating the corresponding functional by functions u_λ, defined by Aubin. We can cite some articles, such as [2], [3], [6], [7], [8] and [12] .

This work aims to study with problems related to the equation (Y), although, as we shall see, with different methods from those used by Yamabe, these results were obtained in [14]. The equation (1) was studied simultaneously by Djadli [7] (in the Riemannian context), where he considered the case $1 < q < 2^* - 1$, and by Gonçalves and Alves [5] (in the Euclidean context), where they considered the case $0 < q < 2^* - 1$. In both cases, they have used the Aubin's functions to perform a condition that, like [1], was needed to be imposed. Here we use the methods of [7] and [5] to study the case $0 < q < 1$, in the riemannian setting.

The Main Theorem is

Theorem 1.1. Let (M, g) be a compact $n-$dimensional riemannian manifold with $n > 6$. Then the equation $\Delta u + a(x)u = f(x)u^{2^*-1} + h(x)u^q$, where $0 < q < 1$, admits a regular positive solution u, if (h_1) , (h_2) and (h_3), holds.

The conditions (h_1), (h_2) and (h_3) will be presented and discussed in the next section.

2 The equation and the conditions

Let (M, g) be a compact $n-$dimensional riemannian manifold, where $n \geq 3$. Consider the equation (1)

$$\Delta u + a(x)u = f(x)u^{2^*-1} + h(x)u^q,$$

where $0 < q < 1$.

Suppose that $a(x)$ is such that there is $\lambda > 0$ satisfying

$$\frac{1}{2}\int_M (|\nabla u|^2 + au^2)dV \geq \lambda \|u\|^2_{H_1^2} \quad \forall\ u \in H_1^2.$$

Let $\lambda_o = \frac{1}{2} \inf_{u \in H_1^2 u \not\equiv 0} \frac{\int_M (|\nabla u|^2 + au^2)dV}{\|u\|^2_{H_1^2}}$.

Remark 1: *In the sequence we will use the notations* (h_1), (h_2) *and* (h_3) *to designate three different hipotheses.*

The first one, (h_1), refers to the function $a(x)$:

(h_1) $\lambda_o > 0$.

In order to define the two other hypotheses, (h_2) and (h_3), we need some considerations.

According to Hebey and M. Vaugon [11] there is $C > 0$ such that $\forall\, u \in H_1^2(M)$,

$$\left(\int_M |u|^{2^*} dV\right)^{1/2^*} \leq K(n,2)\left(\int_M |\nabla u|^2 dV\right)^{1/2} + C\left(\int_M u^2 dV\right)^{1/2} \tag{2}$$

where $K(n,2)$ is the best constant for the inequality (2).

Let $C_o = \inf\{C > 0$ such that (2) holds $\forall\, u \in H_1^2\}$.

Therefore (see [9])

$$\left(\int_M |u|^{2^*} dV\right)^{1/2^*} \leq K(n,2)\left(\int_M |\nabla u|^2 dV\right)^{1/2} + C_o\left(\int_M u^2 dV\right)^{1/2} \tag{3}$$

From this

$$\left(\int_M |u|^{2^*} dV\right)^{1/2^*} \leq C_1\|u\|_{H_1^2}, \quad \forall\, u \in H_1^2, \tag{4}$$

where $C_1 = \max\{K(n,2), C_o\}$.

Thus, we obtain $\forall\, u \in H_1^2$,

$$\int_M f(u^+)^{2^*} dV \leq \sup_M |f| \int_M |u|^{2^*} dV \leq \sup_M |f|(C_1)^{2^*}\|u\|_{H_1^2}^{2^*} \tag{5}$$

where $u^+ = \max\{u, 0\}$.

Analogously, by (4) and by Hölder's inequality

$$\begin{aligned}\int_M h(u^+)^{q+1} dV &\leq \sup_M |h| \int_M |u|^{q+1} dV \\ &\leq \sup_M |h| vol(M)^{[2^*-(q+1)]/2^*}\left(\int_M |u|^{2^*} dV\right)^{(q+1)/2^*} \\ &\leq \sup_M |h| C_2 \|u\|_{H_1^2}^{(q+1)}\end{aligned} \tag{6}$$

where $C_2 = vol(M)^{[2^*-(q+1)]/2^*}(C_1)^{(q+1)}$ and $vol(M) =$ volume of M.

Taking $\alpha = \dfrac{C_2.\sup|h|}{q+1}$ and $\beta = \dfrac{(C_1)^{2^*}.\sup|f|}{2^*}$, we can consider the second hypotheses:

(h_2) $f > 0$ and $h > 0$ such that

$$\alpha\left[\frac{\alpha(1-q)}{\beta(2^*-2)}\right]^{(q-1)/[2^*-(q+1)]} + \beta\left[\frac{\alpha(1-q)}{\beta(2^*-2)}\right]^{(2^*-2)/[2^*-(q+1)]} < \lambda_o.$$

To define the last hypotheses, let us consider $x_o \in M$ such that $f(x_o) = \max\limits_M f$. Thus, the third condition is:

(h_3) $\dfrac{2R(x_o)}{n-4} - \dfrac{8(n-1)a(x_o)}{(n-2)(n-4)} > \dfrac{\Delta f(x_o)}{f(x_o)}$

where $R(x)$ is the scalar curvature of g at x.

Remark 2: *From now on we will consider* $n > 6$.

3 Auxiliary lemmas

To proof the Theorem 1 we need some considerations and two lemmas.

Lemma 3.1. *Let* $0 < q < 1$, $2^* = 2n/(n-2)$, $A > 0$ *and* $B > 0$. *For each*
$k \in \mathbb{N}$ *let consider* $A(k)$, $B(k)$ *and* $C(k)$ *real numbers such that* $A(k) \longrightarrow A$,
$B(k) \longrightarrow B$ *and* $C(k) \longrightarrow 0$, *when* $k \to \infty$, *with* $C(k) > 0$, $\forall\ k \in \mathbb{N}$. *Define*

$$F(t,k) = A(k).t^2 - B(k).t^{2^*} - C(k).t^{q+1}.$$

Then for a large enough k, *there exist* $t_k > 0$ *such that*

$$F(t_k,k) = \max_{t\geq 0} F(t,k) > 0$$

with the aditional property that, if $t_o = \left(\dfrac{2A}{2^*B}\right)^{1/(2^*-2)}$, *then* $t_k \longrightarrow t_o$ *when* $k \to \infty$.

Moreover, if $A(k) = A + O(1/k)$, $B(k) = B + O(1/k)$ *and* $C(k) = O(1/k)$, *then*
$t_k = t_o + O(1/k)$.

Proof of Lemma 1:

Take a big enough k such that $A(k), B(k) > 0$.

Hence, $\lim\limits_{t\to\infty} F(t,k) = -\infty$. Therefore there is $t_k \geq 0$ such that $F(t_k,k) = \max\limits_{t\geq 0} F(t,k)$.

On the other hand,

$$F(t,k) = t^2 \left[A(k) - B(k).t^{2^*-2} - C(k).t^{q-1}\right].$$

Let us define

$$g(t,k) = B(k).t^{2^*-2} + C(k).t^{q-1}.$$

Since $2^* - 2 > 0$, $q - 1 < 0$ and $B(k), C(k) > 0$ we have that

$$\lim_{t\to 0^+} g(t,k) = +\infty \quad \text{and} \quad \lim_{t\to\infty} g(t,k) = +\infty.$$

Then, there exist $s_k > 0$, such that

$$g(s_k,k) = \min_{t>0} g(t,k) > 0,$$

where s_k is given by $g^{'}(s_k,k) = 0$, namely,

$$B(k)\,(2^*-2)\,(s_k)^{2^*-3} = (1-q)C(k)(s_k)^{q-2}$$

and then,

$$s_k = \left[\frac{(1-q)C(k)}{(2^*-2)B(k)}\right]^{1/[2^*-(q+1)]}.$$

Whence

$$\begin{aligned} g(s_k,k) =& B(k)\left[\frac{(1-q)C(k)}{(2^*-2)B(k)}\right]^{(2^*-2)/[2^*-(q+1)]} \\ &+ C(k)\left[\frac{(1-q)C(k)}{(2^*-2)B(k)}\right]^{(q-1)/[2^*-(q+1)]} \\ =&[B(k)]^{(1-q)/[2^*-(q+1)]}[C(K)]^{(2^*-2)/[2^*-(q+1)]}R \end{aligned}$$

where $R = \left[\dfrac{(1-q)}{(2^*-2)}\right]^{(2^*-2)/[2^*-(q+1)]} + \left[\dfrac{(1-q)}{(2^*-2)}\right]^{(q-1)/[2^*-(q+1)]} > 0.$

And with this, as $B(k) \longrightarrow B > 0$ and $C(k) \longrightarrow 0$, when $k \to \infty$, we obtain

$$\lim_{k\to\infty} g(s_k, k) = 0.$$

Thus, as $A(k) \longrightarrow A > 0$, when $k \to \infty$, we have that, for a large enough k,

$$g(s_k, k) < A(k).$$

Therefore,

$$\sup_{t\geq 0} F(t,k) \geq F(s_k, k) = (s_k)^2 \left[A(k) - g(s_k, k)\right] > 0.$$

Since $F(0,k) = 0$, we have that $F(t_k, k) = \sup_{t\geq 0} F(t,k) > 0$ with $t_k > 0$.

Now, as $0 < q < 1 < 2 < 2^*$, for a small t we have that $F(t,k) < 0$, what tell us that there is $\epsilon > 0$ such that $t_k > \epsilon$ for big k.

Analogously, as $\lim_{t\to\infty} F(t,k) = -\infty$, there exist $T > 0$ such that $t_k < T$ for big k.

Then, under a subsequence, $t_k \longrightarrow t_o$, for some $t_o > 0$.

On the other hand, as

$$0 = F'(t_k, k) = 2A(k)t_k - 2^*B(k).(t_k)^{2^*-1} - (q+1)C(k).(t_k)^q,$$

$A(k) \longrightarrow A > 0$, $B(k) \longrightarrow B > 0$, $C(k) \longrightarrow 0$ and $t_k \longrightarrow t_o$, obtemos $2At_o = 2^*B(t_o)^{2^*-1}$. Therefore,

$$t_o = \left(\frac{2A}{2^*B}\right)^{1/(2^*-2)} .$$

To prove the second part of the lemma, we can suppose that $t_k = t_o + \theta_k$, where $\theta_k \longrightarrow 0$ when $k \to \infty$.

Now, as $2A(k) = 2^*B(k).(t_k)^{2^*-2} + (q+1)C(k).(t_k)^{q-1}$

and by using that $A(k) = A + O(1/k)$, $B(k) = B + O(1/k)$, $C(k) = O(1/k)$ and the fact that (t_k) is bounded, we obtain

$$2A = 2^*B.(t_k)^{2^*-2} + O(1/k).$$

Consequently,

$$2A + O(1/k) = 2^*B.(t_o+\theta_k)^{2^*-2} = 2^*B.(t_o)^{2^*-2}\left(1+\frac{\theta_k}{t_o}\right)^{2^*-2}.$$

As $2A = 2^*B(t_o)^{2^*-2}$ and $\left(1+\frac{\theta_k}{t_o}\right)^{2^*-2} = 1 + \frac{2^*-2}{t_o}\theta_k + o(\theta_k)$ we obtain

$$\theta_k = O(1/k).$$

■

Let consider the functional

$$J : H_1^2 \longrightarrow \mathbb{R} \qquad \text{given by}$$

$$J(u)= \frac{1}{2}\int_M (|\nabla u|^2 + au^2)dV - \frac{1}{2^*}\int_M f(u^+)^{2^*}dV - \frac{1}{q+1}\int_M h(u^+)^{q+1}dV. \tag{7}$$

Thus, $J \in C^1(H_1^2)$ and

$$\langle J'(u), v\rangle= \int_M (\nabla u\nabla v + auv)dV - \int_M f(u^+)^{2^*-1}vdV - \int_M h(u^+)^q vdV, \tag{8}$$

for $u, v \in H_1^2$.

Lemma 3.2. *Assume that* $f > 0$, $h > 0$ *and* (h_3). *Then there is* $u_o \geq 0$, $u_o \not\equiv 0$ *such that*

$$0 < \sup_{t\geq 0} J(tu_o) < \frac{1}{nK(n,2)^n(\sup f)^{(n-2)/2}}.$$

Proof of Lemma 2:

Consider, for each $k \in \mathbb{N}^*$,

$$\psi_k(x) = \begin{cases} \left(\frac{1}{k} + \frac{1-\cos(\rho r(x))}{\rho^2}\right)^{1-\frac{n}{2}} - \left(\frac{1}{k} + \frac{1-\cos(\rho\delta)}{\rho^2}\right)^{1-\frac{n}{2}} & \text{if } x \in B_\delta(x_o) \\ 0 & \text{if } x \notin B_\delta(x_o) \end{cases}$$

where $r(x) = d_g(x, x_o)$ (the distance function from x to x_o with respect to the metric g), $R(x_o) = n(n-1)\rho^2$, $R(x_o)$ is the scalar curvature of g at x_o, and $B_\delta(x_o)$ is the geodesic ball with center at x_o and radius δ, δ is such that $|\rho|\delta \leq \pi$ is smaller than the injectivity radius of M. In the above expressions we use the convention that if $R(x_o) < 0$, $cos(\rho r) = cosh(i\rho r)$ and if $R(x_o) = 0$, $\dfrac{1-cos(\rho r)}{\rho^2} = \dfrac{r^2}{2}$.

We define also, for two positive real numbers ν and η such that $\nu - \eta > 1$

$$I_\nu^\eta = \int_0^\infty (1+\tau)^{-\nu}\tau^\eta d\tau$$

that have the following properties

$$I_{\nu+1}^\eta = \frac{\nu-\eta-1}{\nu} I_\nu^\eta \quad \text{and} \quad I_{\nu+1}^{\eta+1} = \frac{\eta+1}{\nu-\eta-1} I_{\nu+1}^\eta .$$

Moreover, if $\gamma \in \mathbb{R}^+$

$$\lim_{k\to\infty}\left[\int_0^\gamma (\tau+\frac{1}{k})^{-\nu}\tau^\eta d\tau \ - \ k^{\nu-\eta-1} I_\nu^\eta\right] \text{ is a finite number, if } \nu-\eta-1>0.$$

Remark 3: *In the following computations we will use that* $\omega_n = 2^{n-1}\omega_{n-1} I_n^{\frac{n}{2}-1}$, *where* ω_{n-1} *is the volume of the unitary ball in* $\mathbb{R}^n$, *and* $K(n,2)^2 = \dfrac{4}{n(n-2)(\omega_n)^{2/n}}$ *(see [1] and [10]).*

Let $v_k(x) = \left[\dfrac{n(n-2)}{k}\right]^{(n-2)/4} \psi_k(x)$.

According to the Aubin's development [1], Djadli [7] or Druet [8] we have that

$$
\begin{aligned}
\int_M |\nabla v_k|^2 dV &= \left[\frac{n(n-2)}{k}\right]^{(n-2)/2} \int_M |\nabla \psi_k|^2 dV \\
&= \left[\frac{n(n-2)}{k}\right]^{(n-2)/2} \frac{(n-2)^2}{4} 2^{n/2} k^{(n-2)/2} \times \\
&\quad \omega_{n-1} I_n^{n/2} \Big\{ 1 - \frac{n(n+2)\rho^2}{4(n-4)} \frac{1}{k} + o(\frac{1}{k}) \Big\} \\
&= [n(n-2)]^{(n-2)/2} \frac{(n-2)^2}{4} 2^{n/2} \omega_{n-1} I_n^{n/2} \Big\{ 1 - \frac{n(n+2)\rho^2}{4(n-4)} \frac{1}{k} \\
&\quad + o(\frac{1}{k}) \Big\}, \qquad (9)
\end{aligned}
$$

$$
\begin{aligned}
\int_M a(v_k)^2 dV &= \left[\frac{n(n-2)}{k}\right]^{(n-2)/2} \int_M a(\psi_k)^2 dV \\
&= \left[\frac{n(n-2)}{k}\right]^{(n-2)/2} \frac{(n-2)(n-1)}{n(n-4)} 2^{(n+2)/2} k^{(n-2)/2} \times \\
&\quad \omega_{n-1} I_n^{n/2} \Big\{ a(x_o) \frac{1}{k} + o(\frac{1}{k}) \Big\} \\
&= [n(n-2)]^{(n-2)/2} \frac{(n-2)(n-1)}{n(n-4)} 2^{(n+2)/2} \times \\
&\quad \omega_{n-1} I_n^{n/2} \Big\{ a(x_o) \frac{1}{k} + o(\frac{1}{k}) \Big\}, \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
\int_M f(v_k)^{2^*} dV &= \left[\frac{n(n-2)}{k}\right]^{n/2} \int_M f(\psi_k)^{2^*} dV \\
&= \left[\frac{n(n-2)}{k}\right]^{n/2} 2^{(n-2)/2} k^{n/2} \times \\
&\quad \omega_{n-1} I_n^{\frac{n}{2}-1} \Big\{ f(x_o) - \frac{1}{k} \left(\frac{\Delta f(x_o)}{(n-2)} + \frac{f(x_o) n \rho^2}{4} \right) + o(\frac{1}{k}) \Big\} \\
&= [n(n-2)]^{n/2} 2^{(n-2)/2} \omega_{n-1} I_n^{\frac{n}{2}-1} \times \\
&\quad f(x_o) \Big\{ 1 - \frac{1}{k} \left(\frac{\Delta f(x_o)}{f(x_o)(n-2)} + \frac{n\rho^2}{4} \right) + o(\frac{1}{k}) \Big\} \qquad (11)
\end{aligned}
$$

and

$$\begin{aligned}\int_M (v_k)^{2^*} dV &= \left[\frac{n(n-2)}{k}\right]^{n/2} \int_M (\psi_k)^{2^*} dV \\ &= \left[\frac{n(n-2)}{k}\right]^{n/2} 2^{(n-2)/2} k^{n/2} \omega_{n-1} I_n^{\frac{n}{2}-1} \left\{1 - \frac{1}{k}\frac{n\rho^2}{4} + o(\frac{1}{k})\right\} \\ &= [n(n-2)]^{n/2}\, 2^{(n-2)/2} \omega_{n-1} I_n^{\frac{n}{2}-1} \left\{1 - \frac{1}{k}\frac{n\rho^2}{4} + o(\frac{1}{k})\right\}. \quad (12)\end{aligned}$$

By using (12), we obtain that

$$\begin{aligned}\|v_k\|_{2^*}^{-2} &= \left(\int_M (v_k)^{2^*} dV\right)^{-2/2^*} \\ &= \left(\int_M (v_k)^{2^*} dV\right)^{(2-n)/n} \\ &= [n(n-2)]^{(2-n)/2} \left[2^{(n-2)/2} \omega_{n-1} I_n^{\frac{n}{2}-1}\right]^{(2-n)/n} \times \\ &\quad \left\{1 + \frac{1}{k}\frac{(n-2)\rho^2}{4} + o(\frac{1}{k})\right\} \quad (13)\end{aligned}$$

by using (9) and (10)

$$\begin{aligned}\int_M [|\nabla v_k|^2 + a(v_k)^2] dV &= [n(n-2)]^{(n-2)/2}\, 2^{n/2} \frac{(n-2)^2}{4} \omega_{n-1} I_n^{n/2} \left\{1 + \frac{1}{k} \times \right. \\ &\quad \left. \left(\frac{a(x_o)8(n-1)}{n(n-2)(n-4)} - \frac{n(n+2)\rho^2}{4(n-4)}\right) + o(\frac{1}{k})\right\}. \quad (14)\end{aligned}$$

By (13), (14) and the Remark 3 we obtain that

$$\begin{aligned}\frac{\int_M [|\nabla v_k|^2 + a(v_k)^2] dV}{\|v_k\|_{2^*}^2} &= \frac{1}{K(n,2)^2}\left\{1 + \frac{1}{k}\left(\frac{a(x_o)8(n-1)}{n(n-2)(n-4)} - \frac{n(n+2)\rho^2}{4(n-4)}\right.\right. \\ &\quad \left.\left. + \frac{(n-2)}{4}\rho^2\right) + o(\frac{1}{k})\right\}. \quad (15)\end{aligned}$$

Analogously, we can obtain that

$$\frac{\int_M f(v_k)^{2^*} dV}{\|v_k\|_{2^*}^{2^*}} = f(x_o) - \frac{1}{k}\frac{\Delta f(x_o)}{(n-2)} + o(\frac{1}{k}). \quad (16)$$

Statement 1

$$\frac{\displaystyle\int_M h(v_k)^{q+1}dV}{\|v_k\|_{2^*}^{q+1}} \longrightarrow 0 \textit{ when } k \to \infty.$$

Indeed, by (12)

$$\begin{aligned}
\|v_k\|_{2^*}^{q+1} &= \left[\int_M (v_k)^{2^*}dV\right]^{(q+1)/2^*} \\
&= \left[[n(n-2)]^{n/2}\, 2^{(n-2)/2}\omega_{n-1}I_n^{\frac{n}{2}-1}\right]^{(q+1)/2^*} \times \\
&\qquad \left\{1-\frac{1}{k}\frac{n\rho^2}{4}+o(\frac{1}{k})\right\}^{(q+1)/2^*} \\
&\overset{k\to\infty}{\longrightarrow} \left[[n(n-2)]^{n/2}\, 2^{(n-2)/2}\omega_{n-1}I_n^{\frac{n}{2}-1}\right]^{(q+1)/2^*}. \qquad (17)
\end{aligned}$$

Namely, $\left(\|v_k\|_{2^*}^{q+1}\right)^{-1}$ is bounded in $\mathbb{R}$.

On the other hand,

$$(v_k)^{q+1} = \left[\frac{n(n-2)}{k}\right]^{(q+1)(n-2)/4} (\psi_k)^{q+1} \overset{k\to\infty}{\longrightarrow} 0 \text{ a.e. in } M$$

and $(v_k)^{q+1}$ is bounded in $L^{2^*/(q+1)}$, since by (12)

$$\int_M [(v_k)^{q+1}]^{2^*/(q+1)}dV = \int_M (v_k)^{2^*}dV \overset{k\to\infty}{\longrightarrow} [n(n-2)]^{n/2}\, 2^{(n-2)/2}\omega_{n-1}I_n^{\frac{n}{2}-1}.$$

Therefore, (see [4])

$$(v_k)^{q+1} \rightharpoonup 0 \quad \text{in} \quad L^{2^*/(q+1)}.$$

What give us

$$\int_M h(v_k)^{q+1}dV \longrightarrow 0 \text{ when } k \to \infty. \qquad (18)$$

Consequently, by (17) and (18) we obtain the Statement 1. □

Now we can estimate

$$J\left(t\frac{v_k}{\|v_k\|_{2^*}}\right) = \frac{t^2}{2}\frac{\int_M [|\nabla v_k|^2 + a(v_k)^2]dV}{\|v_k\|_{2^*}^2}$$
$$-\frac{t^{2^*}}{2^*}\frac{\int_M f(v_k)^{2^*}dV}{\|v_k\|_{2^*}^{2^*}} - \frac{t^{q+1}}{q+1}\frac{\int_M h(v_k)^{q+1}dV}{\|v_k\|_{2^*}^{q+1}}.$$

Taking for each $k \in \mathbb{N}^*$

$$A(k) = \frac{1}{2}\frac{\int_M [|\nabla v_k|^2 + a(v_k)^2]dV}{\|v_k\|_{2^*}^2},$$

$$B(k) = \frac{1}{2^*}\frac{\int_M f(v_k)^{2^*}dV}{\|v_k\|_{2^*}^{2^*}}$$

and

$$C(k) = \frac{1}{q+1}\frac{\int_M h(v_k)^{q+1}dV}{\|v_k\|_{2^*}^{q+1}}$$

we obtain, by (15), (16) and from Statement 1, that

$$A(k) \overset{k\to\infty}{\longrightarrow} A = \frac{1}{2K(n,2)^2} > 0,$$

$$B(k) \overset{k\to\infty}{\longrightarrow} B = \frac{f(x_o)}{2^*} > 0$$

and

$$C(k) \overset{k\to\infty}{\longrightarrow} 0 \quad \text{with } C(k) > 0.$$

Then, using Lemma 1, there is $t_k > 0$, for a big enough k, such that

$$
\begin{aligned}
0 < \sup_{t\geq 0} J\left(t\frac{v_k}{\|v_k\|_{2^*}}\right) &= J\left(t_k\frac{v_k}{\|v_k\|_{2^*}}\right) \\
&= \frac{(t_k)^2}{2K(n,2)^2} - \frac{(t_k)^{2^*}f(x_o)}{2^*} + \frac{1}{k}\left\{\frac{(t_k)^2}{2K(n,2)^2}\left[\frac{a(x_o)8(n-1)}{n(n-2)(n-4)}\right.\right. \\
&\left.\left. - \frac{n(n+2)\rho^2}{4(n-4)} + \frac{(n-2)\rho^2}{4}\right] + \frac{(t_k)^{2^*}\Delta f(x_o)}{2^*(n-2)}\right\} + o(\frac{1}{k}) \\
&- \frac{(t_k)^{q+1}}{q+1}\frac{\int_M h(v_k)^{q+1}dV}{\|v_k\|_{2^*}^{q+1}} \\
&< \frac{(t_k)^2}{2K(n,2)^2} - \frac{(t_k)^{2^*}f(x_o)}{2^*} \\
&+ \frac{1}{k}\left\{\frac{(t_k)^2}{2K(n,2)^2}\left[\frac{a(x_o)8(n-1)}{n(n-2)(n-4)} - \frac{n(n+2)\rho^2}{4(n-4)} + \frac{(n-2)\rho^2}{4}\right]\right. \\
&\left. + \frac{(t_k)^{2^*}\Delta f(x_o)}{2^*(n-2)}\right\} + o(\frac{1}{k}).
\end{aligned}
$$

By using that $R(x_o) = n(n-1)\rho^2$, we obtain

$$
\begin{aligned}
0 < \sup_{t\geq 0} J\left(t\frac{v_k}{\|v_k\|_{2^*}}\right) < \frac{(t_k)^2}{2K(n,2)^2} - \frac{(t_k)^{2^*}f(x_o)}{2^*} - \frac{1}{k}\left\{\frac{(t_k)^2R(x_o)}{n(n-4)K(n,2)^2}\right. \\
\left. - \frac{a(x_o)4(n-1)(t_k)^2}{n(n-2)(n-4)K(n,2)^2} - \frac{\Delta f(x_o)(t_k)^{2^*}}{2n}\right\} + o(\frac{1}{k}).
\end{aligned}
$$

Now,

$$A(k) = \frac{1}{2K(n,2)^2} + O(\frac{1}{k}),$$

$$B(k) = \frac{f(x_o)}{2^*} + O(\frac{1}{k})$$

and

$$C(k) = O(\frac{1}{k})$$

we have that

$$t_k = t_o + O(\frac{1}{k}),$$

where

$$t_o \;=\; \left(\frac{2A}{2^*B}\right)^{1/(2^*-2)} .$$

Thus,

$$\frac{(t_k)^2R(x_o)}{n(n-4)K(n,2)^2} \;-\; \frac{a(x_o)4(n-1)(t_k)^2}{n(n-2)(n-4)K(n,2)^2} \;-\; \frac{\Delta f(x_o)(t_k)^{2^*}}{2n}$$

$$= \frac{(t_o)^2R(x_o)}{n(n-4)K(n,2)^2} - \frac{a(x_o)4(n-1)(t_o)^2}{n(n-2)(n-4)K(n,2)^2} - \frac{\Delta f(x_o)(t_o)^{2^*}}{2n} + O(\frac{1}{k}).$$

And, as

$$t_o \;=\; \left(\frac{1}{f(x_o)K(n,2)^2}\right)^{1/(2^*-2)} ,$$

$$\frac{(t_o)^2R(x_o)}{n(n-4)K(n,2)^2} \;-\; \frac{a(x_o)4(n-1)(t_o)^2}{n(n-2)(n-4)K(n,2)^2} \;-\; \frac{\Delta f(x_o)(t_o)^{2^*}}{2n} \;>\; 0$$

if, and only if,

$$\frac{R(x_o)}{n(n-4)K(n,2)^2} \;-\; \frac{a(x_o)4(n-1)}{n(n-2)(n-4)K(n,2)^2} \;-\; \frac{\Delta f(x_o)}{2nf(x_o)K(n,2)^2} \;>\; 0$$

or, equivalently

$$\frac{2R(x_o)}{(n-4)} \;-\; \frac{a(x_o)8(n-1)}{(n-2)(n-4)} \;>\; \frac{\Delta f(x_o)}{f(x_o)}.$$

Then,

$$0 < \sup_{t\geq 0} J\left(t\frac{v_k}{\|v_k\|_{2^*}}\right) < \frac{(t_k)^2}{2K(n,2)^2} - \frac{(t_k)^{2^*}f(x_o)}{2^*} - \frac{1}{k}\left\{\frac{(t_o)^2R(x_o)}{n(n-4)K(n,2)^2}\right.$$
$$\left. - \frac{a(x_o)4(n-1)(t_o)^2}{n(n-2)(n-4)K(n,2)^2} - \frac{\Delta f(x_o)(t_o)^{2^*}}{2n}\right\} + o(\frac{1}{k}).$$

Thus, by (h_3) and for a big enough k

$$-\frac{1}{k}\left\{\frac{(t_o)^2R(x_o)}{n(n-4)K(n,2)^2} \;-\; \frac{a(x_o)4(n-1)(t_o)^2}{n(n-2)(n-4)K(n,2)^2} \;-\; \frac{\Delta f(x_o)(t_o)^{2^*}}{2n}\right\}$$
$$+\; o(\frac{1}{k}) \;<\; 0.$$

Therefore,

$$
\begin{aligned}
0 \; < \; \sup_{t \geq 0} J\left(t \frac{v_k}{\|v_k\|_{2^*}} \right) \; &< \; \frac{(t_k)^2}{2K(n,2)^2} - \frac{(t_k)^{2^*} f(x_o)}{2^*} \\
&\leq \; \frac{(t_o)^2}{2K(n,2)^2} - \frac{(t_o)^{2^*} f(x_o)}{2^*} \\
&= \; \frac{1}{nK(n,2)^n (\sup f)^{(n-2)/2}}
\end{aligned}
$$

it concludes the proof of the lemma. ■

4 Proof of the Main result

Using the estimates we can now prove the Theorem 1.

For $u \in H_1^2$, by (h_1), (5) and (6), we have that

$$
\begin{aligned}
J(u) \; &\geq \; \lambda_o \|u\|_{H_1^2}^2 \; - \; \frac{\sup f (C_1)^{2^*}}{2^*} \|u\|_{H_1^2}^{2^*} \; - \; \frac{\sup h(C_2)}{q+1} \|u\|_{H_1^2}^{q+1} \\
&= \; \lambda_o \|u\|_{H_1^2}^2 \; - \; \alpha \|u\|_{H_1^2}^{q+1} \; - \; \beta \|u\|_{H_1^2}^{2^*} \\
&= \; \|u\|_{H_1^2}^2 \left[\lambda_o \; - \; \alpha \|u\|_{H_1^2}^{q-1} \; - \; \beta \|u\|_{H_1^2}^{2^*-2} \right].
\end{aligned}
$$

Let

$$
Q(s) \; = \; \alpha s^{q-1} \; + \; \beta s^{2^*-2}, \quad s \; > \; 0.
$$

As

$$
\lim_{s \to 0^+} Q(s) \; = \; +\infty \text{ and } \lim_{s \to \infty} Q(s) \; = \; +\infty,
$$

there is $s_o > 0$ such that

$$
Q(s_o) \; = \; \min_{s > 0} Q(s).
$$

Then,

$$
Q'(s_o) \; = \; 0
$$

where

$$
s_o \; = \; \left[\frac{\alpha(1-q)}{\beta(2^* - 2)} \right]^{1/[2^* - (q+1)]}.
$$

Thus, by (h_2)

$$Q(s_o) = \alpha\left[\frac{\alpha(1-q)}{\beta(2^*-2)}\right]^{(q-1)/[2^*-(q+1)]} + \beta\left[\frac{\alpha(1-q)}{\beta(2^*-2)}\right]^{(2^*-2)/[2^*-(q+1)]} < \lambda_o.$$

Therefore,

$$J(u) \geq (s_o)^2\,[\lambda_o - Q(s_o)] = \eta > 0,$$

when

$$\|u\|_{H_1^2} = s_o = \rho > 0.$$

Since for any $u_o \geq 0,\ u_o \not\equiv 0,$

$$\lim_{t\to\infty} J(tu_o) = -\infty$$

taking u_o like in the Lemma 2, we can take a big enough t_o such that $J(t_o u_o) < 0.$

Taking

$$v = t_o u_o \text{ and } \mathcal{B} = \left\{b \in C([0,1], H_1^2) \text{ such that } b(0) = 0 \text{ and } b(1) = v\right\},$$

we have satisfied the hypotheses of the Mountain Pass Theorem. Then there is a sequence $(u_j) \in H_1^2$ such that

$$J(u_j) \longrightarrow c \tag{19}$$

and

$$J^{'}(u_j) \longrightarrow 0 \text{ strongly in } \left(H_1^2\right)^{'} \tag{20}$$

where $c = \inf\limits_{b\in\mathcal{B}} \sup\limits_{0\leq t\leq 1} J(b(t)).$

And, by Lemma 2

$$0 < c < \frac{1}{nK(n,2)^n(\sup f)^{(n-2)/2}}. \tag{21}$$

Statement 2 (u_j) *is bounded in* H_1^2.

Indeed, as

$$J(u_j) = c + o(1) \tag{22}$$

and

$$\left\langle J^{'}(u_j), u_j \right\rangle \;=\; \|u_j\|_{H_1^2} o(1) \tag{23}$$

it follows by (7), (8), (22) and (23) that

$$\begin{aligned}
J(u_j) - \frac{1}{2}\left\langle J^{'}(u_j), u_j \right\rangle = &\frac{1}{2}\int_M (|\nabla u_j|^2 + a(u_j)^2)dV - \frac{1}{2^*}\int_M f((u_j)^+)^{2^*}dV \\
&- \frac{1}{q+1}\int_M h((u_j)^+)^{q+1}dV \;-\; \frac{1}{2}\int_M (|\nabla u_j|^2 \;+\; a(u_j)^2)dV \\
&+ \frac{1}{2}\int_M f((u_j)^+)^{2^*}dV \;+\; \frac{1}{2}\int_M h((u_j)^+)^{q+1}dV \\
=&\frac{1}{n}\int_M f((u_j)^+)^{2^*}dV \;+\; \frac{q-1}{2(q+1)}\int_M h((u_j)^+)^{q+1}dV \\
=&C \;+\; \|u_j\|_{H_1^2}.o(1)
\end{aligned}$$

where C represents a positive constant.

The above expression give us that

$$\frac{1}{n}\int_M f((u_j)^+)^{2^*}dV \;=\; \frac{(1-q)}{2(q+1)}\int_M h((u_j)^+)^{q+1}dV \;+\; C \;+\; \|u_j\|_{H_1^2} o(1).$$

As $2^* > q+1$ and M is compact, given $\epsilon > 0$ there is $C_\epsilon > 0$ such that

$$h(x)t^{q+1} \;\leq\; \epsilon t^{2^*} \;+\; C_\epsilon, \;\; \forall\, x \in M \;\text{ and }\; \forall\, t \geq 0. \tag{24}$$

Therefore,

$$\frac{\inf f}{n}\int_M ((u_j)^+)^{2^*}dV \;\leq\; \frac{(1-q)\epsilon}{2(q+1)}\int_M ((u_j)^+)^{2^*}dV \;+\; C \;+\; \|u_j\|_{H_1^2} o(1).$$

Namely,

$$\left[\frac{\inf f}{n} \;-\; \frac{(1-q)\epsilon}{2(q+1)}\right]\int_M ((u_j)^+)^{2^*}dV \;\leq\; C \;+\; \|u_j\|_{H_1^2} o(1).$$

Taking a small enough ϵ such that

$$\frac{\inf f}{n} \;-\; \frac{(1-q)\epsilon}{2(q+1)} \;>\; 0,$$

We conclude that

$$\int_M ((u_j)^+)^{2^*} dV \ \leq \ C \ + \ \|u_j\|_{H_1^2} o(1). \tag{25}$$

Therefore, by (h_1), (22), (24) and (25), we obtain

$$\begin{aligned} \frac{1}{2}\lambda_o \|u_j\|^2_{H_1^2} \ &\leq \ \frac{1}{2}\int_M (|\nabla u_j|^2 \ + \ a(u_j)^2) dV \\ &= \ \frac{1}{2^*}\int_M f((u_j)^+)^{2^*} dV \ + \ \frac{1}{q+1}\int_M h((u_j)^+)^{q+1} dV \ + \ C \\ &\leq \ C\int_M ((u_j)^+)^{2^*} dV \ + \ C \ \leq \ C \ + \ \|u_j\|_{H_1^2} o(1) \end{aligned}$$

and with which we conclude the Statement 2. □

Then, by Statement 2, the compact embedded $H_1^2 \hookrightarrow L^s$ for $1 \leq s < 2^*$ and the fact that H_1^2 is reflexive, there is a subsequence (u_j) of (u_j) (we will use the same notation here and for the sequence) and $u \in H_1^2$ such that

$$u_j \rightharpoonup u \ \text{ in } H_1^2, \tag{26}$$

$$u_j \longrightarrow u \ \text{ in } L^s, \quad 1 \leq s < 2^* \tag{27}$$

and

$$u_j \longrightarrow u \ \text{ a.e. in } M. \tag{28}$$

Note that, by (28)

$$((u_j)^+)^{2^*-1} \longrightarrow (u^+)^{2^*-1} \ \text{ a.e. in } M.$$

By the continuous embedded $H_1^2 \hookrightarrow L^{2^*}$ and the Statement 2, we know that

$$((u_j)^+)^{2^*-1} \text{ is bounded in } L^{2^*/(2^*-1)}.$$

Then, (see [3]) under a subsequence

$$((u_j)^+)^{2^*-1} \rightharpoonup (u^+)^{2^*-1}. \tag{29}$$

Analogously,

$((u_j)^+)^q \longrightarrow (u^+)^q$ a.e. in M and $((u_j)^+)^q$ is bounded in $L^{1/q}$ (note that $0 < q < 1$). This last conclusion follows by the embedded $H_1^2 \hookrightarrow L^1$ and by Statement 2.

Thus, taking a subsequence

$$((u_j)^+)^q \rightharpoonup (u^+)^q \text{ in } L^{1/q}. \tag{30}$$

Substituting (u_j) in (3.7) we obtain $\forall\, v \in H_1^2$,

$$\begin{aligned}\langle J'(u_j), v\rangle = \int_M (\nabla u_j.\nabla v + a u_j v)dV - \int_M f((u_j)^+)^{2^*-1} v dV \\ - \int_M h((u_j)^+)^q v dV.\end{aligned} \tag{31}$$

Taking $j \to \infty$, in (31), and using (20), (26), (27), (29) and (30), we get

$$\int_M (\nabla u.\nabla v + auv)dV = \int_M f(u^+)^{2^*-1} v dV + \int_M h(u^+)^q v dV,\ \forall v \in H_1^2.$$

Namely, u is a weak solution for equation

$$\Delta u + a(x)u = f(x)(u^+)^{2^*-1} + h(x)(u^+)^q.$$

As $f > 0$ and $h > 0$, by the weak comparison principle, we conclude that $u \geq 0$. Thus, u satisfies the equation

$$\Delta u + a(x)u = f(x)u^{2^*-1} + h(x)u^q.$$

By a Global Elliptic Regularity Theorem $u \in C^\infty(M)$.

And, by the Strong Maximum Principle, $u \equiv 0$ or $u > 0$. Our goal now is to show that $u > 0$.

Let us suppose that $u \equiv 0$.

By (22) we have that

$$\frac{1}{2}\int_M (|\nabla u_j|^2 + a(u_j)^2)dV - \frac{1}{2^*}\int_M f((u_j)^+)^{2^*} dV - \frac{1}{q+1}\int_M h((u_j)^+)^{q+1} dV$$
$$= c + o(1).$$

Using (27) in the above equation, we obtain

$$\lim_{j\to\infty} \frac{1}{2}\int_M |\nabla u_j|^2 dV \;-\; \lim_{j\to\infty} \frac{1}{2^*}\int_M f((u_j)^+)^{2^*} dV \;=\; c. \qquad (32)$$

By using (23) and the Statement 2

$$\lim_{j\to\infty} \int_M |\nabla u_j|^2 dV \;=\; \lim_{j\to\infty} \int_M f((u_j)^+)^{2^*} dV \;=\; l \;\geq\; 0. \qquad (33)$$

By (32) and (33)

$$l \;=\; nc. \qquad (34)$$

On the other hand, by (2)

$$\begin{aligned}\left(\int_M f((u_j)^+)^{2^*} dV\right)^{1/2^*} &\leq (\sup f)^{1/2^*} \|u_j\|_{2^*} \\ &\leq (\sup f)^{1/2^*} K(n,2) \left(\int_M |\nabla u_j|^2 dV\right)^{1/2} \\ &\quad + C\left(\int_M (u_j)^2 dV\right)^{1/2}.\end{aligned}$$

Taking $j \to \infty$ in the above expression and using (27), (33) and $u \equiv 0$, we obtain

$$l^{1/2^*} \;\leq\; (\sup f)^{1/2^*} K(n,2) l^{1/2}.$$

Thus,

$$l \;\geq\; \frac{1}{K(n,2)^n (\sup f)^{(n-2)/2}}.$$

And, by (34)

$$c \;\geq\; \frac{1}{nK(n,2)^n (\sup f)^{(n-2)/2}}$$

what give us a contradiction with (21).

Therefore, $u > 0$ is a regular solution of equation (1), this concludes the proof of Theorem 1. ■

References

[1] AUBIN, T., *Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire.* J. de Math. Pure et App., vol. **55** (1976), p. 269-296.

[2] AZORERO, J. G., & ALONSO, I. P., *Existence and nonuniqueness for the p-Laplacian: nonlinear eigenvalues.* Comm. in Partial Diff. Equations, vol. **12** (1987), p. 1389-1430.

[3] BRÉZIS, H., & NIRENBERG, L., *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents.* Comm. on Pure and App. Math., vol. **36** (1983), p. 437-477.

[4] BRÉZIS, H., *Analyse Foncionnelle: Théorie et applications*, Coll. Math. App. pour la Mairise, Masson, Paris (1993).

[5] GONÇALVES, J. V., & ALVES, C. O. *Existence of positive solutions for m-Laplacian equations in $\mathbb{R}^N$ involving critical Sobolev exponents.* Nonlinear Analysis, Theory, Methods and Applications, vol. **32**, n. 1 (1998), p. 57-70.

[6] DEMEGEL, F., & HEBEY, E. *On some nonlinear equations involving the p-Laplacian with critical Sobolev growth.* Adv. in Diff. Equations, vol. **3**, n. 4 (1998), p. 533-574.

[7] DJADLI, Z. *Nonlinear elliptic equations with critical Sobolev exponent on compact riemannian manifolds.* Calc. of Variat. and Partial Diff. Eq., vol. **8** (1999), p. 293-326.

[8] DRUET, O. *Generalized scalar curvature type equations on compact riemannian manifolds.* Proc. of the Royal Society of Edinburgh, vol. **130 A** (2000), p. 767-788.

[9] DRUET, O., & HEBEY, E. *The AB program in geometric analysis: sharp Sobolev inequalities and related problems.* Memoirs of the AMS, vol. **160** (2002), n. 761.

[10] HEBEY, E.. *Scalar curvature type problems in riemannian geometry.* Notes from Lectures at the University of Roma 3 (1999).

[11] HEBEY, E., & VAUGON, M. *From best constants to critical functions.* Mathematische Zeitschrift, vol. **237** (2001), p. 737-767.

[12] MIYAGAKI, O. H. *On a class of semilinear elliptic problems in* $\mathbb{R}^n$ *with critical growth.* Nonlinear Analysis, Theory, Meth. and App., vol. **29**, n. 7 (1997), p. 773-781.

[13] SCHOEN, R. *Conformal deformation of a riemannian metric to constant scalar curvature.* J. of Diff. Geom., vol. **20** (1984), p. 479-495.

[14] SILVA, C. R. *Algumas Equações Diferenciais Não Lineares em Variedades Riemannianas Compactas.* Doctoral thesis-UnB, June, (2004).

[15] TRÜDINGER, N. S. *Remarks concerning the conformal deformation of riemannian structures on compact manifolds.* Ann. Scuola Norm. Sup. Pisa, vol. **3**, n. 22 (1968), p. 265-274.

[16] YAMABE, H. *On a deformation of riemannian structures on compact manifolds.* Osaka Math. J., vol. **12** (1960), p. 21-37.

Carlos Rodrigues da Silva
Instituto de ciências exatas e
da terra, CAMPUS Universi–
dade do Araguaia, Universi–
dade federal de Mato Grosso,
Brazil
carlosro@ufmt.br