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On harmonic diffeomorphisms from conformal annuli to Riemannian annuli

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Abstract

In this paper we construct a harmonic diffeomorphism from $\mathbb{C}^*$ onto the quotient of the hyperbolic plane $\mathbb{H}^2$ by the group $< \psi >$ generated by some parabolic or hyperbolic isometry ψ. It draws its inspiration to a large extent from [CR]. The proof we give here is based on the theory of minimal surfaces: we construct an entire minimal graph $\Sigma \subset (\mathbb{H}/ < \psi >) \times \mathbb{R}$ over $\mathbb{H}/ < \psi >$, such that Σ is conformally $\mathbb{C}^*$. This solves the problem since the vertical projection from Σ onto $\mathbb{H}/ < \psi >$ is a harmonic diffeomorphism when Σ is minimal. In the main part of the article, we focus on the case where ψ is a parabolic isometry; we quickly explain then how it works for a hyperbolic isometry. We then show there is no harmonic diffeomorphism from the once punctured disk $\mathbb{D}^*$ (or $\mathbb{H}/ < \psi >$ when ψ is a parabolic translation) onto the cylinder $\mathbb{S}^1 \times \mathbb{R}$ with the flat metric.

1 Introduction

There is now a good understanding of the existence of harmonic diffeomorphisms from one simply connected Riemann surface onto a complete

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simply connected Riemannian surface. E. Heinz proved in 1952 that there exists no harmonic diffeomorphism from the disk onto the complex plane $\mathbb{C}$ with the euclidean metric. R. Schoen and S.-T. Yau generalized this by proving $\mathbb{C}$ may be replaced by a simply connected Riemannian surface of non-negative curvature in the theorem of Heinz (cf. [SY]). It is unknown if $\mathbb{C}$ may be replaced by a complete simply connected parabolic surface.

P. Collin and H. Rosenberg constructed harmonic diffeomorphisms from $\mathbb{C}$ onto the hyperbolic plane $\mathbb{H}$, using the theory of minimal graphs (cf. [CR]).

We remark that the existence of a harmonic diffeomorphism depends on the conformal type of the domain and the metric on the range. We refer to the book [SY] by R. Schoen and S.-T. Yau for an introduction to this theory.

In this paper we consider the question of the existence of harmonic diffeomorphisms between Riemannian surfaces of annular topological type.

We construct a harmonic diffeomorphism from the once punctured complex plane $\mathbb{C}^*$ onto the quotient of the hyperbolic plane $\mathbb{H}$ by a hyperbolic or parabolic isometry.

We also prove there is no harmonic diffeomorphism from the once punctured disk $\mathbb{D}^*$ onto the cylinder $\mathbb{S}^1 \times \mathbb{R}$, with the flat metric.

To obtain such a harmonic diffeomorphism, we construct entire minimal graphs $\Sigma \subset (\mathbb{H}/ < \psi >) \times \mathbb{R}$, over $\mathbb{H}/ < \psi >$, which are conformally $\mathbb{C}^*$ (here, ψ is a hyperbolic or parabolic isometry of $\mathbb{H}$). The vertical projection of such a graph then yields a harmonic diffeomorphism when Σ is minimal.

To obtain such Σ, we must look for unbounded graphs. For if Σ were bounded and conformally $\mathbb{C}^*$, the harmonic height function would extend to the puncture of $\mathbb{C}^*$, hence be constant, so Σ would be a slice $\mathbb{H}/ < \psi >$, which has the conformal type of $\mathbb{D}^*$ or some $\{0 < r_1 < |z| < r_2\}$.

To construct unbounded minimal graphs over $\mathbb{H}/ < \psi >$, we use the theory of ideal Scherk graphs developped in [CR]: more precisely, we work on a fundamental domain of the hyperbolic plane under the action of ψ, where we define the notion of a pseudo-Scherk graph. Applying rotations by π about some geodesics of the boundary, we then get a Scherk graph over the quotient $\mathbb{H}/<\psi>$, that is a graph over a ring domain bounded by some ideal polygon, which takes values $\pm\infty$ on the polygonal boundary. We prove that such a graph is conformally $\mathbb{C}^*$.

We then construct a sequence of Scherk graphs in $(\mathbb{H}/<\psi>) \times \mathbb{R}$, denoted by $(\Sigma_n)_n$, such that the polygonal domains D_n on which they are defined exhaust $\mathbb{H}/<\psi>$; to do this, given a domain D_n such that ∂D_n is an ideal polygon, we extend it into a domain D_{n+1} by attaching pairs of quadrilaterals to all of the sides of its boundary. While choosing these quadrilaterals, we have to be careful about three things:

- there must be Scherk graphs over the new domain;
- the sequence $(D_n)_n$ has to be an exhaustion of $\mathbb{H}/<\psi>$; to ensure this condition, we define the notion of *regular* quadrilaterals, and show that if the quadrilaterals we add are "almost" regular, then the sequence D_n is actually an exhaustion;
- the new quotient Scherk graph Σ_{n+1} has to be as "close" as possible to the graph Σ_n, so that the conformal properties of both graphs over the domain D_n vary little.

We then get an entire minimal graph over $\mathbb{H}/<\psi>$ by repeating the previous extension process and then taking a subsequence converging to an entire minimal graph Σ_∞. To prove that the conformal type of Σ_∞ is still $\mathbb{C}^*$, we proceed as follows: we choose some domain $\mathscr{A}$ in $\mathbb{H}/<\psi>$ such that $\mathscr{A}$ is conformally $\mathbb{D}^*$, and we write the remaining part of the graph as the disjoint union of an infinite number of compact annuli whose conformal modulus is at least one: to do this, we use the fact that for each $n \geq 0$, the quotient Scherk graph over D_{n+1} is conformally $\mathbb{C}^*$, so it is possible to choose an annulus in $\restriction \Sigma_{n+1} D_{n+1} - D_n$ with conformal modulus greater than one, which vertically projects onto an annulus $\mathscr{A}_{n+1} - \mathscr{A}_n$. Since the conformal properties of two successive quotient Scherk graphs over the intersection of their domains of definition change little, the annulus in the limit graph Σ_∞ defined over $\mathscr{A}_{n+1} - \mathscr{A}_n$ still has conformal modulus > 1. We then conclude by Grötzsch lemma that the limit graph has the conformal type of $\mathbb{C}^*$.

2 Construction of minimal graphs over a polygonal domain in the quotient

2.1 Preliminaries

2.1.1 Complex Geometry

References for this part are [V], [Al], [FM].

Let$f$ be a $\mathscr{C}^1$ homeomorphism from one region of the complex plane to another. We denote:

$$f_z := \frac{1}{2}(f_x - i f_y), \qquad f_{\overline{z}} := \frac{1}{2}(f_x + i f_y).$$

The quantity $D_f := \frac{|f_z|+|f_{\overline{z}}|}{|f_z|-|f_{\overline{z}}|}$ (≥ 1) is called the *dilatation* at the point z. From a geometrical point of view, it corresponds to the ratio of the major axis of the ellipses to which circles are mapped by the differential of f at z. We also define $d_f := \frac{|f_{\overline{z}}|}{|f_z|}$ related to D_f by:

$$D_f = \frac{1+d_f}{1-d_f}, \qquad d_f = \frac{D_f - 1}{D_f + 1}.$$

The mapping f is conformal at z if and only if $D_f = 1$, $d_f = 0$.

Definition 2.1 (Quasiconformal maps)**.** *The mapping f is said to be quasiconformal if D_f is bounded; it is K-quasiconformal if $D_f \leq K$. It is equivalent to $d_f \leq k := \frac{K-1}{K+1}$.*

Let Γ be a familiy of curves in the plane. Each $\gamma \in \Gamma$ shall be a countable union of open arcs, closed arcs or closed curves, and every closed subarc shall be rectifiable. We introduce the geometric quantity $\lambda(\Gamma)$, called the *extremal length* of Γ, which is a sort of average minimal length. Its importance lies in the fact that it is invariant under conformal mappings, and quasi-invariant under quasi-conformal mappings (the latter means that it is multiplied by a bounded factor).

A function ρ, defined in the whole plane, will be called *allowable* if it satisfies the following conditions:

1. $\rho \geq 0$ and measurable;
2. $A(\rho) = \iint \rho^2 dxdy \neq 0, \infty$.

For such a ρ, set:

$$L_\gamma(\rho) = \int_\gamma \rho dz$$

if ρ is measurable on γ and $L_\gamma(\rho) = \infty$ otherwise. We introduce:

$$L(\rho) = \inf_{\gamma \in \Gamma} L_\rho(\gamma)$$

and

$$\lambda(\Gamma) = \sup_\rho \frac{L(\rho)^2}{A(\rho)}$$

for all allowable ρ.

The modulus of an annulus

Let D be a doubly connected region in the finite plane with C_1 the bounded and C_2 the unbounded component of the complement. We say the closed curve γ in D separates C_1 and C_2 if γ has non-zero winding number about the points of C_1. Let Γ be the family of closed curves in D which separate C_1 and C_2.

Definition 2.2 (Modulus)**.** *With the previous notations, we define the modulus of D to be $M(D) := \lambda(\Gamma)$.*

Consider for example the annulus $D = \{r_1 \leq |z| \leq r_2\}$. We get successively

$$\begin{aligned}
L(\rho) &\leq \int_0^{2\pi} \rho(re^{i\theta}) r d\theta, \\
\frac{L(\rho)}{r} &\leq \int_0^{2\pi} \rho d\theta, \\
L(\rho) \log\left(\frac{r_2}{r_1}\right) &\leq \iint \rho dr d\theta, \\
L(\rho)^2 \log^2\left(\frac{r_2}{r_1}\right) &\leq 2\pi \log\left(\frac{r_2}{r_1}\right) \iint \rho^2 r dr d\theta, \\
\frac{L(\rho)^2}{A(\rho)} &\leq \frac{2\pi}{\log\left(\frac{r_2}{r_1}\right)}.
\end{aligned}$$

Moreover we get the equality for $\rho = \frac{1}{2\pi r}$. Indeed, for any $\gamma \in \Gamma$, we have:

$$1 \leq |\mathrm{ind}(\gamma, 0)| = \frac{1}{2\pi} \left| \int_\gamma \frac{dz}{z} \right| \leq \frac{1}{2\pi} \int_\gamma \frac{|dz|}{|z|} = L_\rho(\gamma),$$

so $L(\rho) = 1$ and $A(\rho) = \frac{1}{2\pi} \log\left(\frac{r_2}{r_1}\right)$. Thus we conclude that $M(D) = \frac{1}{2\pi} \log\left(\frac{r_2}{r_1}\right)$. ■

Suppose now that all $\gamma \in \Gamma$ are contained in a region Ω and let ϕ be a K-quasiconformal mapping of Ω on Ω'. Let Γ' be the image set of Γ. Then we have:

Theorem 2.1.

$$K^{-1}\lambda(\Gamma) \leq \lambda(\Gamma') \leq K\lambda(\Gamma).$$

Proof. For a given $\rho(z)$ define $\rho'(\zeta) = 0$ outside Ω' and

$$\rho'(\zeta) = \frac{\rho}{|\phi_z| - |\phi_{\overline{z}}|} \circ \phi^{-1}$$

in Ω'. Then:

$$\int_{\gamma'} \rho' |d\zeta| \geq \int_{\gamma} \rho |dz|,$$

$$\iint \rho'^2 d\xi d\eta = \iint_{\Omega} \rho^2 \frac{|\phi_z| + |\phi_{\overline{z}}|}{|\phi_z| - |\phi_{\overline{z}}|} dxdy \leq KA(\rho).$$

This proves $\lambda' \geq K^{-1}\lambda$, and the other inequality follows by considering the inverse. ∎

Corollary 2.1. $\lambda(\Gamma)$ is a conformal invariant.

Theorem 2.2 (Grötzsch Lemma)**.** Let D be an annulus $D := \{z : r_1 < |z| < r_2\}$ and $D_1, \ldots, D_n$ be non-overlapping doubly connected domains that separate the boundary components of ∂D. If $M(D)$ stands for the modulus of D with respect to the family of curves that separate its boundary components, then:

$$M(D) \geq \sum_{j=1}^{n} M(D_j).$$

Proof. The metric $\rho := \frac{|dz|}{2\pi|z|}$ is extremal for $M(D)$ and, simultaneously, allowable for $M(D_j)$. Then:

$$M(D) = \frac{1}{2\pi} \log\left(\frac{r_2}{r_1}\right) = \iint_D (\rho(z))^2 d\sigma_z \geq \sum_{j=1}^{n} \iint_{D_j} (\rho(z))^2 d\sigma_z \geq \sum_{j=1}^{n} M(D_j).$$

∎

2.1.2 Minimal surfaces

We begin with some properties established in [JS], [NR] and [CR]. Let D be a domain of the hyperbolic plane. By solution in D we mean a solution of the minimal surface equation in the domain D.

Theorem 2.3 (Compactness Theorem)**.** Let $\{u_n\}_n$ be a uniformly bounded sequence of solutions in D. Then a subsequence converges uniformly on compact subsets of D to a solution in D.

Theorem 2.4 (Monotone Convergence Theorem)**.** Let $\{u_n\}_n$ be a monotone sequence of solutions in D. If the sequence $\{|u_n|\}_n$ is bounded at one point of D, then there is a nonempty open set $U \subset D$, called the *convergence set*, such that $\{u_n\}$ uniformly converges to a solution on compact subsets of U, and uniformly diverges on compact subsets of $D - U =: V$, where V is called the *divergence set*.

Now let Γ be an ideal polygon of $\mathbb{H}$ (i.e. Γ is a polygon all of whose vertices lie at $\partial_\infty \mathbb{H}$). Assume that Γ has a finite number of vertices, and that it is composed of geodesic sides $(A_i)_{i=1,\ldots,l}$, $(B_j)_{j=1,\ldots,l'}$, and convex arcs $(C_k)_{k=1,\ldots,l''}$, where the C_k are assumed to be convex with respect to the domain D bounded by Γ. We also assume that no two of the A_i (or B_j) have a vertex in common, that D is simply connected, and that ∂D $(= \Gamma)$, together with the vertices, is homeomorphic to $\mathbb{S}^1$.

Theorem 2.5 (Divergence Structure Theorem)**.** Let $\{u_n\}$ be a monotone sequence of solutions in D, each u_n continuous on $\overline{D}$. If the divergence set V is nonempty, then $\mathring{V} \neq \emptyset$, and ∂V is composed of ideal geodesics among the A_i and B_j, convex arcs among the C_k, and interior ideal geodesics $\gamma \subset D$ joining two vertices of ∂D. No two interior geodesics γ_1, γ_2 of ∂V have the same vertex at infinity.

Now let $\{u_n\}$ be a sequence defined in D satisfying the hypothesis of the Divergence Structure Theorem. For each n, we define on D the vector field $X_n := \frac{\nabla u_n}{W_n}$, where $W_n^2 := 1 + |\nabla u_n|^2$. For $\mathscr{W} \subset D$, and α a boundary arc of $\mathscr{W}$, we define the flux of u_n across α to be

$$F_n(\alpha) = \int_\alpha \langle X_n, \nu \rangle ds;$$

here, α is oriented as the boundary of $\mathscr{W}$ and ν is the outer conormal to $\mathscr{W}$ along α. More generally, for any solution u in D and an oriented arc α, we write F_u the flux of the associated field $X := \frac{\nabla u}{W}$, where $W^2 := 1 + |\nabla u|^2$.

Theorem 2.6 (Flux Theorem)**.** Let $\mathscr{W}$ be a domain in D. Then:

1. If $\partial\mathscr{W}$ is a compact cycle, then $F_n(\partial\mathscr{W}) = 0$ (Stokes Theorem + $\operatorname{div}(X_n) = 0$ for a minimal solution).

2. If $\mathscr{W} \subset U$ and α is a compact arc of $\partial\mathscr{W}$ on which the u_n diverge to $+\infty$, then α is a geodesic and

$$\lim_{n\to\infty} F_n(\alpha) = |\alpha|.$$

If the u_n diverge to $-\infty$ on α, then α is a geodesic and

$$\lim_{n\to\infty} F_n(\alpha) = -|\alpha|.$$

3. If $\mathscr{W} \subset V$, and the u_n remain uniformly bounded in α, then

$$\lim_{n\to\infty} F_n(\alpha) = -|\alpha|, \ \textit{if } u_n \to +\infty \textit{ on } V,$$

and

$$\lim_{n\to\infty} F_n(\alpha) = |\alpha|, \ \textit{if } u_n \to -\infty \textit{ on } V.$$

When we establish existence theorems for unbounded boundary data, we need to know solutions take on the boundary values prescribed. This is guaranteed in our situation by the following result.

Theorem 2.7 (Boundary Values Lemma)**.** Let D be a domain and let C be a compact convex arc in ∂D. Suppose $\{u_n\}$ is a sequence of solutions in D that converges uniformly on compact subsets of D to a solution u in D. Assume each u_n is continuous in $D \cup C$ and that the boundary values of u_n on C converge uniformly to a function f on C. Then u is continuous in $D \cup C$ and u equals f on C.

Theorem 2.8 (Generalized Maximum Principle)**.** Let D be a domain with ∂D an ideal geodesic polygon. Let $\mathscr{U} \subset D$ be a domain and $u, v \in \mathscr{C}^0(\overline{\mathscr{U}})$, two solutions of the minimal surface equation in $\mathscr{U}$ with $u \leq v$ on $\partial\mathscr{U}$. Then $u \leq v$ in $\mathscr{U}$.

We end this section by stating the existence theorem for compact domains.

Theorem 2.9 (Existence Theorem)**.** Let $\mathscr{W}$ be a bounded domain with $\partial\mathscr{W}$ a Jordan curve. Assume there is a finite set $E \subset \partial\mathscr{W}$ and $\partial\mathscr{W} - E$ is composed of convex arcs. Then there is a solution to the Dirichlet problem in $\mathscr{W}$ taking on arbitrarily prescribed continuous data on $\partial\mathscr{W} - E$. The arcs need not be strictly convex.

2.2 The Dirichlet problem on domains bounded by ideal polygons

We are interested in solving some Jenkins-Serrin type's problem in the quotient $\mathbb{H}/ < \psi >$; to do this, we work in a fundamental domain Δ of $\mathbb{H}$ under the action of ψ. Given a domain $D \subset \Delta$ bounded by some ideal polygon Γ, we state necessary and sufficient conditions on the "lengths " of the geodesic sides of Γ which ensure the existence of a minimal graph defined over D and satisfying some special Dirichlet condition. We then explain how to get by a geometric constuction a minimal surface defined over the quotient $D/ < \psi >$, and assuming values $\pm\infty$ on the boundary.

Let then $\Gamma := \partial D$ be an ideal polygon of $\mathbb{H}$. Assume that Γ has a finite number of vertices at infinity, and that it is composed of geodesic sides $(A_i)_{i=1,\dots,l}$, $(B_j)_{j=1,\dots,l'}$, and convex arcs $(C_k)_{k=1,\dots,l''}$, where the C_k are assumed to be convex towards D. As previously for the Flux Theorem, we also suppose that no two of the A_i (or B_j) have a vertex in common, that D is simply connected, and that Γ, together with the vertices, is homeomorphic to $\mathbb{S}^1$.

We make an important assumption on the sides of Γ: when a convex arc C in Γ has a point $a_i \in \partial_\infty\mathbb{H}$ as a vertex, then the other arc α of Γ having a_i as a vertex is asymptotic to C at a_i, which means that for a sequences $p_n \in \alpha$ such that $\lim\limits_{n\to\infty} p_n = a_i$, one has $\lim\limits_{n\to\infty} dist_{\mathbb{H}}(p_n, C) = 0$. This assumption is what is needed to assure that the Generalized Maximum Principle (Theorem 2.8) holds in D.

The theorem we are interested in gives conditions on the "lengths " of the edges of Γ which enable us to construct some minimal graph over D, such that the corresponding function equals $+\infty$ on each A_i, $-\infty$ on each B_j, and 0 on the C_k. But before we can state it, we have to define what we exactly mean by the "length " of geodesics of $\mathbb{H}$.

At each vertex a_i of Γ, place a horocycle H_i; do this so $H_i \cap H_j = \emptyset$ if $i \neq j$. Let F_i be the convex horodisk with boundary H_i (cf. Figure 1).

Each A_i meets exactly two horodisks. Denote by $\tilde{A}_i$ the compact arc of A_i which is the part of A_i outside the two horodisks ($\partial \tilde{A}_i$ is two points, each on a horocycle); we define the *truncated length* $|A_i|$ to be the distance between these horocycles, i.e. the length of $\tilde{A}_i$. Define $\tilde{B}_i$ and $|B_i|$ in the same way.

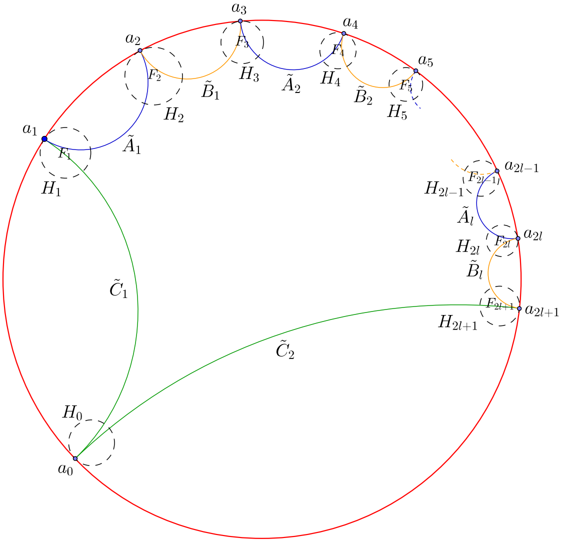


Figure 1: Horocycles and truncated polygon

Now define:

$$a(\Gamma) = \sum_{i=1}^{l} |A_i|,$$

$$b(\Gamma) = \sum_{i=1}^{l'} |B_i|.$$

Definition 2.3 (Inscribed polygons)**.** *An ideal geodesic polygon $\mathscr{P}$ is said to be inscribed in D if the vertices of $\mathscr{P}$ are among the vertices of Γ; $\mathscr{P}$ is a simple closed polygon whose edges are either interior in D or equal to edges of Γ.*

We can extend to an inscribed polygon $\mathscr{P}$ the quantities defined previously for Γ and a choice of horocycles at its vertices: in the same way, we set $a(\mathscr{P}) := \sum_{A_i \in \mathscr{P}} |A_i|$, $b(\mathscr{P}) := \sum_{B_j \in \mathscr{P}} |B_j|$, and we define the truncated length $|\mathscr{P}|$ to be the length of the part of $\mathscr{P}$ outside the horodisks.

Remark: The utilization of the truncated perimeter $|\mathscr{P}|$ gives rise to quantities associated to $\mathscr{P}$ which are independent of the choice of the horocycles at each vertex. This will allow us to check some lengths' inequalities for a choice of horocycles *a priori*. Indeed on the one hand, at a vertex of $\mathscr{P}$ with a side A_i (necessarily unique from our hypotheses) the quantity $|\mathscr{P}| - 2a(\mathscr{P})$ does not depend on the choice of the horocycles at that point. On the other hand, for the remaining vertices, this quantity increases arbitrarily for a choice of "small " enough horocycles. By symmetry, the same is true for the quantity $|\mathscr{P}| - 2b(\mathscr{P})$.

We can now state the result:

Theorem 2.10. Let Γ be as described above and let f be continuous on the convex arcs C_k of Γ (we assume there are such arcs). Then there is a unique solution in D which is $+\infty$ on each A_i, $-\infty$ on each B_j, and f on each C_k, if and only if

$$|\mathscr{P}| - 2a(\mathscr{P}) > 0$$

and

$$|\mathscr{P}| - 2b(\mathscr{P}) > 0$$

for all inscribed polygons $\mathscr{P}$ in Γ.

Definition 2.4. *An inscribed polygon satisfying the previous conditions is called admissible.*

Before we prove the theorem, we remark that by our previous discussion, the sign of the quantities which appear in the left side of the two inequalities does not depend on the choice we make of small enough horocycles.

Proof of the theorem: Let us rename the edges of Γ by setting: $C_{l''+j} := B_j,\ j = 1, \dots, l'$. Let $\tilde{f}$ be the function defined on the $C_k,\ k = 1, \dots, l'+l''$ in the following way:

$$\restriction \tilde{f} C_k := \begin{cases} \max(\restriction f C_k, 0) \text{ for } k \leq l'', \\ 0 \text{ for } k > l''. \end{cases}$$

Notice that $(A_i)_{i=1,\dots,l},\ (C_k)_{k=1,\dots,l'+l''}$ and $\tilde{f}$ satisfy the hypotheses of Lemma 2.1, which is stated below, so there exists a solution u^+ such in D equal to $+\infty$ on the $(A_i)_{i=1,\dots,l}$ and to $\tilde{f}$ on the $(C_k)_{k=1,\dots,l'+l''}$, i.e.:

$$\begin{cases} \restriction u^+ A_i,\ i = 1, \dots, l & = +\infty, \\ \restriction u^+ B_j,\ j = 1, \dots, l' & = 0, \\ \restriction u^+ C_k,\ k = 1, \dots, l'' & = \max(\restriction f C_k, 0). \end{cases}$$

Similarly, let u^- be the solution in D such that

$$\begin{cases} \restriction u^- A_i,\ i = 1, \dots, l & = 0, \\ \restriction u^- B_j,\ j = 1, \dots, l' & = -\infty, \\ \restriction u^- C_k,\ k = 1, \dots, l'' & = \min(\restriction f C_k, 0). \end{cases}$$

Let v_n be the solution such that: $\begin{cases} \restriction v_n A_i,\ i = 1, \dots, l & = n, \\ \restriction v_n B_j,\ j = 1, \dots, l' & = -n, \\ \restriction v_n C_k,\ k = 1, \dots, l'' & = -n \wedge f \wedge n, \end{cases}$
where $-n \wedge f \wedge n$ is f truncated above by n and below by $-n$.

Then by the Generalized Maximum Principle,

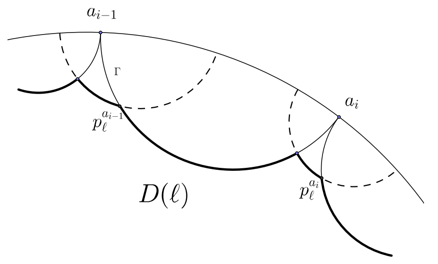
$$u^- \leq v_n \leq u^+ \text{ in } D.$$

Therefore, the sequence $\{v_n\}$ is uniformly bounded on compact subsets of D so that a subsequence converges uniformly on compact sets to a solution v in D. By the Boundary Values Lemma, v takes on the desired boundary values on Γ. ■

Let us now prove the following lemma, which we needed to establish Theorem 2.10 (compared to Theorem 2.10, there are no $B's$ here, and the data on the $C's$ are supposed to be bounded below):

Lemma 2.1. Let Γ be an ideal polygon composed of geodesic sides $(A_i)_{i=1,\dots,m}$ and of convex arcs $(C_k)_{k=1,\dots,m'}$ asymptotic to their neighbors. Suppose also that $2a(\mathscr{P}) < |\mathscr{P}|$ for all inscribed polygons $\mathscr{P}$ in Γ, and that f is bounded below. Then there exists a solution u in the domain D bounded by Γ, equal to $+\infty$ on the A_i, and f on the C_k. (Of course, a similar result holds if Γ is composed of geodesic sides $(B_j)_{j=1,\dots,m}$ and convex arcs $(C_k)_{k=1,\dots,m'}$, $2b(\mathscr{P}) < |\mathscr{P}|$ for all inscribed polygons $\mathscr{P}$ in Γ, and f is bounded above; in that case, the solution u equals $-\infty$ on the B_j and f on the C_k.)

Proof. We first construct an exhaustion of D by compact convex disks $D(\ell)$ on which we solve some Dirichlet problem. To do this, we proceed as follows: given two contiguous arcs $\alpha, \beta \in \Gamma$ with a common vertex a_i, we choose a sequence $(p_\ell^{a_i})_\ell$ of points of α such that $\lim\limits_{\ell\to\infty} p_\ell^{a_i} = a_i$. Let then $\gamma_{a_i}(\ell)$ be associated geodesic arcs joining the $p_\ell^{a_i}$ to β; from the hypothesis we made on the convex arcs C_k, we know that $\lim\limits_{\ell\to\infty} |\gamma_{a_i}(\ell)| = 0$. We denote by $\Gamma(\ell)$ the polygon consisting of the $(\gamma_{a_i}(\ell))_{a_i}$ and of the parts of the initial arcs bounded by two successive $\gamma_{a_i}(\ell)$, and we denote by $D(\ell)$ the disk bounded by $\Gamma(\ell)$.



In each $D(\ell)$, let $u_n(\ell)$ be the solution defined by its boundary values on $\partial D(\ell)$:

$$\begin{cases} \restriction u_n(\ell)\tilde{A}_i, \; i = 1, \ldots, m & = n, \\ \restriction u_n(\ell)\gamma_{a_i}(\ell) & = 0, \\ \restriction u_n(\ell)\tilde{C}_k, \; k = 1, \ldots, m' & = \min(n, f). \end{cases}$$

For $\ell \leq \ell'$, $u_n(\ell')$ is everywhere above $u_n(\ell)$ on $\Gamma(\ell)$, so by the Maximum Principle we conclude that $u_n(\ell) \leq u_n(\ell')$ on $D(\ell)$. We deduce that locally, $(u_n(\ell))_\ell$ is an increasing bounded sequence and therefore, by the Monotone Convergence Theorem, it converges to a solution u_n in D that equals n on the A_i and $\min(n, f)$ on the C_k.

By the Generalized Maximum Principle, $(u_n)_n$ is a monotone increasing sequence; moreover, it is uniformly bounded in a neighborhood of the C_k by the Boundary Values Lemma.

Suppose that the sequence $(u_n)_n$ has a nonempty divergence set V; then let P be a connected component of V, bounded by some inscribed polygon $\mathscr{P}$, whose edges are geodesics joining vertices of Γ. A flux calculation contradicts the hypothesis $2a(\mathscr{P}) < |\mathscr{P}|$.

Then let the u_n converge to a solution u in D; as expected, the latter equals $+\infty$ on the A_i and f on the C_k from the Boundary Values Lemma, which concludes the proof of the lemma.

We now examine the flux calculation in more detail.

$\star$ *First suppose* $V = D$, so $\partial V = \Gamma$.

We fix ℓ and for $n \geq 0$, we consider the function u_n on $D(\ell)$. Let $X_n = \frac{\nabla u_n}{W_n}$, where $W_n^2 = 1 + |\nabla u_n|^2$. By the Flux Theorem, the flux of X_n along

$\partial D(\ell) = \Gamma(\ell)$ is zero. For each arc $\alpha \subset \Gamma(\ell)$, let $F_n(\alpha) = \int_\alpha \langle X_n, \nu \rangle \, ds$, where ν is the outer conormal to $D(\ell)$ along $\partial D(\ell)$.

Then the flux of X_n along $\partial D(\ell)$ yields:

$$0 = F_n(\tilde{A}_1) + F_n(\tilde{A}_2) + \cdots + F_n(\tilde{A}_m) + F_n(\tilde{C}_1) + \cdots + F_n(\tilde{C}_{m'}) + \sum_{a_i} F_n(\gamma_{a_i}(\ell)),$$

where the $\gamma_{a_i}(\ell)$ are small geodesic arcs in $\partial D(\ell)$. Now the u_n diverge uniformly to infinity on compact subsets of D and are bounded on $C_1 \cup C_2 \cup \ldots \cup C_{m'}$, so that:

$$\forall \; k = 1, \ldots, m', \; \lim_{n \to \infty} F_n(\tilde{C}_k) = -|\tilde{C}_k|.$$

Also $F_n(\alpha) \leq |\alpha|$ for any arc $\alpha \subset \partial D(\ell)$. Hence letting $n \to \infty$, we obtain:

$$|\gamma(\ell)| + |\tilde{A}_1| + \cdots + |\tilde{A}_m| \geq |\tilde{C}_1| + \cdots + |\tilde{C}_{m'}|,$$

where $|\gamma(\ell)|$ is the length of all the arcs $\gamma_{a_i}(\ell)$ in $\partial D(\ell)$. Adding $|\tilde{A}_1| + \cdots + |\tilde{A}_l|$ to both sides yields $2a(\mathscr{P}) \geq |\mathscr{P}|$, since $|\gamma(\ell)| \to 0$ when $\ell \to \infty$, *which is impossible.*

$\star$ It remains to show that $\{V \neq D, \, V \neq \emptyset\}$ is impossible too. *Suppose this were the case.* Fix ℓ and consider $V \cap D(\ell) = V(\ell)$. Then $V(\ell)$ is bounded by interior geodesic arcs $\alpha_1, \alpha_2, \ldots$, some arcs $\tilde{A}_{i_1}, \tilde{A}_{i_2}, \ldots, \tilde{C}_{k_1}, \tilde{C}_{k_2}, \ldots$ of $\partial D(\ell)$, and some small geodesic arcs $\gamma_{a_{p_1}}, \gamma_{a_{p_2}}, \ldots$ (all of these arcs depend on ℓ).

The flux of X_n along $\partial V(\ell)$ equals zero; hence:

$$\sum F_n(\tilde{A}_i) + \sum F_n(\tilde{C}_k) + \sum F_n(\alpha_m) + \sum F_n(\gamma_{a_p}) = 0,$$

where the sums are taken over all the arcs in $\partial V(\ell)$. The arcs $\alpha_1, \alpha_2, \ldots,$ are interior arcs of D so on each α_m,

$$\lim_{n \to \infty} F_n(\alpha_m) = -|\alpha_m|.$$

Similarly, for the $\tilde{C}_k$ in $\partial V(\ell)$,

$$\lim_{n \to \infty} F_n(\tilde{C}_k) = -|\tilde{C}_k|.$$

Let $|\gamma|$ be the total length of the small geodesic arcs in $\partial V(\ell)$, so $|\gamma| \geq \sum F_n(\gamma_{a_p})$. Similarly for the arcs $\tilde{A}_i$ we have: $a(\mathscr{P}) \geq \sum F_n(\tilde{A}_i)$.

We let $n \to \infty$; then the previous flux equality yields:

$$a(\mathscr{P}) + |\gamma| \geq |\alpha| + |C|$$

where $|\alpha|$ is the sum of the lengths of the arcs α_m and $|C|$ is the sum of the lengths of the arcs C_k in $\mathscr{P}$. Adding $a(\mathscr{P})$ to both sides we get:

$$2a(\mathscr{P}) + |\gamma| \geq |\mathscr{P}|.$$

Since $|\gamma| \to 0$ when $\ell \to \infty$ *we also get a contradiction* here, so the lemma is proved. ■

3 Quotient Scherk graphs

3.1 Construction of Quotient Scherk graphs

Definition 3.1 (Pseudo-Scherk polygon).

*Let Γ be an ideal polygon of $\mathbb{H}$, and suppose Γ has an even number of sides, with distinguished sides $(A_i)_{i=1,\dots,l}$, $(B_j)_{j=1,\dots,l}$, $(C_k)_{k=1,2}$, ordered clockwise in the following way: $C_1, A_1, B_1, A_2,$
$B_2, \dots, A_l, B_l, C_2$ (cf Figure 2). Then the polygon Γ is said to be a pseudo-Scherk polygon if all its inscribed polygons $\mathscr{P}$ are admissible.*

Remark that since the C_k are geodesics, they are in particular convex and satisfy the property to be asymptotic to their contiguous arcs.

Definition 3.2 (Pseudo-Scherk graph). *A graph $\Sigma = \{(x, u(x)), x \in D\}$, where D is a domain bounded by a geodesic polygon $(C_1, A_1, B_1, A_2, B_2, \dots,$ $A_l, B_l, C_2)$, is said to be a pseudo-Scherk graph if u assumes values $+\infty$ on the A_i, $-\infty$ on the B_j, and 0 on the C_k.*

Example 3.1. Given a pseudo-Scherk polygon $\Gamma = \partial D$, Theorem 2.10 states there exists a unique solution in D such that the graph of u is a pseudo-Scherk graph.

Let ψ be a parabolic isometry of $\mathbb{H}$ whose fixed point we denote by $a_0 \in \partial_\infty \mathbb{H}$; ψ leaves invariant the horocycles passing through a_0. Let p be any point of $\mathbb{H}$; then the complete geodesics $(a_0 p)$ and $(a_0 \psi(p))$ bound a fundamental domain of $\mathbb{H}$ under the action of ψ, which we denote by Δ. Let us call a_1 the second intersection point of $(a_0 p)$ and $\partial_\infty \mathbb{H}$, and a_3 the second intersection point of the segment bisector of $[p\psi(p)]$ and $\partial_\infty \mathbb{H}$ (cf Figure 3). Then Δ splits into two parts Δ^+ and Δ^- bounded by the complete geodesics $(a_0 p)$ and $(a_0 a_3)$ (resp. $(a_0 a_3)$ and $(a_0 \psi(p))$).

Let us now state a lemma which extends the usual triangle inequality to triangles with eventually some vertices at infinity:

Lemma 3.1 (Triangle Inequality at Infinity)**.** For any triangle with vertices p, q and r (ideal or not), and small enough pairwise disjoint horocycles

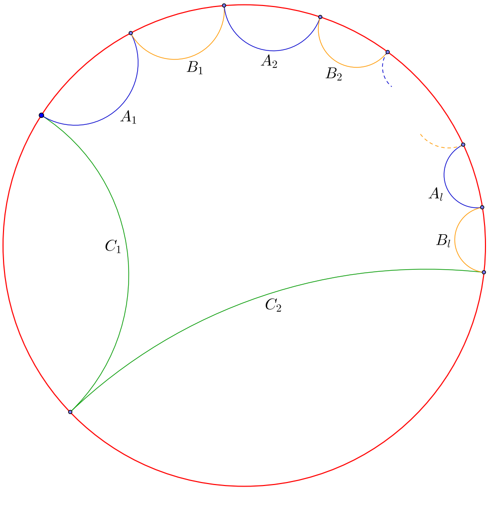


Figure 2: Pseudo-Scherk polygon

placed at the vertices at infinity, $|pq| \leq |pr| + |rq|$. If p and q are in $\partial_\infty \mathbb{H}$, and $r \in \mathbb{H}$, the inequality is true independently of disjoint horocycles placed at p and q.

Proof. If $r \in \partial_\infty \mathbb{H}$, then the inequality is true for small enough horocycles placed at r. If r is in $\mathbb{H}$, and if p (or q) is at infinity, then the geodesic (rp) (or (rq)) is asymptotic to (pq) at p (or q). Then $\forall \alpha > 0$, there exist horocycles small enough so that $|pq| < |pr| + |rq| + \alpha$. However the quantity $|pr| + |rq| - |pq|$ does not depend on the horocycles placed at p and q if any. Then, for $\alpha \to 0^+$, we get the triangle inequality at infinity: $0 \leq |pr| + |rq| - |pq|$.

In the particular case where $r \in \mathbb{H}$, $p, q \in \partial_\infty \mathbb{H}$, denote by H_p and H_q the horocycles placed at p and q respectively. If r is in the convex side of one of these horocycles, H_p say, so that $|pr| = 0$, change H_q until touching H_p (on the geodesic (pq)). The triangle inequality becomes $0 \leq |rq|$, which is true. If r is outside the horocycles, change one of the horocycles, say H_p, until $r \in H_p$, and use the previous computation. ∎

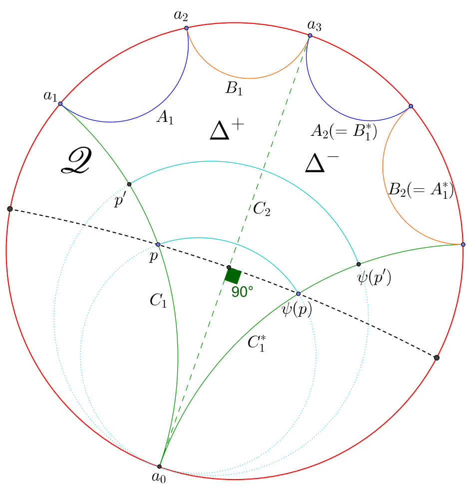


Figure 3: The fundamental domain Δ and the quadrilateral $\mathscr{Q}$

We then have the following Lemma:

Lemma 3.2. If $a_2 \in (a_1 a_3)$, let us define $C_1 = (a_0 a_1)$, $A_1 = (a_1 a_2)$, $B_1 = (a_2 a_3)$, $C_2 = (a_3 a_0)$. Then for any such a_2, the quadrilateral $\mathscr{Q} = (C_1, A_1,$

$B_1, C_2)$ is a pseudo-Scherk polygon.

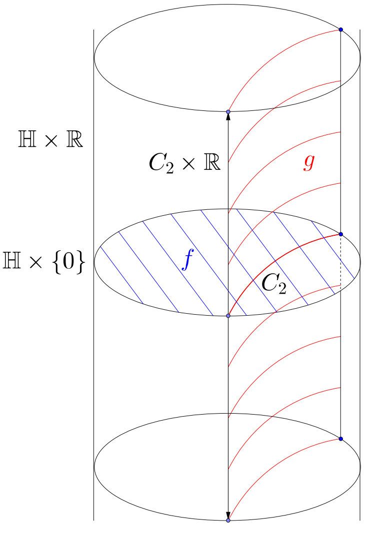
Proof. Since the polygon $\mathscr{Q}$ is a quadrilateral, there are few inscribed polygons for which we have to check they are admissible.

$\star$ The Triangle Inequality at Infinity (Lemma 3.1) yields that the three inscribed triangles $\mathscr{P}_1 = (a_0a_1a_2)$, $\mathscr{P}_2 = (a_1a_2a_3)$ and $\mathscr{P}_3 = (a_2a_3a_0)$ are admissible. Let us check it for the first one; from Lemma 3.1, we have the following inequality: $|a_1a_2| < |a_1a_0| + |a_0a_2|$. Adding the quantity $|a_1a_2|$ to each side, we get that $2a(\mathscr{P}_1) < |\mathscr{P}_1|$. The other inequality is clear.

$\star$ Let us now check that the inequalities hold for the whole quadrilateral $\mathscr{Q}$. The Triangle Inequality at Infinity yields: $|a_1a_2| < |a_1a_0| + |a_0a_2|$. But we also have: $|a_0a_2| < |a_0a_3| + |a_3a_2|$ so that we get: $|a_1a_2| < |a_1a_0| + |a_0a_3| + |a_3a_2|$. We then add $|a_1a_2|$ to both sides to obtain: $2a(\mathscr{Q}) < |\mathscr{Q}|$. In the same way, we can check that $2b(\mathscr{Q}) < |\mathscr{Q}|$. ∎

Since $\mathscr{Q}$ is a pseudo-Scherk polygon, we know there exists a unique u defined on the domain D bounded by $\mathscr{Q}$ whose graph Σ is a pseudo-Scherk graph.

Let us now recall that the rotation by π about some geodesic $\gamma \times \{0\} \in \mathbb{H} \times \{0\}$ is the composition of the reflection g with respect to $\gamma \times \mathbb{R}$ (for a point in $\mathbb{H} \times \{t\}$, it consists in the reflection with respect to $\gamma \times \{t\}$; for instance, in the disk model of $\mathbb{H}$, the latter is an inversion with respect to the circle corresponding to $\gamma \times \{t\}$) and of the reflection $f : (x,t) \mapsto (x, -t)$.



By applying a rotation by π to Σ about the geodesic C_2, we obtain from Schwarz Reflection Principle a minimal graph, which is a pseudo-Scherk Graph Σ' on the whole domain bounded by the polygon $(C_1, A_1, B_1, A_2, B_2, C_1^*)$, where we denote by (C_1^*) the image of (C_1) by the rotation. Moreover, by continuing the rotations about the new geodesic boundaries passing through a_0 that arise, we get a minimal graph which is invariant under the action of ψ, that we can also see as a minimal graph over the quotient space $\mathbb{H}/ < \psi >$.

Remarks: This construction generalizes to all pseudo-Scherk polygons inscribed in the domain Δ^+, whose sides C_1 and C_2 correspond respectively to $(a_0 p)$ and $(a_0 a_3)$. We thus get a minimal graph over polygonal domains of the quotient space $\mathbb{H}/ < \psi >$; let us call such a graph a *quotient Scherk graph*. For example, we will construct in the sequel an exhaustion of Δ^+ by pseudo-Scherk polygons, to which correspond pseudo-Scherk graphs; applying to those graphs the same construction as what we did for the quadrilateral $\mathscr{Q}$, we then get a sequence of quotient Scherk graphs, whose corresponding pseudo-Scherk polygons define an exhaustion of $\mathbb{H}/<\psi>$.

Moreover, by the Triangle Inequality, we see that it suffices to check the conditions on the lengths of edges for inscribed polygons which do not have a_0 as a vertex. In fact, the C_k do not intervene at all, which seems to be rather logical, since those edges vanish when we go to the quotient. More precisely, given a pseudo-Scherk polygon $(C_1, A_1, B_1, \ldots, A_l, B_l, C_2) \in \Delta^+$, let us denote by $A_1^*, B_1^*, \ldots, A_l^*, B_l^*$ the images of the previous edges by a rotation by π about C_2. We define the polygon $\overline{\Gamma} := (\overline{A_1}, \ldots, \overline{B_l}, \overline{B_l^*}, \ldots, \overline{A_1^*})$ in $\mathbb{H}/<\psi>$. Then we can check the lengths conditions of Theorem 2.10 directly in the quotient space, for polygons inscribed in $\overline{\Gamma}$: indeed, since ψ is an isometry, $H/<\psi>$ is naturally endowed with the quotient metric.

3.2 Conformal type

The quotient Scherk graphs we obtain by this construction are topologically annuli; indeed, since the various u are continuous, these graphs are naturally homeomorphic to the polygonal domains of $\mathbb{H}/<\psi>$ on which they are defined, which are annuli. Then from the classification of annuli, we know that these graphs are conformally equivalent either to $\mathbb{D}^*$, $\mathbb{C}^*$ or $A_r = \{z \in \mathbb{C} \mid r < |z| < 1\}$ for some $r \in]0,1[$.

The following proposition gives the answer:

Proposition 3.1. *The conformal type of a quotient Scherk graph is $\mathbb{C}^*$.*

Proof. Let $\Gamma = (C_1 = (a_0, a_1), A_1, B_1, \ldots, A_l, B_l, C_2 = (a_{2l-1}, a_{2l} = a_0))$ be a pseudo-Scherk polygon inscribed in Δ^+, to which we associate as previously a minimal function u defined on the convex hull D of the quotient $\overline{\Gamma}$ of Γ, and its quotient Scherk graph Σ. In the quotient space, the C_k disappear, and we denote by $\overline{A_i}$ and $\overline{B_j}$ the edges of $\overline{\Gamma}$.

As a minimal graph, Σ is stable, and since it is complete, it has uniformly bounded curvature by Schoen's curvature estimates (cf. [CM] for example).

Then there is exists $\delta > 0$ such that for each $x \in \Sigma$, Σ is a graph in a neighborhood of x over $D_\delta(x) \subset T_x\Sigma$, where $D_\delta(x)$ is the disk of radius δ in the tangent space $T_x\Sigma$ of Σ at x, centered at the origin of $T_x\Sigma$. Moreover, this local graph has bounded geometry, independently of $x \in \Sigma$. Let $G_\delta(x)$ denote this local graph.

For $p \in D$, we denote by $\Sigma_\delta(p)$ the local graph $G_\delta(x)$ translated vertically so that $x = (p, u(p))$ goes to height zero (such a translation is an isometry, so $\Sigma_\delta(p)$ is still a minimal graph).

Let γ be one of the $\overline{A_i}$ or $\overline{B_j}$, and let $q \in \gamma$, $p_n \in D$ such that $\lim\limits_{p_n \to \infty} = q$. We claim the local surfaces $\Sigma_\delta(p_n)$ converge uniformly to $\gamma_\delta(q) \times [-\delta, \delta]$, where $\gamma_\delta(q)$ is the interval of length 2δ centered at q.

First, observe that the tangent planes to $\Sigma_\delta(p_n)$ at p_n must converge to the vertical plane Q tangent to $\gamma \times \mathbb{R}$ at q. *Suppose it were not the case*; then by compactness we can extract a subsequence $(q_n)_n$ from $(p_n)_n$ such that the $T_{q_n}\Sigma_\delta(q_n)$ converge to a plane $P \ni q$ distinct from Q. Since the graphs $\Sigma_\delta(q_n)$ have uniformly bounded geometry, there exists a rank n_0 such that for $n \geq n_0$, $\Sigma_\delta(q_n)$ is a graph over the disk of radius $\delta/2$ in P, centered at q, denoted by $P_{\delta/2}(q)$. By the Compactness Theorem, a subsequence of these graphs converges uniformly to a minimal graph F over $P_{\delta/2}(q)$. Since $P \neq Q$ by hypothesis, there are points of F whose projection to $\mathbb{H}/ < \psi >$ lie outside of $\overline{D}$. But the $\Sigma_\delta(q_n)$ converge uniformly to F and so for n large enough, $\Sigma_\delta(q_n)$ would not be a vertical graph over a domain in D, *which is a contradiction.*

Thus the tangent planes at p_n to $\Sigma_\delta(p_n)$ converge to Q. By the fact that the $\Sigma_\delta(p_n)$ have uniformly bounded geometry, the Compactness Theorem enables us to extract a subsequence of these graphs which converges to a minimal graph G over Q. *If G were different from* $\gamma_\delta(q) \times [-\delta, \delta]$, then since both are minimal surfaces, G would not lie on one side of $\gamma \times \mathbb{R}$ from the Maximum Principle, so G would have points near q whose projection to $\mathbb{H}/ < \psi >$ is outside of $\overline{D}$. Then the $\Sigma_\delta(p_n)$ would also have points near q whose projection is outside of $\overline{D}$, *which is impossible.*

Now let $\ell > 0$ and suppose $\gamma(\ell)$ is a segment of γ of length ℓ. By compacity of $\gamma(\ell)$, it is possible to find a finite set of points $\{q^k \in \gamma(\ell)\}_{k=1,\ldots,m}$

such that the disks $D_{\delta/2}(q^k)$ of $\mathbb{H}/<\psi>$ of radius $\delta/2$ form a finite cover of $\gamma(\ell)$. From our previous discussion, we know that for sequences $(q_n^k)_n$ which converge to q^k, the $\Sigma_\delta(q_n^k)$ converge uniformly to $\gamma_\delta(q^k)\times[-\delta,\delta]$. For n large, the graph $\Sigma_n = \bigcup_{k=1,\dots,m} G_\delta(q_n^k)$ is as close as we want for the uniform norm of $\gamma'\times[-\delta,\delta]$, where γ' is a neighborhood of $\gamma(\ell)$ in γ. Therefore, for any $\varepsilon>0$, there exists a height $h=h(\ell,\varepsilon)$ and a tubular neighborhood T of $\gamma(\ell)$ in $\overline{D}$ such that the graph of u over T is ε-close in the $\mathscr{C}^2$-topology to $\gamma(\ell)\times[h,+\infty[$ when γ is an $\overline{A_i}$, and to $\gamma(\ell)\times]-\infty,h]$ when γ is a $\overline{B_j}$.

We denote by $\Sigma(\gamma(\ell))$ this part of Σ, above (or below) height h, that is ε-close to $\gamma(\ell)\times[h,+\infty[$ (or to $\gamma(\ell)\times]-\infty,h]$). As one goes higher (or lower), the $\Sigma(\gamma(\ell))$ converge to $\gamma(\ell)\times\mathbb{R}$. In particular, the horizontal projection of $\Sigma(\gamma(\ell))$ to $\gamma(\ell)\times\mathbb{R}$ is a *quasi-isometry.* (Recall that $\pi:\mathbb{H}\to\mathbb{H}$ is a K-quasi-isometry for some $K>0$ if

$$\forall(z,w),\ \frac{d(z,w)}{K}\le d(\pi(z),\pi(w))\le Kd(z,w);$$

then, the modulus m' of the image of an annulus of modulus m by π satisfies the following inequality (cf. [FM] for instance): $\frac{1}{K}m\le m'\le Km$.)

Now consider a vertex of $\overline{\Gamma}$ and let $\overline{A_i}$ and $\overline{B_i}$ be the edges of $\overline{\Gamma}$ at this vertex. Let H_i be a horocycle at the vertex and $F_i\subset D$ the inside of H_i. By choosing H_i small, we can guarantee that each point of F_i is as close to $\overline{A_i}\cup\overline{B_i}$ as we want. Then for any $\varepsilon>0$, there exists a H_i such that the part of Σ over F_i, denoted $\Sigma(F_i)$, is ε-close to $\overline{A_i}\times\mathbb{R}$ and $\overline{B_i}\times\mathbb{R}$. We choose ε small so that the horizontal projection of $\Sigma(F_i)$ to $\overline{A_i}\times\mathbb{R}$ is a quasi-isometry onto its image.

Let then be $H_1,\dots,H_{2l}$ be small horocycles at each of the vertices of Γ so that $\Sigma(F_i)$ is quasi-isometric to $\overline{A_i}\times\mathbb{R}$.

Let $\tilde{\overline{A_i}}$ and $\tilde{\overline{B_i}}$ denote the compact arcs on each $\overline{A_i}$ and $\overline{B_i}$ outside of each F_i. For $|h|$ large enough, and T a small tubular neighborhood of $\bigcup_{j=1,\dots,l}(\tilde{\overline{A_i}}\cup\tilde{\overline{B_j}})$, each component of the part of Σ over T projects horizontally to $\overline{\tilde{A}_i} \times [h, +\infty[$ or to $\overline{\tilde{B}_j} \times]-\infty, h]$ quasi-isometrically.

Now let K_0 be the part of Σ over $D - \left(T \cup \left(\bigcup_{j=1,\dots,2l} F_i\right)\right)$; K_0 is conformally a punctured disk. Then, choose h_1 large so that $\Sigma \cap (\mathbb{H} \times [-h_1, h_1])$ contains an annulus $K_1 - K_0$, such that $K_1 - K_0$ is compact and has conformal modulus at least one (to do this, we choose an annulus of $\overline{\Gamma} \times \mathbb{R}$ with a modulus sufficiently large, so that its preimage in Σ by the quasi-isometry has a suitable modulus). In the same way, choose $h_2 > h_1$ such that $\Sigma \cap (\mathbb{H} \times [-h_2, h_2])$ contains an annulus $K_2 - K_1$ of conformal modulus at least one. Since the part of Σ outside these K_j converges to quotient geodesic boundary $\overline{\Gamma} \times \mathbb{R}$, such a K_j exists for all j.

To conclude, we write $\Sigma = K_0 \cup \left(\bigcup_{j=1}^{\infty} (K_j - K_{j-1})\right)$, where $K_j - K_{j-1}$ are disjoint annuli of conformal modulus at least one. Then applying Grötzsch Lemma (cf. [V]), we conclude that the conformal modulus of the annulus $\bigcup_{j=1}^{\infty} (K_j - K_{j-1})$ is $+\infty$. Thus Σ, which is topologically an annulus, is conformally $\mathbb{C}^*$. ∎

4 Extension of quotient Scherk graphs

In this part, we explain how to get an exhaustion of the domain Δ^+ by pseudo-Scherk polygons. To do this, we proceed as follows: given a pseudo-Scherk polygon in Δ^+, let $a_m,\ a_{m+1},\ a_{m+2}$ be three vertices such that $[a_m, a_{m+1}]$ belongs to the A_i and $[a_{m+1}, a_{m+2}]$ to the B_j. Then we attach two ideal quadrilaterals E and E' to these edges; we show that we can choose them such that the resulting polygon is still a pseudo-Scherk polygon. We then apply such a construction to all the $A_i's$ and $B_j's$ edges of the initial polygon. We show that if the attached quadrilaterals are chosen regular enough, then given any fixed point O in the initial domain, the extended polygon is a fixed constant farther from O; this shows that the sequence of extended polygons that we get by repeating this process actually exhaust $\mathbb{H}/ < \psi >$. From this sequence of pseudo-Scherk polygons we can get a sequence of quotient Scherk graphs by applying the same construction as what was described previously: we apply a reflection with respect to the geodesic C_2 to each pseudo-Scherk polygon, and we solve some Dirichlet problem with infinite values on the A_i and the B_j.

4.1 Extension of the polygonal domain by quadrilaterals

Let us first fix some notations. Let $(a_0, a_1, a_2, \dots, a_n)$ be $(n+1)$ distinct points of $\mathbb{S}^1$, ordered clockwise, and denote by $P(a_0, a_1, a_2, \dots, a_n)$ the convex hull of the $(n+1)$ points in $\mathbb{H}$ (for the hyperbolic metric).

Let D be a polygonal domain in Δ^+, where $D = P(a_0, \dots, a_{2l-1})$. Let $D_0 = P(b_1, b_2, a_1, b_3,$
$b_4, a_2 \dots, a_{2l} = a_0)$ be the polygonal domain D to which we attach two regular polygons $E = P(a_1, b_1, b_2, a_2)$ and $E' = P(a_2, b_3, b_4, a_3)$; E is attached to the side $[a_1, a_2]$ of D and E' to the side $[a_2, a_3]$, cf. Figure 4.

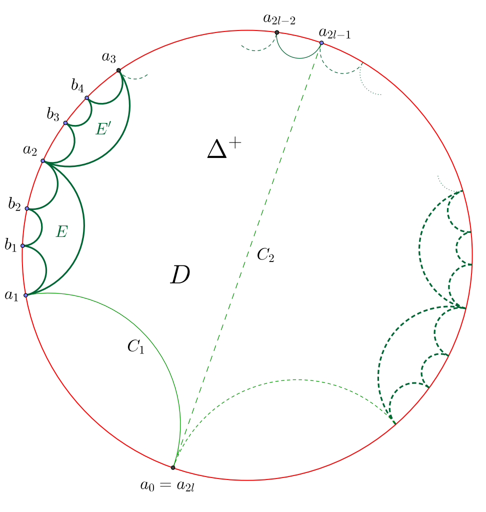


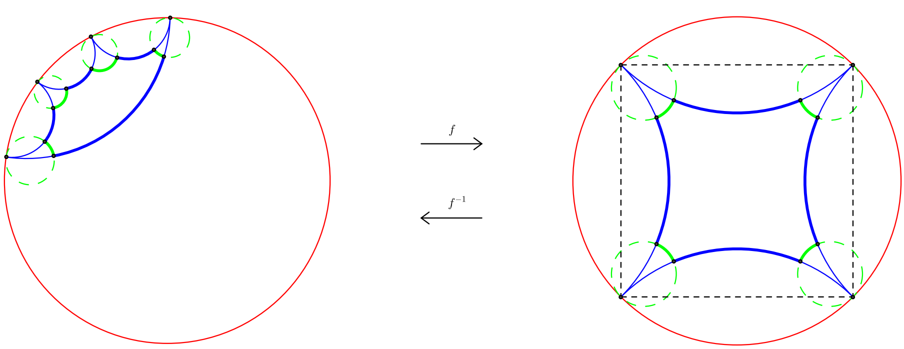
Figure 4: Extension of a pseudo-Scherk polygon in Δ^+

Let $\mathscr{P} = \partial P(a_0, a_1, \dots, a_n)$ be an ideal polygon. As in the previous section, place a horocycle H_i at each vertex d_i, and let $|\mathscr{P}| = \sum_{i=0}^{i=n} |a_i a_{i+1}|$ denote the truncated perimeter, where $a_0 = a_{n+1}$ and $|a_i a_j|$ is the distance between the horocycles H_i and H_j. $|\mathscr{P}|$ represents the total length of arcs of $\mathscr{P}$ exterior to all the horocycles. The quantity $|\mathscr{P}|$, as the distances $|a_i a_j|$, extends naturally to geodesic polygons with vertices in $\mathbb{H}$: place horocycles only at vertices which are at infinity. The same extension can be done for the quantities $a(\mathscr{P})$ and $b(\mathscr{P})$ if the polygon $\mathscr{P}$ comes from a Dirichlet problem.

Definition 4.1 (Regular quadrilateral)**.** *We say $\partial P(a_0, a_1, a_2, a_3)$ is a regular quadrilateral when the cross ratio $\frac{(a_0-a_2)(a_1-a_3)}{(a_1-a_2)(a_0-a_3)}$ equals* 2*.*

Consider a (euclidean) square $\partial P(b_0, b_1, b_2, b_3)$ inscribed in the circle $\partial_\infty \mathbb{H}$. A little computation shows that the cross-ratio $\frac{(b_0-b_2)(b_1-b_3)}{(b_1-b_2)(b_0-b_3)}$ equals 2, so there exists a Moebius transformation f mapping a_i to b_i for $i = 0, \dots, 3$. By choosing for the square horocycles corresponding to euclidean circles of the same radius, we can ensure that at each step, the truncated length of its sides are equal. Since a Moebius transformation is an isometry, we also get horocycles for the regular quadrilateral $\partial P(a_0, a_1, a_2, a_3)$ such that, step-by-step, the following lengths are equal:

$$|a_0, a_1| = |a_1, a_2| = |a_2, a_3| = |a_3, a_0|.$$



Moreover, to be sure that we will be able to construct an exhaustion of Δ^+, we have to choose carefully the regular quadrilaterals we attach to the sides (before we slightly perturb them). Let O be a fixed point in the first polygonal domain (the domain bounded by the quadrilateral $\mathscr{Q}$ with the previous notations). Then we require the regular quadrilaterals E and E' to be symmetric with respect to the geodesic orthogonal to γ and passing through the point O (cf. Figure 5).

The problem with the special regular quadrilaterals described above is that the extension of a pseudo-Scherk polygon by such quadrilaterals is not a pseudo-Scherk polygon any longer. However, we will see that almost all inscribed polygons inscribed in the extended domain D_0 are admissible. Then we can get a genuine extended pseudo-Scherk polygon by a small variation of $(b_i)_{i=2,3}$.

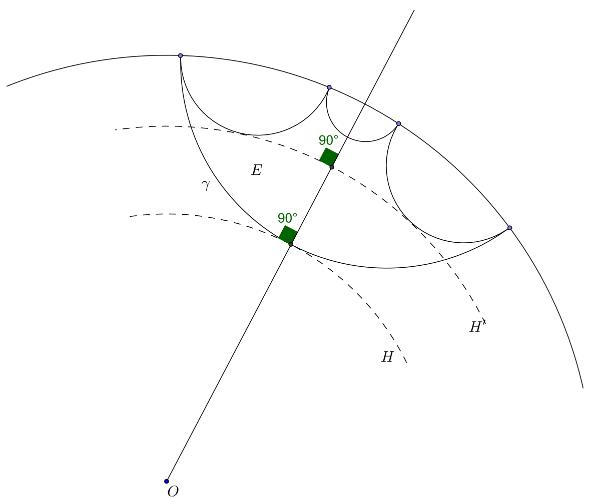


Figure 5: Non-perturbed quadrilateral

Recall that from the above discussion, the regularity of E and E' permits the choice, step by step, of the horocycles such that:

$$|a_1b_1| = |b_1b_2| = |b_2a_2| = |a_2a_1|,$$
$$|a_2b_3| = |b_3b_4| = |b_4a_3| = |a_3a_2|.$$

We see then that the boundaries of E and E' do not satisfy the lengths' conditions of Theorem 2.10. Fortunately, the next result shows that except these two quadrilaterals, each polygon inscribed in the extended domain is admissible:

Lemma 4.1. All the inscribed polygons of D_0 are admissible except the boundaries of E and E'.

Proof. Let $\Gamma_0 = \partial D_0$ be the polygonal boundary of D_0. We only prove the required inequalities for values $+\infty$ on the boundary. Then by symmetry of the problem, we will get the inequalities for the $-\infty$ data on the boundary.

We assume now that $\mathscr{P} = \partial P$ is an inscribed polygon of D_0, $P = \partial P(d_1, \dots, d_n)$ where the $(d_i)_{i=1,\dots,n}$ are vertices of D_0, and moreover that $P \neq D_0$, $P \neq E$, $P \neq E'$, $P \neq D_0 \setminus E$, $P \neq D_0 \setminus E'$ (this is easy to check that the two last polygons are admissible). By a previous remark, it is enough to prove that we have $|\mathscr{P}| - 2a(\mathscr{P}) > 0$ for an a priori choice of disjoint horocycles H_i at d_i ($i = 1, \dots, n$) when the sides A_i where $v = +\infty$ alternate on $\mathscr{P}$, we assume this hypothesis from now on.

Consider $\mathscr{P}' = \partial P$, $P' = P \setminus E'$.

Claim: *if* $|\mathscr{P}'| - 2a(\mathscr{P}') > 0$ *then* $|\mathscr{P}| - 2a(\mathscr{P}) > 0$.

Proof of the claim: Consider $\mathscr{P}'$; if $\mathscr{P}' = \mathscr{P}$ there is nothing to prove, otherwise the geodesic $[b_3, b_4]$ where $v = +\infty$ is in $\mathscr{P}$. For convenience, change the notation so that $\mathscr{P} = \partial P(d_1, b_3, b_4, d_2, \dots, d_n)$. Let $q_1 = [d_1, b_3] \cap [a_2, a_3]$ and $q_2 = [d_2, b_4] \cap [a_2, a_3]$, cf. Figure 6. Notice that if $a_2 \in \mathscr{P}$ (resp. $a_3 \in \mathscr{P}$) then $q_1 = a_2$ and $|a_2q_1| = 0$ by convention (resp. $q_2 = a_3$ and $|a_3q_2| = 0$).

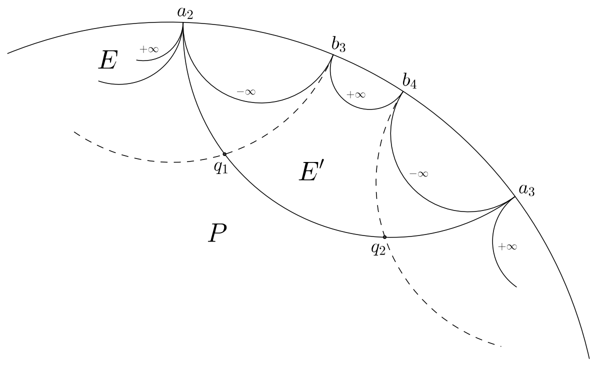


Figure 6: $\mathscr{P}$ and $\mathscr{P}'$

We have the relations:

$$a(\mathscr{P}) = a(\mathscr{P}') + a,$$
$$|\mathscr{P}| = |\mathscr{P}'| - |q_1q_2| + (|q_1b_3| + a + |q_2b_4|).$$

Now, $|\mathscr{P}'| - 2a(\mathscr{P}') > 0$ hence by substitution:

$$\begin{aligned} |\mathscr{P}| - 2a(\mathscr{P}) &> (|q_1b_3| + |q_2b_4|) - (a + |q_1q_2|) \\ &= (|q_1b_3| + |q_2b_4|) - (2a - |a_2q_1| - |a_3q_2|) \\ &= (|a_2q_1| + |q_1b_3| - a) + (|a_3q_2| + |q_2b_4| - a). \end{aligned}$$

By the Triangle Inequality at infinity (or directly if $q_1 = a_2$ or $q_2 = a_3$):

$$|a_2q_1| + |q_1b_3| - a \geq 0$$
$$|a_3q_2| + |q_2b_4| - a \geq 0.$$

Hence, $|\mathscr{P}| - 2a(\mathscr{P}) > 0$ and the claim is proved. ■

So it remains to prove $|\mathscr{P}'| - 2a(\mathscr{P}') > 0$. For that, define $\mathscr{P}'' = \partial P''$, $P'' = P' \setminus E$.

The key point is a flux inequality for $\mathscr{P}''$ coming from the initial solution u defined on D. We have, $P'' \subset D$ and there exists the divergence free field

$X = \frac{\nabla u}{W}$ ($W = (1+|\nabla u|^2)^{1/2}$) on P''. Moreover, on the arcs of $\mathscr{P}''$, $X = \nu$ if $u = +\infty$, $X = -\nu$ if $u = -\infty$, where ν is the outward normal of P''. Let us write $\mathscr{P}''$ as $I_0 \cup I_1 \cup J$: I_0 is the union of all geodesics A_i (where $u = +\infty$) contained in $\mathscr{P}''$ and disjoint from $[a_1, a_2]$, $I_1 = \mathscr{P}'' \cap [a_1, a_2]$ and J the union of the remaining arcs.

The flux of X along $\mathscr{P}'' = \partial P''$ is zero, which yields:

$$0 = a(\mathscr{P}'') + |I_1| + F_u(J) + \rho.$$

Here the flux $F_u(J)$ is taken on the compact part of J outside the horocycles and ρ a residual term corresponding to the flux of X along some parts of horocycles. Next we have the truncated perimeter of $\mathscr{P}''$:

$$|\mathscr{P}''| = a(\mathscr{P}'') + |I_1| + |J|.$$

Adding these last two equalities:

$$|\mathscr{P}''| - 2(a(\mathscr{P}'') + |I_1|) = |J| + F_u(J) + \rho.$$

Remark that the condition on the A_i sides of $\mathscr{P}$ (alternate) yields: on the one hand the quantity we have to estimate $|\mathscr{P}'| - 2a(\mathscr{P}')$ is independent of the choice of horocycles, so this allows us to change this choice as necessary; on the other hand, we have that $P'' \neq D$ and $P'' \neq \emptyset$. For, if $P'' = D$, a careful analysis of the possibilities of inscribed polygons $\mathscr{P}$ with alternate A_i sides (using a_1 and a_2 are vertices of $\mathscr{P}$, cf. Figure 8) leads to $P = D_0$ or $P = D_0 \setminus E$ which are excluded by hypothesis. Similarly, if $P'' = \emptyset$, then $P \subset E$ or $P \subset E'$ and, in this case, the only possibility of an inscribed polygon with alternate A_i sides is E which is excluded too. Therefore J contains interior arcs and as the horocycles at vertices of $\mathscr{P}''$ diverge:

$$\exists c_0 > 0 \text{ so that } |J| + F_u(J) \geq c_0.$$

We can ensure $|\rho| < c_0$ for a suitable choice of horocycles. Hence we get the following flux inequality:

$$|\mathscr{P}''| - 2(a(\mathscr{P}'') + |I_1|) > 0. \tag{1}$$

We have three cases to consider.

Case 1. Suppose $[a_1, b_1] \cup [b_2, a_2] \subset \mathscr{P}$. In this case, $E \subset P'$ and

$$\begin{aligned} a(\mathscr{P}') &= a(\mathscr{P}'') + 2a, \\ |\mathscr{P}'| &= |\mathscr{P}''| + 2a, \\ |I_1| &= a. \end{aligned}$$

The flux inequality (1) directly gives:

$$0 < (|\mathscr{P}'| - 2a) - 2(a(\mathscr{P}') - a) = |\mathscr{P}'| - 2a(\mathscr{P}').$$

Case 2. Suppose only one of the $[a_1, b_1]$ or $[b_2, a_2]$ is contained in $\mathscr{P}$; $[a_1, b_1]$ say. If we denote $I_1 = [a_1, q]$ (cf. Figure 7 below), we have:

$$\begin{aligned} a(\mathscr{P}') &= a(\mathscr{P}'') + a, \\ |\mathscr{P}'| &= |\mathscr{P}''| - |I_1| + a + |b_1 q|. \end{aligned}$$

The flux inequality (1) yields:

$$\begin{aligned} 0 &< (|\mathscr{P}'| + |I_1| - a - |b_1 q|) - 2(a(\mathscr{P}') - a + |I_1|), \\ 0 &< |\mathscr{P}'| - 2a(\mathscr{P}') - |b_1 q| - |I_1| + a. \end{aligned}$$

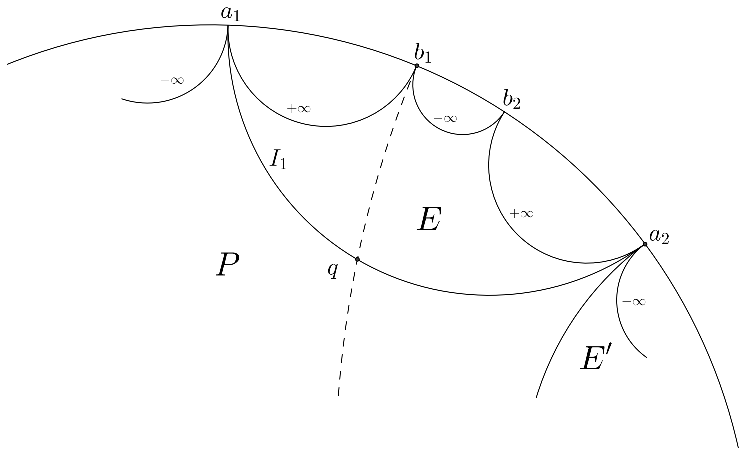


Figure 7: Case 2

Hence, using the Triangle Inequality at infinity, we obtain:

$$|\mathscr{P}'| - 2a(\mathscr{P}') > |b_1q| + |qa_1| - a \geq 0.$$

Case 3. The remaining case is for $P' \subset D$. Then the flux inequality (1) gives directly the result for $\mathscr{P}' = \partial P'$. ■

From the previous lemma, we see that the only obstructions to the existence of an ideal Scherk graph on D_0 come from the polygons E, E', for which we have equalities instead of strict inequalities. In the next lemma, we will ensure by a small perturbation of D_0 that $2a(\mathscr{P}) < |\mathscr{P}|$ and $2b(\mathscr{P}) < |\mathscr{P}|$ for any inscribed polygon $\mathscr{P}$.

Lemma 4.2. There exists $\tau_0 > 0$ such that for all $\tau \in (0, \tau_0]$, there exists v_τ, an ideal Scherk graph on $D_\tau = P(a_1, b_1, b_2(\tau), a_2, b_3(\tau), b_4, a_3, \ldots, a_{2l} = a_0)$ with: $|b_i(\tau) - b_i| \leq \tau$ for $i = 2, 3$.

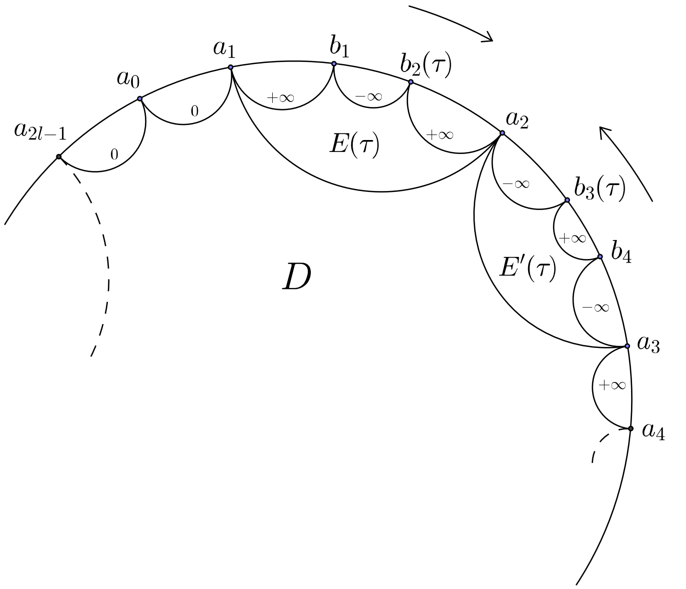


Figure 8: Perturbation of the quadrilaterals $E(\tau)$ and $E'(\tau)$

Proof. Let $\tau > 0$ and consider moving b_2 and b_3 in the direction of a_2, such that $|b_i(\tau) - b_i| \le \tau$ $(i = 2, 3)$, cf. Figure 8. By such variations, the quantities $(|a_1a_2| - |a_2b_2(\tau)| + |b_2(\tau)b_1| - |b_1a_1|)$ and $(|a_3a_2| - |a_2b_3(\tau)| + |b_3(\tau)b_4| - |b_4a_3|)$, which are independent of the choice of horocycles, increase. They are zero for the initial polygons E et E'. Then there exists such variations $b_2(\tau)$ and $b_3(\tau)$ so that, for $\tau > 0$:

$$\varphi(\tau) = |a_1a_2| - |a_2b_2(\tau)| + |b_2(\tau)b_1| - |b_1a_1| > 0,$$
$$\varphi(\tau) = |a_3a_2| - |a_2b_3(\tau)| + |b_3(\tau)b_4| - |b_4a_3| > 0.$$

For this choice of variation, the $\varphi(\tau)$-perturbed polygons $E_\tau = P(b_1, b_2(\tau), a_2, a_1)$ and $E'_\tau = P(a_2, b_3(\tau), b_4, a_3)$ satisfy

$$|\partial E_\tau| - 2a(\partial E_\tau) = |\partial E'_\tau| - 2b(\partial E'_\tau) = \varphi(\tau) > 0.$$

In order to prove the required inequalities for all other inscribed polygons of D_τ, we use the Lemma 4.1. For the inscribed polygons $\mathscr{P}$ of D_0

(except ∂D_0 and those excluded by Lemma 4.1), the inequalities are strict and so, are stable by small enough perturbations of vertices (and attached horocycles). There are a finite number of such admissible polygons; thus there exists $\tau_0 > 0$ such that for all $0 < \tau \leq \tau_0$, and variations $b_2(\tau)$ and $b_3(\tau)$ as above, the conditions of Theorem 2.10 are satisfied. This ensures the existence of v_τ on D_τ. ∎

To conclude this section, we explain how to construct an exhaustion of the domain Δ^+ by pseudo-Scherk polygons. From the initial quadrilateral $\mathscr{Q}$ we can obtain a new pseudo-Scherk polygon by attaching perturbed quadrilaterals $E(\tau)$ and $E'(\tau)$ (τ small) respectively to the sides $[a_1, a_2]$ and $[a_2, a_3]$. Repeat then this construction to the new polygon: to each pair (A_i, B_i), we attach perturbed quadrilaterals (cf Figure 9). A each step, we slightly perturb the quadrilaterals so that we still get a pseudo-Scherk polygon by this extension.

Owing to the special geometry of the regular polygons which we then perturb, at each step the boundary of the new pseudo-Scherk polygon is a fixed constant farther from the fixed point O, so that we actually get a sequence of pseudo-Scherk polygons which exhaust Δ^+.

We then apply a reflection to these pseudo-Scherk polygons with respect to the geodesic C_2, in order to obtain a sequence of pseudo-Scherk polygons which exhaust the fundamental domain Δ.

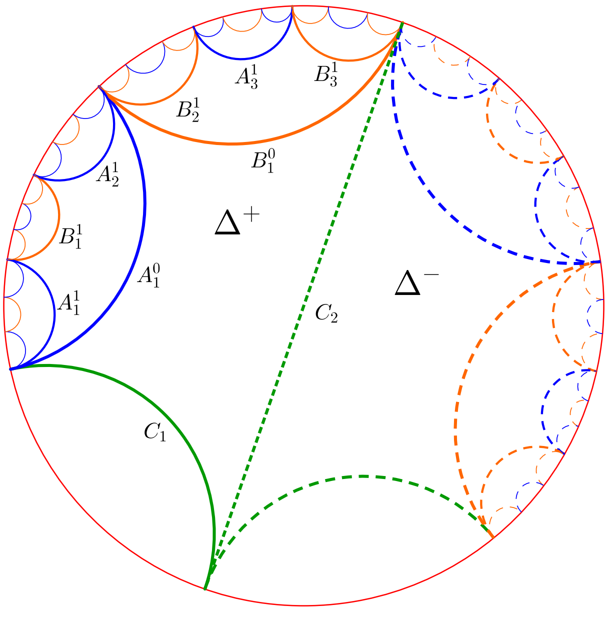


Figure 9: Exhaustion of Δ^+ by pseudo-Scherk polygons

4.2 Conformal issues

In this section, we will measure the differences between the functions corresponding to two successive quotient Scherk graphs on the intersection of their domains of definition. Indeed, to ensure that it is possible to take a subsequence of the previous sequence of quotient Scherk graphs which converges to a quotient Scherk graphs with the good conformal type, namely $\mathbb{C}^*$, we have to check that the conformal properties of two successive graphs vary little. Here, instead of working on a fundamental domain as previously, we work directly in the product $\mathbb{H}/ < \psi > \times \mathbb{R}$. Let us begin with a lemma, which extends the first result of the Flux Theorem to minimal graphs defined over the quotient space:

Lemma 4.3. [Flux in the quotient space] Let Σ be the quotient Scherk graph of a function u defined over a domain D in $\mathbb{H}/ < \psi >$, and $\mathscr{W}$ a domain in D. Let us denote the flux by $F(\partial \mathscr{W}) = \int_{\partial \mathscr{W}} \langle X, \nu \rangle ds$, where

$X = \frac{\nabla u}{W}$, and $W^2 = 1 + |\nabla u|^2$.

Then if $\partial \mathscr{W}$ is a compact cycle, then $F(\partial \mathscr{W}) = 0$.

Proof. If a lift of the cycle is itself a cycle, then, as in the usual case, it comes from Stokes' theorem, for $\operatorname{div}(X) = 0$ since u is a minimal function.

Else, the cycle is a generator of the fundamental group of Σ. Since the flux is homotopically invariant, we can homotop the cycle to a cycle closer to the cusp, so that its length becomes as small as we want. But $|X| \leq 1$, so the flux across the cycle is less than its length, and $F(\partial \mathscr{W}) = 0$ here too. ∎

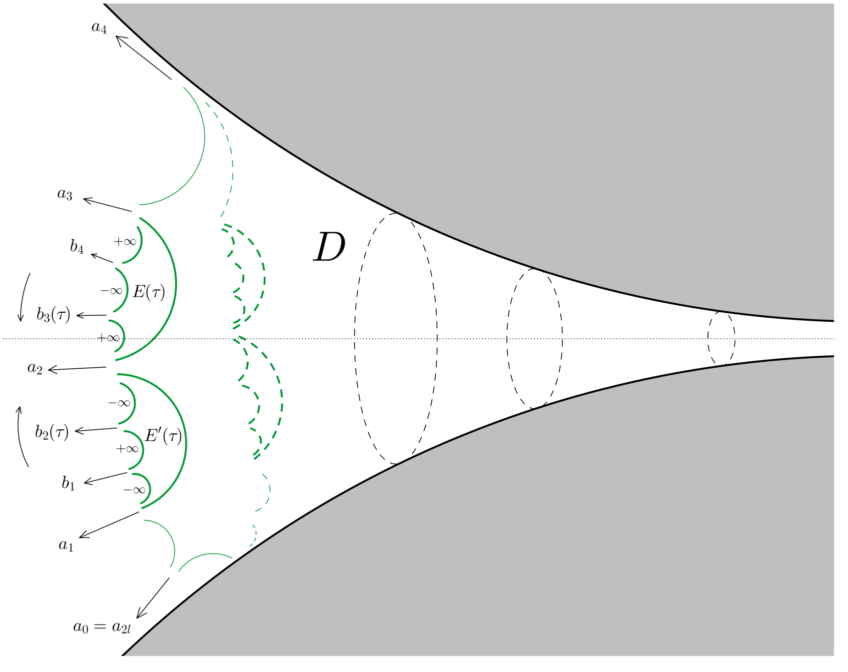


Figure 10: The initial polygonal domain D and its extension by $E(\tau)$ and $E'(\tau)$

The next result we are interested in shows that for small enough perturbations of the regular quadrilerals we attach to the boundaries of a polygonal domain D, the quotient Scherk graph defined over the extended domain D_0 can be chosen as close to the ancient one as we want over their common domain D.

Proposition 4.1. *Let u be a quotient Scherk graph defined on a polygonal domain $D = P(a_1, \dots, a_{2l-1}, a_{2l} = a_0)$ and K a compact of D. Let $D_0 = P(b_1, b_2, a_1, b_3, b_4, a_2, \dots, a_{2l})$ be the polygonal domain to which we attach two regular polygons*

$$E = P(b_1, b_2, a_1, a_0) \text{ and } E' = P(a_1, b_3, b_4, a_2);$$

E is attached to the side (a_0, a_1) and E' to the side (a_1, a_2); cf. Figure 10.

*Then for all $\varepsilon > 0$, there exist $(b_i')_{i=2,3}$ and v a quotient Scherk graph on $P(b_1, b_2', a_1, b_3'$,
$b_4, a_2, \dots, a_{2l})$ such that:*

$$|b_i' - b_i| \leq \varepsilon \text{ and } \|v - u\|_{\mathscr{C}^2(K)} \leq \varepsilon.$$

Proof. Recall that D denotes the original domain, on which both u and v_τ are defined. The main step of the proof consists in establishing that $\lim_{\tau\to 0} \nabla v_\tau|_D = \nabla u$. For that, consider $X_\tau = \dfrac{\nabla v_\tau}{W_\tau}$ and $X = \dfrac{\nabla u}{W}$ (with $W_\tau = (1 + |\nabla v_\tau|^2)^{1/2}$, $W = (1 + |\nabla u|^2)^{1/2}$). We know that $\operatorname{div}(X_\tau) = 0$ and $\operatorname{div}(X) = 0$ since u is a minimal function. To get $\lim_{\tau\to 0} \nabla v_\tau|_D = \nabla u$, we will prove that $\lim_{\tau\to 0} X_\tau|_D = X$.

In order to study $X - X_\tau$ on the interior of D, we consider the flux of $X - X_\tau$ along a level curve of $u - v_\tau$ through an interior point p. *Suppose that this level curve were compact*; the associated points in the graphs of u and v_τ correspond to two parallel curves c and c_τ respectively. Then by a vertical translation of height $(u - v_\tau)(p)$ of the graph of v_τ (this translation is an isometry), we get two minimal surfaces which agree on the curve c, so they are equal, *which is impossible* since u equals ∞ on (a_0, a_1) whereas v_τ is bounded on compact parts of this geodesic. Therefore this level curve is non compact, and thus, it goes to the boundary of D. We then create a cycle by adding small horocycle arcs which join the connected component of level curve containing p to ∂D (cf Figure 11). We know from Lemma 4.3 that the flux along this cycle is zero. We first show that the flux of $X - X_\tau$ along the part of ∂D is small; then, since the flux along the horocycles is small too, we get that the flux along the part of the level curve in the cycle is small. Finally, by contraposition, we get that the tangent planes of u and v_τ are close, which implies that $\| X(p) - X_\tau(p) \|$ is small.

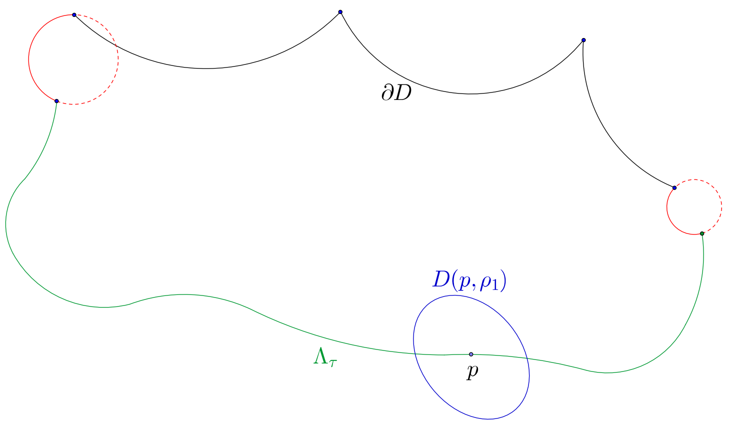


Figure 11: Cycle constructed from a level curve of $u - v_\tau$

We show in this paragraph that the flux of $X - X_\tau$ on any arc of ∂D is as small as we want. On the boundary, let ν be the outer pointing normal to ∂D. On $\partial D \backslash \{(a_0,a_1) \cup (a_1,a_2)\}$, v_τ and u take the same values, namely $\pm\infty$. Hence $X_\tau = X = \pm\nu$. On the other side, consider the boundary of the domain E_τ truncated by horocycles. Denote the four horocycle arcs by $\tilde{\gamma}$. By Lemma 4.3, the flux of X_τ yields:

$$0 = |a_0b_1| - |b_1b_2(\tau)| + |b_2(\tau)a_1| + \int_{[a_0',a_1']} \langle X_\tau, (-\nu) \rangle \, ds + F_{v_\tau}(\tilde{\gamma});$$

the integral is on $[a_0', a_1']$, the compact part of (a_0, a_1) joining the horocycles. Then

$$0 = -\varphi(\tau) + \int_{[a_0',a_1']} (1 - \langle X_\tau, \nu\rangle)\, ds + F_{v_\tau}(\tilde{\gamma}).$$

For a diverging sequence of nested horocycles, we get the convergence of the integral on the whole geodesic and the equality (the first inequality comes from the fact that $X = \nu$ on $[a_0, a_1]$, since $u = +\infty$):

$$\int_{(a_0,a_1)} \langle X - X_\tau, \nu\rangle\, ds = \int_{(a_0,a_1)} (1 - \langle X_\tau, \nu\rangle)\, ds = \varphi(\tau).$$

In the same way, on (a_1, a_2) we get a convergent integral

$$\int_{(a_1,a_2)} \langle -X + X_\tau, \nu\rangle\, ds = \int_{(a_1,a_2)} (1 + \langle X_\tau, \nu\rangle)\, ds = \varphi(\tau).$$

Then we get for any family α of disjoint arcs of ∂D

$$\left|\int_\alpha \langle X - X_\tau, \nu\rangle\, ds\right| \leq \int_{(a_0,a_1)\cup(a_1,a_2)} |\langle X - X_\tau, \nu\rangle|\, ds = 2\varphi(\tau). \tag{2}$$

We now study the flux of $X - X_\tau$ along the connected part of a level curve of $u - v_\tau$ containing the point p. The graphs Σ and Σ_τ corresponding respectively to u and v_τ are stable, complete and satisfy uniform curvature estimates. Let us denote by n_τ the normal to Σ_τ pointing down, and by $B((p, v_\tau(p)), \rho)$ the ball of radius ρ, centered at $(p, v_\tau(p)) \in \mathbb{H}/ < \psi > \times\mathbb{R}$. Then:

$$\forall \mu > 0,\ \exists \rho > 0 \text{ (independent of } \tau) \text{ such that } \forall p \in D :$$
$$q \in \Sigma_\tau \cap B((p, v_\tau(p)), \rho) \Longrightarrow \| n_\tau(q) - n_\tau(p) \| \leq \mu.$$

We have the same estimates for Σ.

Fix any $\mu > 0$ and $p \in D$; by continuity, there exists a $\rho_1 \leq \rho/2$ (independent of τ) such that $\forall q \in D(p, \rho_1)$ (the disk of $\mathbb{H}/ < \psi >$ with center p and radius ρ_1), we have: $|u(q) - u(p)| \leq \rho/2$.

Assume now that $\| n(p) - n_\tau(p) \| \geq 3\mu$. Let $\Omega_\tau(p)$ be the connected component of $\{u - v_\tau > u(p) - v_\tau(p)\}$ with p in its boundary, and Λ_τ the component of $\partial\Omega_\tau$ containing p. Λ_τ, as level curve of $u - v_\tau$, is piecewise smooth. Above $\Lambda_\tau \cap D(p, \rho_1)$, there are two parallel curves ($u = v_\tau + constant$): $\sigma \subset \Sigma$ and $\sigma_\tau \subset \Sigma_\tau$. Moreover on σ:

$$\forall q \in \Lambda_\tau \cap D(p, \rho_1),\ |(q, u(q)) - (p, u(p))| \leq \rho_1 + \rho/2 \leq \rho$$

Hence $\| n(q) - n(p) \| \leq \mu$ by the above estimate.

By a vertical translation of height $(v_\tau(p) - u(p))$:

$$(q, v_\tau(q)) \in \sigma_\tau \text{ and } v_\tau(q) - v_\tau(p) = u(q) - u(p).$$

Then

$$\| (q, v_\tau(q)) - (p, v_\tau(p)) \| \leq \rho \text{ and } \ \| n_\tau(q) - n_\tau(p) \| \leq \mu.$$

Combining the two last estimates with the assumption on the normals at p, the triangle inequality yields:

$$\forall q \in \Lambda_\tau \cap D(p, \rho_1),\ \| n(q) - n_\tau(q) \| \geq \| n(p) - n_\tau(p) \| - 2\mu \geq \mu.$$

Apply Lemma 4.4 below to conclude

$$\int_{\Lambda_\tau \cap D(p,\rho_1)} \langle X - X_\tau, \eta \rangle \, ds \geq \frac{\rho_1 \mu^2}{2}.$$

Remark that $\langle X - X_\tau, \eta \rangle$ is non negative outside the isolated points where $\nabla(u - v_\tau) = 0$ (for Λ_τ is part of a level curve of $u - v_\tau$); then for all compact arcs $\beta \subset \Lambda_\tau$, containing $\Lambda_\tau \cap D(p, \rho_1)$ we have:

$$\int_\beta \langle X - X_\tau, \eta \rangle \, ds \geq \frac{\rho_1 \mu^2}{2}. \tag{3}$$

As explained above, Λ_τ is non compact in D, so its two infinite branches go close to ∂D. Then there exists a connected compact part β of Λ_τ, containing $\Lambda_\tau \cap D(p, \rho_1)$, and two arcs $\tilde{\gamma}$ in D small enough and joining the extremities of β to ∂D. Eventually truncating by a family of horocycles $\tilde{\gamma}'$, the Flux Formula for $X - X_\tau$ yields:

$$0 = \int_\beta \langle X - X_\tau, (-\eta) \rangle \, ds + \int_\alpha \langle X - X_\tau, \nu \rangle \, ds + F_{u - v_\tau}(\tilde{\gamma} \cup \tilde{\gamma}'),$$

where α is contained in ∂D and $\tilde{\gamma}$ is contained in the horocycles and correctly oriented. Using (2) and (3) we obtain:

$$\frac{\rho_1\mu^2}{2} \leq 2\varphi(\tau) + F_{u-v_\tau}(\tilde{\gamma}\cup\tilde{\gamma}').$$

When the length of $\tilde{\gamma}\cup\tilde{\gamma}'$ goes to zero, we conclude:

$$\frac{\rho_1\mu^2}{2} \leq 2\varphi(\tau).$$

Hence, we get by contraposition:

$$\varphi(\tau) \leq \frac{\rho_1\mu^2}{4} \Longrightarrow \| X(p) - X_\tau(p) \| \leq \| n(p) - n_\tau(p) \| \leq 3\mu.$$

This gives precisely the behavior of X_τ and ∇v_τ for τ close to zero.

After the renormalization $v_\tau(p_0) = u(p_0)$ for a fixed $p_0 \in D$, we have $\lim\limits_{\tau\to 0} v_\tau|_D = u$. The convergence being uniform and $\mathscr{C}^\infty$ on compact sets, for τ small enough we can ensure $\| v_\tau - u \|_{\mathscr{C}^2(K)} \leq \varepsilon$. ∎

Lemma 4.4. Let w and w' be two minimal graphs of $\mathbb{H}/<\psi>\times\mathbb{R}$, above a domain $\Omega\subset\mathbb{H}/<\psi>$, and n and n' their respective normals. Then at any regular point of $w'-w$:

$$\langle X'-X, \eta\rangle_{\mathbb{H}/<\psi>} \geq \frac{\| n'-n \|^2}{4} \geq \frac{|X'-X|^2}{4},$$

where X (resp. X') is the projection of n (resp. n') on $\mathbb{H}/<\psi>$ and $\eta = \dfrac{\nabla(w'-w)}{|\nabla(w'-w)|}$ orients the level curve at this regular point.

Proof. We write $X = \frac{\nabla w}{W}$, $X' = \frac{\nabla w'}{W'}$ (the normals point down). We have (cf. [CK]):

$$\begin{aligned}\langle X'-X, \nabla w' - \nabla w\rangle_{\mathbb{H}/<\psi>} &= \langle n'-n, W'n' - Wn\rangle_{\mathbb{H}/<\psi>\times\mathbb{R}} \\ &= (W+W')\big(1 - \langle n, n'\rangle_{\mathbb{H}/<\psi>\times\mathbb{R}}\big) \\ &= (W+W')\frac{\| n'-n \|^2}{2}.\end{aligned}$$

Also $\dfrac{|\nabla w' - \nabla w|}{W'+W} \leq \dfrac{|\nabla w'|}{W'} + \dfrac{|\nabla w|}{W} \leq 2.$

Hence:

$$\langle X' - X, \eta \rangle = \frac{W + W'}{|\nabla(w' - w)|} \frac{\| n' - n \|^2}{2} \geq \frac{\| n' - n \|^2}{4}.$$

The last inequality of Lemma 4.4 simply arises by projection. ∎

4.3 Final construction

We can now state the main result of this paper, namely the existence of a harmonic diffeomorphism from $\mathbb{C}^*$ onto $\mathbb{H}/ < \psi >$ with the hyperbolic metric. For a minimal graph $\Sigma \subset (\mathbb{H}/ < \psi >) \times \mathbb{R}$ defined over a domain D, the projection $p : \Sigma \to D$ gives rise to a harmonic diffeomorphism, so the following theorem gives the expected result, since Σ is conformally $\mathbb{C}^*$ and $D = \mathbb{H}/ < \psi >$ here.

Theorem 4.1. In $(\mathbb{H}/ < \psi >) \times \mathbb{R}$, there exist entire minimal graphs over $\mathbb{H}/ < \psi >$ which are conformally $\mathbb{C}^*$.

Proof. Let $\mathscr{A}$ be a non-compact annulus of $\mathbb{H}/ < \psi >$ containing the cusp ($\mathscr{A}$ is conformally a once-punctured disk).

In a first step, we recursively use Proposition 3.1 and Proposition 4.1 to construct a sequence of Scherk functions u_n defined over ideal polygonal domains D_n which exhaust $\mathbb{H}/ < \psi >$, together with an exhaustion of $(\mathbb{H}/ < \psi >) - \mathscr{A}$ by compact annuli $\mathscr{A}_n \subset D_n$ with $\mathscr{A}_n \subset \mathring{\mathscr{A}}_{n+1}$, satisfying the following:

i) $\| u_{n+1} - u_n \|_{\mathscr{C}^2(\mathscr{A}_n)} < \varepsilon_n$, for some sequence $\varepsilon_n > 0$, with $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$,

ii) For each j, $0 \leq j < n$, the conformal modulus of the annulus in the graph of u_n over the domain $\mathscr{A}_{j+1} - \mathscr{A}_j$ is greater than one.

For that, let ε_n be a sequence of positive real numbers such that $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$.

We choose D_0 to be the polygonal domain bounded by the image in the quotient space of $\mathscr{Q} \cup \mathscr{Q}^*$, where $\mathscr{Q}$ is the quadrilateral defined above, and $\mathscr{Q}^*$ is its image by the reflection with respect to the geodesic C_2. Let then u_0 be the Scherk function defined over D_0, and $\mathscr{A}_0$ be some compact annulus in $D_0 - \mathscr{A}$.

Now assume that $(D_j, u_j, \mathscr{A}_j)$ are constructed for $0 \leq j \leq n$ and satisfy the properties **i)** and **ii)** we require above.

Using Proposition 4.1, attach perturbed elementary quadrilaterals E_τ, E'_τ, to *all* of the pairs of sides of ∂D_n, to obtain an ideal Scherk graph u_{n+1} over an enlarged polygonal domain D_{n+1}. The E_τ, E'_τ are attached successively to the pairs of sides of ∂D_n; the parameter τ of each pair attached depends on the previous expanded polygons.

Moreover, in the Proposition 4.1, we can choose ε small enough to get u_{n+1} as close as we want to u_n in the $\mathscr{C}^2$-topology on the compact $\mathscr{A}_n$, so that properties **i)** and **ii)** are satisfied. For we can easily ensure that first $\| u_{n+1} - u_n \|_{C^2(\mathscr{A}_n)} < \varepsilon_n$, and secondly, as for u_n, the graph of u_{n+1} over each annulus $\mathscr{A}_{j+1} - \mathscr{A}_j$, $0 \leq j < n$, has conformal modulus greater than one, since the closer the graphs are, the closer are the conformal moduli.

Let us denote by Σ the graph of u_{n+1} over $\mathring{\mathscr{A}}_n$. Now, since the graph of u_{n+1} over D_{n+1} is conformally $\mathbb{C}^*$ by Proposition 3.1, there is a compact annulus Σ' in this graph satisfying:

- Σ' contains Σ in its interior, and
- the conformal modulus of the annulus $\Sigma' - \Sigma$ is greater than one.

Then define $\mathscr{A}_{n+1}$ to be the vertical projection of Σ'; eventually enlarge Σ' in order that $\mathscr{A}_{n+1}$ has its boundary in a tubular neighborhood of radius one of ∂D_{n+1}. By the above construction, this $\mathscr{A}_{n+1}$ satisfies property **ii)**. Then the sequence is constructed and the argument will be complete if we prove that the $\mathscr{A}_n$ exhaust $\mathbb{H}/<\psi>$.

For each n, using the particular geometry of the perturbed quadrilateral we attach to all of the sides of ∂D_n, we get that the boundary of D_{n+1} is a fixed constant farther from the fixed point O we have chosen in K_0 (if the quadrilateral we add were regular, it would be at least $\ln(1+\sqrt{2})$;

cf. Figure 5, where the equidistant curves H and H' are two parallel horocycles). Hence ∂D_n diverge to infinity with n. But we constructed $\mathscr{A}_n$ so that $\partial \mathscr{A}_n$ is uniformly close to ∂D_n, so that we actually get an exhaustion.

Now the second step: we let n tend to infinity. We obtain an entire graph Σ_∞ of a minimal function u, since at any $x \in \mathbb{H}/ < \psi >$, the $u_n(x)$ form a Cauchy sequence. Since on each $\mathscr{A}_{j+1} - \mathscr{A}_j$, the u_n converge uniformly to u in the $\mathscr{C}^2$-topology, the modulus of the part of Σ_∞ over $\mathscr{A}_{j+1} - \mathscr{A}_j$ is at least one. Hence, by Grötzsch Lemma (cf. [V]), the conformal type of the graph of the limit graph Σ_∞ is $\mathbb{C}^*$. ■

4.4 Hyperbolic case

Let us now examine what happens in the case where ψ is a hyperbolic isometry of $\mathbb{H}$: ψ has two fixed points $x_1, x_2 \in \partial_\infty \mathbb{H}$, so that the geodesic (x_1, x_2) is invariant under ψ. The equidistant curves from (x_1, x_2) are invariant too. Let p be any point on (x_1, x_2); the geodesics going through p and $\psi(p)$ and orthogonal to (x_1, x_2) bound a fundamental domain Δ (we have drawn its boundary in bold in Figure 12) that we split in two parts Δ^+ and Δ^- as previously.

In order to get an entire minimal graph on $\mathbb{H}/ < \psi >$, we first construct a pseudo-Scherk graph on the quadrilateral of Figure 13; then we construct an exhaustion of Δ^+ by a sequence of pseudo-Scherk polygons obtained by extending the initial quadrilateral $\mathscr{Q}$ with almost regular quadrilaterals that we add to the new sides that occur. As for the parabolic case, the conformal type of the corresponding quotient Scherk graphs is $\mathbb{C}^*$ (we can use Grötzsch Lemma to show that such graphs have two ends). By rotations, we finally get a sequence of quotient Scherk graphs one of whose subsequences converges to an entire minimal graph on $\mathbb{H}/ < \psi >$, and thus the result follows.

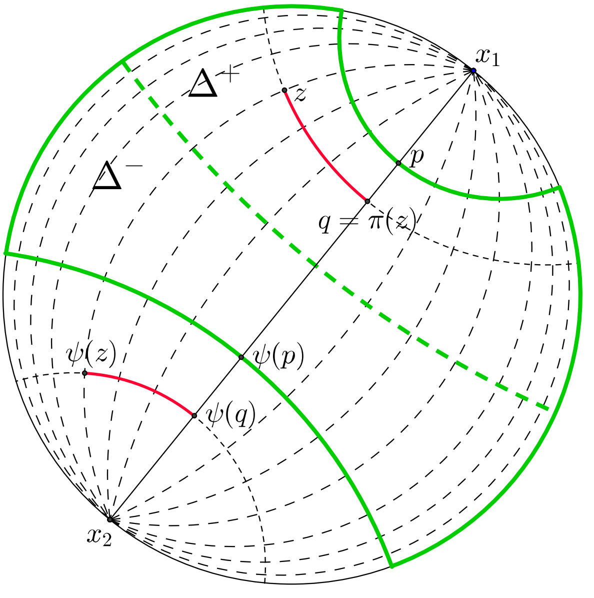


Figure 12: Fundamental domain for the hyperbolic isometry ψ

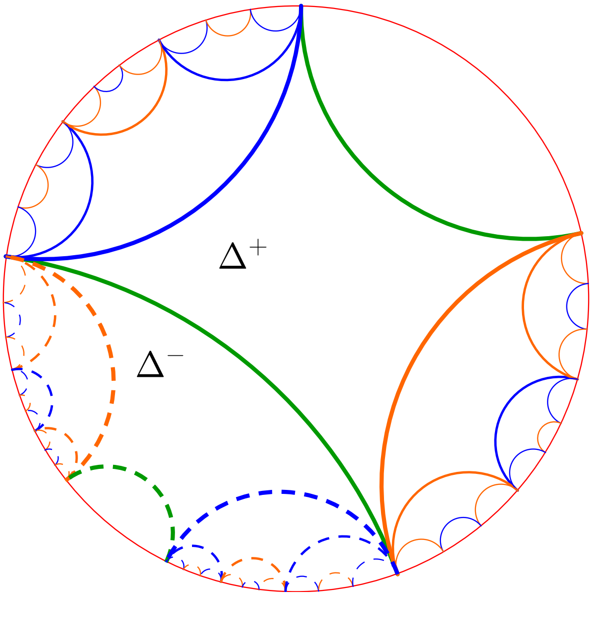


Figure 13: Extension of quotient Scherk graphs in $\mathbb{H}/ < \psi >$

5 Non-existence of a harmonic diffeomorphism from $\mathbb{D}^*$ onto $\mathbb{S}^1 \times \mathbb{R}$

In this section, we will show that there exists no harmonic diffeomorphism from $\mathbb{H}/ < \psi >$ onto $(\mathbb{S}^1 \times \mathbb{R}, |d\zeta|^2)$, where ψ is parabolic and $|d\zeta|^2$ denotes the euclidian metric. First remark that since this property only depends on the conformal type of $\mathbb{H}/ < \psi >$, it amounts to showing that there exists no harmonic diffeomorphism from $\mathbb{D}^*$ (the once punctured disk) onto $(\mathbb{S}^1 \times \mathbb{R}, |du|^2)$.

We will use the following theorem to get the desired result:

Theorem 5.1 (Cheng-Yau's Gradient Estimate)**.** Let M^m be a complete Riemannian manifold of dimension m. Assume that the geodesic ball $D_r(p) \cap \partial M = \emptyset$. Suppose that the Ricci curvature on $D_r(p)$ is bounded from below by:

$$\mathscr{R}_{ij} \geq -(m-1)R$$

for some constant $R \geq 0$. If h is a positive function defined on $D_r(p) \subset M$ satisfying

$$\Delta h = -\mu h$$

for some constant $\mu \geq 0$, then there exists a constant C depending on m such that:

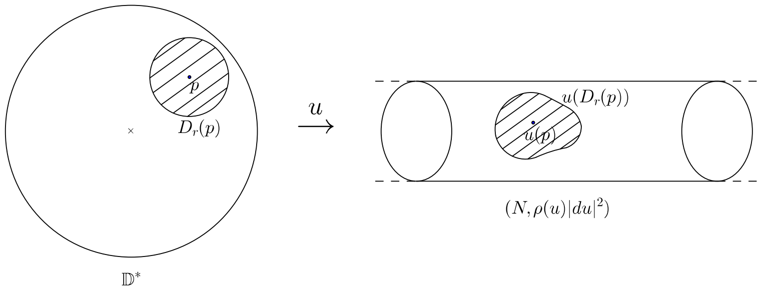
$$\frac{|\nabla h|^2}{h^2}(x) \leq \frac{(4(m-1)^2 + 2\varepsilon)R}{4 - 2\varepsilon} + C\left(4(1+\varepsilon^{-1})r^{-2} + \mu\right),$$

for all $x \in D_{r/2}(p)$ and for any $\varepsilon < 2$.

Then the result will follow from the next theorem[1]:

Theorem 5.2. There exists no harmonic diffeomorphism from $\mathbb{D}^*$ onto $(N, \rho(\zeta)|d\zeta|^2)$ where N is an oriented surface without boundary and $\rho(\zeta)|d\zeta|^2$ is a complete metric of non-negative curvature.

[1] In fact, Cohn-Vossen's Theorem tells us that for the cylinder $\mathbb{S}^1 \times \mathbb{R}$ endowed with a complete metric of non-negative curvature K, we have $|\int_{\mathbb{S}^1 \times \mathbb{R}} K ds| < \infty$, and moreover $\int_{\mathbb{S}^1 \times \mathbb{R}} K ds \leq 2\pi\chi(\mathbb{S}^1 \times \mathbb{R}) = 0$, so that here $K = 0$.



Proof. Suppose that $u : \mathbb{D}^* \to (N, \rho(\zeta)|d\zeta|^2)$ *is a harmonic diffeomorphism*, where both surfaces are endowed with complex coordinates. As usual, we denote:

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \ \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Let $|\partial u(z)| := \sqrt{\rho(u(z))}\left|\frac{\partial u}{\partial z}(z)\right|$ and $|\overline{\partial} u(z)| := \sqrt{\rho(u(z))}\left|\frac{\partial u}{\partial \overline{z}}(z)\right|$; we also define the *energy density* $|du|^2$ by:

$$|du(z)|^2 = \sum_{i=1,2} \rho(u(z))\left[\left(\frac{\partial u^i}{\partial x}(z)\right)^2 + \left(\frac{\partial u^i}{\partial y}(z)\right)^2\right] = \rho(u(z))\sum_{i=1,2}|\nabla u^i(z)|^2,$$

where we set $\nabla u^i = \begin{pmatrix} \frac{\partial u^i}{\partial x} & \frac{\partial u^i}{\partial y} \end{pmatrix}$. Then we have the following expressions:

$$|du|^2 = 2(|\partial u|^2 + |\overline{\partial} u|^2),$$
$$J(u) = |\partial u|^2 - |\overline{\partial} u|^2,$$

where $J(u)$ is the jacobian of u.

First remark that since u is supposed to be a diffeomorphism, $J(u)$ never vanishes so we can assume $J(u) > 0$, i.e. $|\partial u|^2 > |\overline{\partial} u|^2 \geq 0$.

We now choose the following metric on $\mathbb{D}^*$: $\lambda := |\partial u|^2|dz|^2$. Then we calculate its Gaussian curvature:

$$K_\lambda = -\frac{1}{2}\Delta_\lambda \log(|\partial u|^2) = -\Delta_\lambda \log(|\partial u|).$$

From the Bochner formula (cf. [SY]) we also get $\Delta_\lambda \log(|\partial u|) = -K_\rho J(u) + K_\lambda$, so that

$$K_\lambda = \frac{1}{2} K_\rho J(u) \geq 0 \text{ since the curvature of } \rho(\zeta)|d\zeta|^2 \text{ is positive.}$$

Let $\gamma : I \to \mathbb{D}^*$ be a curve parametrized by arc length (for the euclidian metric), with $\gamma = \gamma^1 + i\gamma^2$; we calculate its length for the metric λ:

$$L_\lambda(\gamma) = \int_I \left| \frac{d\gamma}{dt}(t) \right|_\lambda dt = \int_I \left| \frac{d\gamma}{dt}(t) \right| |\partial u(\gamma(t))| dt = \int_I |\partial u(\gamma(t))| dt.$$

We have the following inequalities:

$$\begin{aligned}
L_\rho(u(\gamma)) &= \int_I \left| \frac{d(u(\gamma(t)))}{dt} \right|_\rho dt \\
&= \int_I \left| \frac{\partial u}{\partial x}(\gamma(t)) \frac{d\gamma^1}{dt}(t) + \frac{\partial u}{\partial y}(\gamma(t)) \frac{d\gamma^2}{dt}(t) \right| \sqrt{\rho(\gamma(t))} dt \\
&= \int_I \left| \frac{\partial u}{\partial z}(\gamma(t)) \left(\frac{d\gamma^1}{dt}(t) + i \frac{d\gamma^2}{dt}(t) \right) + \frac{\partial u}{\partial \overline{z}}(\gamma(t)) \left(\frac{d\gamma^1}{dt}(t) - i \frac{d\gamma^2}{dt}(t) \right) \right| \sqrt{\rho(\gamma(t))} dt \\
&\leq \int_I \left(|\partial u(\gamma(t))| \left| \frac{d\gamma}{dt}(t) \right| + |\overline{\partial} u(\gamma(t))| \left| \frac{d\gamma}{dt}(t) \right| \right) dt \\
&\leq 2 \int_I |\partial u(\gamma(t))| dt \quad (\text{since } J(u) > 0) \\
&= 2 L_\lambda(\gamma).
\end{aligned}$$

We thus get:

$$L_\rho(u(\gamma)) \leq 2 L_\lambda(\gamma). \tag{4}$$

Let $r > 0$, $p \in \mathbb{D}^*$; we denote $D_r(p)$ the euclidian ball with center p and radius r and $R := dist_\rho(u(p), \partial u(D_r(p)))$ (> 0 since u is a diffeomorphism).

From inequality (4) we get:

$$\begin{aligned}
dist_\lambda(p, \partial(D_r(p))) &= \inf_{\gamma:\ p \to (q \in \partial D_r(p))} L_\lambda(\gamma) \\
&\geq \frac{1}{2} \inf_{\gamma:\ p \to (q \in \partial D_r(p))} L_\rho(u(\gamma)) \\
&\geq \frac{1}{2} \inf_{\gamma':\ u(p) \to (q' \in \partial u(D_r(p)))} L_\rho(\gamma') \\
&= \frac{R}{2},
\end{aligned}$$

so that $D_r(p) \supset D_\lambda(p, \frac{R}{2})$ (the latter is the geodesic ball for the metric λ).

Let us now check the hypotheses in order to apply the gradient estimate:

- $(\mathbb{D}^*, \lambda)$ is geodesically complete.
  Indeed, let $(x_n)_n$ be a Cauchy sequence in $(\mathbb{D}^*, dist_\lambda)$; for $n, p \in \mathbb{N}$, we have:
  $$dist_\rho(u(x_n), u(x_{n+p})) \leq 2 dist_\lambda(x_n, x_{n+p});$$
  since (x_n) is of Cauchy, the right side can be made as small as we want for n large enough, and thus, $(u(x_n))_n$ is of Cauchy too, so converges since N is complete. But u is invertible, thus $(x_n)_n$ is also convergent, and $(\mathbb{D}^*, \lambda)$ is complete.
- $D_\lambda(p, \frac{R}{2}) \cap \partial\mathbb{D}^* = \emptyset$.
- the Gaussian curvature K_λ is positive.
- $h : (x, y) \mapsto 1 + x$ is harmonic in $D_r(p)$ and positive ($\Delta_\lambda = \frac{1}{|\partial u|^2} \Delta_{eucl}$).

From the gradient estimate, we get that there exists a constant C such that for any $x \in D_{\frac{R}{4}}(p)$ and for all $0 < \varepsilon < 2$:

$$\frac{|\nabla_\lambda h|^2}{h^2}(x) \leq C \left((1 + \varepsilon^{-1}) \left(\frac{R}{4} \right)^{-2} \right).$$

We take $\varepsilon = 1$, and we denote $C' = \sqrt{128C}$. Then for any $x \in D_{\frac{R}{4}}(p)$:

$$|\nabla_\lambda h|(x) \leq \frac{C'}{R}.$$

Since $|\nabla_\lambda h| = \frac{1}{|\partial u|^2}|\nabla_{eucl} h| = \frac{1}{|\partial u|^2}$, we finally have:

$$|du(p)|^2 \geq |\partial u(p)|^2 \geq C'^{-1} R,$$

so if we take r large in order that $\partial D_r(p)$ goes to $\partial \mathbb{D}^*$, then R tends to infinity, so $|du(p)| = +\infty$ and *we actually get a contradiction.* ■

Once this complete λ-metric is constructed, one can avoid the use of Theorem 5.1 as follows.

Assume indeed that u is a such diffeomorphism; as in the previous proof, we can construct a complete metric λ on $\mathbb{D}^*$ with non-negative curvature K_λ. Then from Cohn-Vossen's Theorem, we know that $|\int_{\mathbb{D}^*} K_\lambda ds| < \infty$. But a theorem of Huber (cf. [H]) states that a complete Riemannian surface of finite total curvature is conformally equivalent to a compact Riemann surface punctured in a finite number of points. Here this implies that $(\mathbb{D}^*, \lambda)$ is conformally equivalent to $\mathbb{C}^*$, *which is a contradiction.* ■

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