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Minimal graphs in $\widetilde{PSL}_2(\mathbb{R}, \tau)$

Abigail Folha **Carlos Peñafiel**

Abstract

In this paper we study the existence and non-existence of vertical and horizontal graphs in the space $\widetilde{PSL}_2(\mathbb{R}, \tau)$. We prove that there is no entire horizontal minimal graph. On the other hand, there are entire vertical minimal graphs having prescribed continuous boundary values.

1 Introduction

The space $\widetilde{PSL}_2(\mathbb{R}, \tau)$ is a family of simply connected homogeneous tri-dimensional manifold having four dimensional isometry group, indexed by τ. When $\tau = 0$, $\widetilde{PSL}_2(\mathbb{R}, \tau)$ is the product space $\mathbb{M} \times \mathbb{R}$, where $\mathbb{M}$ is the hyperbolic two dimensional space. The aim of this article is to extend some results on $\mathbb{M} \times \mathbb{R}$ about minimal graphs to the family $\widetilde{PSL}_2(\mathbb{R}, \tau)$.

First, we give a notion of horizontal graph. Using some minimal surfaces on $\widetilde{PSL}_2(\mathbb{R}, \tau)$ invariant by one-paremter isometry group, we prove that there is no entire horizontal minimal graph. The non-existence of such graphs on $\mathbb{M} \times \mathbb{R}$ was proved by R. Sa Earp [5, Theorem 1.1]. Moreover, we prove that there is no horizontal minimal graph over bounded domains whose boundary satisfies a geometric condition, see Theorem 5.4.

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Also, we deal with vertical graphs. In $\mathbb{R}^3$, Bernstein's Theorem says that complete minimal graphs are planes. After that, B. Nelli and H. Rosenrberg proved that in $\mathbb{M} \times \mathbb{R}$, there is an entire vertical minimal graph having prescribed boundary values, see [3, Theorem 4]. We extend this result to $\widetilde{PSL}_2(\mathbb{R}, \tau)$ using some Scherk type surfaces, see Theorem 6.1.

The paper is organized as follows. Section 2 is devoted to fix some notations which will be used along the text. In Section 3, we give some properties of the space $\widetilde{PSL}_2(\mathbb{R}, \tau)$. We describe the isometries of $\widetilde{PSL}_2(\mathbb{R}, \tau)$ and we give some geometric information of an inversion on $\widetilde{PSL}_2(\mathbb{R}, \tau)$. In Section 4, we define horizontal and vertical graphs. In Section 5, we prove the non-existence of entire horizontal minimal graph as well as the non-existence of horizontal minimal graphs defined over some bounded domains. Finally, in Section 6 we prove the existence and uniqueness of entire minimal graphs having prescribed continuous boundary value.

2 Notation

We will fix some notations which will be used on this work.

- $\mathbb{H}^2$ denotes the half-plane model for the two dimensional hyperbolic plane.
- $\mathbb{D}^2$ denotes the disk model for the two dimensional hyperbolic plane.
- $\beta(x_1, x_2) \subset \mathbb{H}^2,\ x_1 < x_2$ denotes the complete geodesic in the hyperbolic plane joining $(x_1, 0)$ and $(x_2, 0)$
- $\gamma(x_1) \subset \mathbb{H}^2$ denotes the complete geodesic $\{(x_1, y) \in \mathbb{H}^2\}$.
- $\Lambda(x_1, x_2) = \{(x, 0) \in \partial_\infty \mathbb{H}^2; x_1 \leq x \leq x_2\}$
- $\Upsilon(x_1, x_2) = \{(x, 0) \in \partial_\infty \mathbb{H}^2; x \leq x_1 \text{ or } x \geq x_2\}$
- $\Gamma(x_1) = \{(x, 0) \in \partial_\infty \mathbb{H}^2; x \geq x_1\}$

- $\beta_\theta(x_1,x_2)$ denotes the equidistant curve to $\beta(x_1,x_2)$ making angle $\theta, 0<\theta<\pi$ with the positive semi x-axis at the point $(x_2,0)$. Observe that for $0<\theta<\pi/2$, $\beta_\theta(x_1,x_2)$ is contained at the unbounded region whose boundary is $\beta(x_1,x_2)\cup\Upsilon(x_1,x_2)$. For $\pi/2<\theta<\pi$, $\beta_\theta(x_1,x_2)$ is contained at the unbounded region whose boundary is $\beta(x_1,x_2)\cup\Lambda(x_1,x_2)$.

- $\gamma_\theta(x_1)$ is the equidistant curve to $\gamma(x_1)$ making angle $\theta,\ 0<\theta<\pi$, with the positive semi x-axis.

- $D_1(x_1,x_2)$ denotes the unbounded domain whose boundary is $\beta(x_1,x_2)\cup\ \Lambda(x_1,x_2)$.

- $D_2(x_1,x_2) := \mathbb{H}^2 - (D_1(x_1,x_2)\cup\beta(x_1,x_2))$.

3 The space $\widetilde{PSL}_2(\mathbb{R},\tau)$

The group $PSL_2(\mathbb{R})$ is the isometry group of the hyperbolic plane $\mathbb{M}$. Such isometry group can be identified with the unit tangent bundle $T_1\mathbb{M}$ as follows. Fix a point $(p_0,v_0)\in T_1\mathbb{M}$, for each $(p,v)\in T_1\mathbb{M}$ there is a unique isometry $g\in PSL_2(\mathbb{R})$ such that $g(p,v)=(p_0,v_0)$. Thus we have identified each isometry g with a point (p,v) in $T_1\mathbb{M}$. Furthermore, the space $T_1\mathbb{M}$ is diffeomorphic to the product $\mathbb{M}\times\mathbb{S}^1$ (where $\mathbb{S}^1$ is the unit circle). With this identification, let $\tau\in\mathsf{R}$, the space $PSL(\mathbb{R},\tau)$ is the total space of a fibration over $\mathbb{M}$ whose fibers are circles and whose fibration has bundle curvature τ. Let $\widetilde{PSL}_2(\mathbb{R},\tau)$ be the universal cover of $PSL(\mathbb{R},\tau)$. For each fixed τ, the space $\widetilde{PSL}_2(\mathbb{R},\tau)$ is a homogenous Riemannian manifold having 4-dimensional isometry group, this manifold is one of the eight Thurston's geometries, for more details see [1].

We consider the following Riemannian submersion π from the space $\widetilde{PSL}_2(\mathbb{R},\tau)$ into the hyperbolic space $\mathbb{M}$. In Euclidean coordinates, this Riemannian submersion is given by

$$\begin{array}{rccc} \pi: & \widetilde{PSL}_2(\mathbb{R},\tau) & \to & \mathbb{M} \\ & (x,y,t) & \mapsto & (x,y), \end{array}$$

and the induced metric on $\widetilde{PSL}_2(\mathbb{R},\tau)$ is

$$g := ds^2 = \lambda^2(dx^2+dy^2) + \left(2\tau\left(\frac{\lambda_y}{\lambda}dx - \frac{\lambda_x}{\lambda}dy\right) + dt\right)^2,$$

where, either $\lambda = 2(1-(x^2+y^2))^{-1}$ when $\mathbb{M} = \mathbb{D}^2$ or $\lambda = y^{-1}$, when $\mathbb{M} = \mathbb{H}^2$.

For each $p \in \mathbb{M}$, the fiber $\pi^{-1}(p)$ is difeomorphic to the real line. The unitary vector field tangent to $\pi^{-1}(p)$ is called the vertical vector field and denoted by E_3. Since translation along the fibers are isometries, E_3 is a Killing field. Also we call a field X an horizontal field, if X is orthogonal to E_3.

The hyperbolic space $\mathbb{M}$ has metric $d\sigma = \lambda^2(dx^2+dy^2)$, thus an orthonomal frame is given by $\{e_1 = \lambda^{-1}\partial_x, e_2 = \lambda^{-1}\partial_y\}$. Denoting by E_1 and E_2 the horizontal lifts of e_1 and e_2, respectively, an orthonormal frame in $\widetilde{PSL}_2(\mathsf{R},\tau)$ is given by $\{E_1, E_2, E_3\}$, where

$$\begin{aligned} E_1 &= \lambda^{-1}\partial_x - 2\tau\lambda^{-2}\lambda_y\partial_t \\ E_2 &= \lambda^{-1}\partial_y + 2\tau\lambda^{-2}\lambda_x\partial_t \\ E_3 &= \partial_t \end{aligned}$$

for more details see [1].

We can consider the half-plane model or the disk model for the hyperbolic plane $\mathbb{M}$. For each model of $\mathbb{M}$ we obtain a model for $\widetilde{PSL}_2(\mathbb{R},\tau)$. When we take the disk model for $\mathbb{M}$ we call the correspondent model for $\widetilde{PSL}_2(\mathbb{R},\tau)$, *the cylinder model*. On the other hand, if we consider the half-plane model for the space $\mathbb{M}$, we call the correspondent model for $\widetilde{PSL}_2(\mathbb{R},\tau)$, *the half-space model*.

When $\tau \neq 0$, the space $\widetilde{PSL}_2(\mathbb{R},\tau)$ is not a product space, so, without no further assumptions the horizontal lift $\pi^{-1}(\alpha) \subset \widetilde{PSL}_2(\mathbb{R},\tau)$ of a curve $\alpha \subset \mathbb{M}$ is not necessarily contained in a slice $\mathbb{M}\times\{t_0\} \subset \widetilde{PSL}_2(\mathbb{R},\tau)$. The next lemma describe which are the curves in $\mathbb{M}$ whose horizontal lift is contained in a slice.

Lemma 3.1. Let α be a connected curve in $\mathbb{M}$ and denote by $\widetilde{\alpha}$ its horizontal lift to $\widetilde{PSL}_2(\mathbb{R},\tau)$. Assume that $\widetilde{\alpha}$ lie in some slice $\{t = t_0\}$, then

1. α is contained in a complete geodesic passing through the origin $(0,0)$ of $\mathbb{D}^2$, if we consider the disk model.
2. α is contained in $\{x = x_0\}$ for some $x_0 \in \mathbb{R}$, if we consider the half-plane model $\mathbb{H}^2$.

Proof. Let $\alpha(s) = (x(s), y(s)) \subset \mathbb{M}$ be a connected curve with $s \in I$, I is an open interval. Its horizontal lift $\widetilde{\alpha}(s) = (x(s), y(s), t(s))$ has velocity vector $\widetilde{\alpha}'(s)$ which in the orthonormal frame is given by

$$\widetilde{\alpha}'(s) = \lambda x' E_1 + \lambda y' E_2 + [t' + 2\tau\lambda^{-1}(x'\lambda_y - y'\lambda_x)]E_3$$

Since $\widetilde{\alpha}$ is the horizontal lift of α, we must have $g(\widetilde{\alpha}', E_3) = 0$, that is

$$t' = 2\tau\lambda^{-1}(y'\lambda_x - x'\lambda_y) \tag{3.1}$$

We describe the solutions of this ordinary differential equation (ODE) for each model of $\mathbb{M}$.

1. For the disk model $\mathbb{M} = \mathbb{D}^2$, the solution to the ODE (3.1) is
$$t(s) = t_0 + \int_{s_0}^{s} 2\tau\lambda(\mu)[x(\mu)y'(\mu) - x'(\mu)y(\mu)]d\mu$$
We obtain
$$t(s) = t_0 + \int_{\alpha} 2\tau\lambda(xdy - ydx)$$
In order to $\widetilde{\alpha}$ be contained in $\{t = t_0\}$ we should have $xdy - ydx = 0$, that is, $y(s) = \pm a x(s)$, $a > 0$, $\quad s \in I$.
2. For the half-plane model $\mathbb{M} = \mathbb{H}^2$, the solution to the ODE (3.1) is
$$t(s) = t_0 + \int_{s_0}^{s} 2\tau\lambda(\mu)x'(\mu)d\mu$$
In order to $\widetilde{\alpha}$ stays in the slice $\{t = t_0\}$ we should have $x'(s) \equiv 0$, that is, $x(s) = \text{constant}$, $s \in I$. ∎

Now we construct an isometry between the two models of $\widetilde{PSL}_2(\mathbb{R},\tau)$. Let $\widetilde{PSL}_{(2,\mathbb{H}^2)}(\mathbb{R},\tau)$ denotes the half-space model for the space $\widetilde{PSL}_2(\mathbb{R},\tau)$ and let $\widetilde{PSL}_{(2,\mathbb{D}^2)}(\mathbb{R},\tau)$ denotes the cylinder model for the space $\widetilde{PSL}_2(\mathbb{R},\tau)$.

We consider the isometry $\phi:\mathbb{D}^2\to\mathbb{H}^2$ given by

$$\phi(x,y)=\left(\frac{-2y}{(1-x^2)^2+y^2},\frac{1-x^2-y^2}{(1-x)^2+y^2}\right),$$

we will construct an isometry

$$J(x,y):\widetilde{PSL}_{(2,\mathbb{D}^2)}(\mathbb{R},\tau)\to\widetilde{PSL}_{(2,\mathbb{H}^2)}(\mathbb{R},\tau)$$

having the form $J(x,y,t)=(\phi(x,y),h(x,y,t))$. In order to obtain such isometry, observe that the orthonormal frame $\{E_1,E_2,E_3\}$ of $\widetilde{PSL}_{(2,\mathbb{D}^2)}(\mathbb{R},\tau)$ should be send on an orthonormal frame $\{V_1,V_2,V_3\}$ of $\widetilde{PSL}_{(2,\mathbb{H}^2)}(\mathbb{R},\tau)$, that is, for any point $(x,y,t)\in\widetilde{PSL}_{(2,\mathbb{D}^2)}(\mathbb{R},\tau)$ the frame

$$\begin{aligned}V_1&=D_{(x,y,t)}J(E_1)\\V_2&=D_{(x,y,t)}J(E_2)\\V_3&=D_{(x,y,t)}J(E_3)\end{aligned}$$

is an orthonormal frame of $\widetilde{PSL}_{(2,\mathbb{H}^2)}(\mathbb{R},\tau)$, where $D_{(x,y,t)}J(E_i)$, $i=1,2,3$ denotes the derivative of J at the point (x,y,t). Direct computations imply that $h(x,y,t)=t+f(x,y)$ and f satisfies the second order partial differential equation

$$xf_x(x,y)+yf_y(x,y)+\frac{8\tau y}{(1-x)^2+y^2}=0.$$

Using polar coordinates, we obtain (up to a constant)

$$f(x,y)=-8\tau\arctan\left(\frac{y}{1-x}\right).$$

Thus, we have the following Lemma.

Lemma 3.2. The diffeomorphism

$$J:\widetilde{PSL}_{(2,\mathbb{D}^2)}(\mathbb{R},\tau)\to\widetilde{PSL}_{(2,\mathbb{H}^2)}(\mathbb{R},\tau)$$

given by

$$J(x,y,t)=\left(\frac{-2y}{(1-x^2)^2+y^2},\frac{1-x^2-y^2}{(1-x)^2+y^2},t-8\tau\arctan\left(\frac{y}{1-x}\right)\right)$$

is an isometry between $\widetilde{PSL}_{(2,\mathbb{D}^2)}(\mathbb{R},\tau)$ and $\widetilde{PSL}_{(2,\mathbb{H}^2)}(\mathbb{R},\tau)$.

3.1 Isometries of $\widetilde{PSL}_2(\mathbb{R},\tau)$

We identify the Euclidean space $\mathbb{R}^2$ with the set of complex numbers $\mathbb{C}$, that is, $z=x+iy\approx(x,y)$. With this identification, every point $(x,y,t)\in\widetilde{PSL}_2(\mathbb{R},\tau)$ can be written in the form (z,t), where z is a complex number. The behavior of the isometries of $\widetilde{PSL}_2(\mathbb{R},\tau)$ is given in the following proposition.

Proposition 3.1. *[7, Theorem 9] The isometries of $\widetilde{PSL}_2(\mathbb{R},\tau)$ are given by:*

1. *In the half-space model for $\widetilde{PSL}_2(\mathbb{R},\tau)$*

$$F(z,t)=(f(z),t-2\tau\arg f'+c)$$

or

$$G(z,t)=(-\overline{f}(z),-t+2\tau\arg f'+c)$$

where f is a positive isometry of $\mathbb{H}^2$ and c is a real number.

2. *In the cylinder model for $\widetilde{PSL}_2(\mathbb{R},\tau)$*

$$F(z,t)=(f(z),t-2\tau\arg f'+c)$$

or

$$G(z,t)=(\overline{f}(z),-t+2\tau\arg f'+c)$$

where f is a positive isometry of $\mathbb{D}^2$ and c is a real number.

This proposition motivates the following definition.

Definition 3.3. With the same notation of Proposition 3.1, an isometry $F(z,t)$ is called parabolic, hyperbolic or elliptic if f is a parabolic, hyperbolic or elliptic isometry of the hyperbolic space $\mathbb{M}$, respectively.

3.2 The inversion in $\widetilde{PSL}_2(\mathbb{R},\tau)$

Now we use the Proposition 3.1 to understand the behavior of the isometry in $\widetilde{PSL}_2(\mathbb{R},\tau)$ generated by an inversion in $\mathbb{H}^2$.

Recall that for each $x_1 > 0$, $\beta(-x_1, x_1)$ is the complete geodesic in $\mathbb{H}^2$ joining the points $(-x_1, 0)$ and $(x_1, 0)$. The inversion $I_{x_1} : \mathbb{H}^2 \to \mathbb{H}^2$ with respect to $\beta(-x_1, x_1)$ is given by

$$I_{x_1}(z) = \frac{x_1^2}{\overline{z}} = -\overline{\left(-\frac{x_1^2}{z}\right)} := -\overline{h_{x_1}(z)}.$$

where h_{x_1} is a positive isometry of $\mathbb{H}^2$.

The isometry I_{x_1} maps the complete geodesic $\gamma(x_1) = \{(x_1, y), y > 0\}$ in $\mathbb{H}^2$ in the complete geodesic $\beta(0, x_1)$ joining the points $z = 0$ and $z = x_1$ at the asymptotic boundary of $\mathbb{H}^2$. Moreover, the image by I_{x_1} of the equidistant curve $\gamma_\theta(x_1)$ which makes angle θ with the positive x-axis is the equidistant curve $\beta_{\pi-\theta}(0, x_1)$.

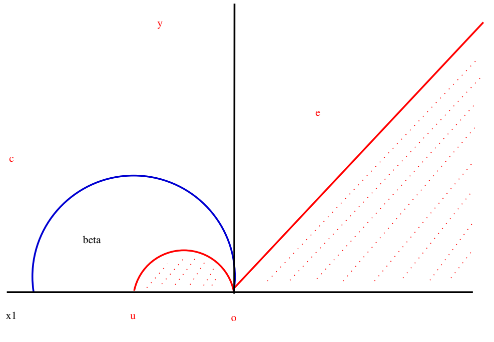

Figure 1: The inversion around $\beta(-x_1, x_1)$ takes $\gamma_\theta(x_1)$ in $\beta_{\pi-\theta}(0, x_1)$.

Lemma 3.4. The argument of h_{x_1} is

$$arg(h'_{x_1})(z) = \arctan(\Theta(z)), \quad \text{where} \quad \Theta(z) = \frac{-2xy}{x^2 - y^2}. \tag{3.2}$$

Proof. Observe that

$$h'_{x_1}(z) \quad = \quad \frac{x_1^2(x^2 - y^2 - 2ixy)}{|z|^4}.$$

Thus the argument of $h'_{x_1}(z)$ is the arc whose tangent is

$$\Theta(z) = \frac{-2xy}{x^2 - y^2}. \qquad \blacksquare$$

Let L_{x_1} denote the isometry in $\widetilde{PSL}_2(\mathbb{R},\tau)$ generated by I_{x_1}. Let us identify $\mathbb{H}^2$ with $\mathbb{H}^2 \times \{t = 0\}$ in $\widetilde{PSL}_{(2,\mathbb{H}^2)}(\mathbb{R},\tau)$. The next lemma explains how is the image of $\gamma_\theta(x_1)$ under the isometry L_{x_1}.

Lemma 3.5. Fix $\theta_0 \in (0, \pi/2)$ and $x_1 > 0$. The image of $\gamma_{\theta_0}(x_1)$ under L_{x_1} is an arc joining the points $(0,0,0)$ and $(0, x_1, -4\tau\theta_0)$ whose projection on $\mathbb{H}^2$ is the curve $\beta_{\theta_0}(0, x_1)$.

Proof. First, observe that

$$\Theta|_{\gamma_{\theta_0}(x_1)}(z) = \frac{-2x^2 \tan\theta_0 + 2xx_1 \tan\theta_0}{x^2(1 - \tan^2\theta) + x_1 \tan^2\theta(2x - x_1)},$$

so

$$\lim_{x\to x_1} \Theta|_{\gamma_{\theta_0}(x_1)}(z) = 0 \quad \text{and} \quad \lim_{x\to+\infty} \Theta|_{\gamma_{\theta_0}(x_1)}(z) = -\tan 2\theta_0. \tag{3.3}$$

Moreover,

$$\pi \circ L_{x_1}(\gamma_{\theta_0}(x_1)) = I_{x_1}(\gamma_{\theta_0}(x_1)) = \beta_{\theta_0}(0, x_1). \qquad \blacksquare$$

4 Graphs in $\widetilde{PSL}_2(\mathbb{R},\tau)$

In this section we will give the definitions of *vertical* and *horizontal* graphs.

Given a domain $\Omega \subset \mathbb{M}$, a section of the Riemannian submersion π is a map

$$s : \Omega \subset \mathbb{M} \longrightarrow \widetilde{PSL}_2(\mathbb{R},\tau)$$

such that $\pi \circ s = id_{\mathbb{M}}|_\Omega$, where $id_{\mathbb{M}}|_\Omega$ is the identity map on $\mathbb{M}$ restrict to Ω.

Definition 4.1. (Vertical graph) A vertical graph in $\widetilde{PSL}_2(\mathbb{R},\tau)$ is the image of a section of the Riemannian submersion $\pi : \widetilde{PSL}_2(\mathbb{R},\tau) \longrightarrow \mathbb{M}$.

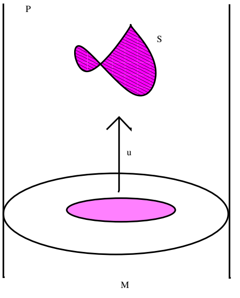

Figure 2: Vertical graph.

Given a domain $\Omega \subset \mathbb{M}$ we also denote by Ω its lift to $\mathbb{M} \times \{0\}$. With this identification, given a function $u : \Omega \to \mathbb{R}$, the set

$$\Sigma(u) = \{(x, y, u(x, y)) \in \widetilde{PSL}_2(\mathbb{R}, \tau); (x, y) \in \Omega\},$$

is a vertical graph which we call the vertical graph of u. Reciprocally, given a vertical graph S, we can find a function $u : \Omega \to \mathsf{R}$ over a domain $\Omega \subset \mathbb{M} \times \{0\}$ such that $S = \Sigma(u)$.

If the vertical graph $\Sigma(u)$ has constant mean curvature (CMC) H, u satisfies the following second order elliptic partial differential equation

$$L_H(u) := div_{\mathbb{H}^2}\left(\frac{\alpha}{W}e_1 + \frac{\beta}{W}e_2\right) - 2H = 0, \tag{4.1}$$

where H is the mean curvature function with respect to the upward pointing normal vector, $W = \sqrt{1 + \alpha^2 + \beta^2}$,

- $\alpha = \dfrac{u_x}{\lambda} + 2\tau\dfrac{\lambda_y}{\lambda^2}$,
- $\beta = \dfrac{u_y}{\lambda} - 2\tau\dfrac{\lambda_x}{\lambda^2}$.

Now we define horizontal graphs.

Definition 4.2. (Horizontal graph) Let $\Omega \subset \{(x,0,t) \in \partial_\infty \mathbb{H}^2 \times \mathbb{R}\}$ be a domain and $y = f(x,t)$ be a smooth positive function, that is $f(x,t) > 0$ for all $(x,t) \in \Omega$. The horizontal graph of f, denoted by $\Sigma(f)$, is the set defined by

$$\Sigma(f) := \{(x, f(x,t), t) \in \widetilde{PSL}_2(\mathbb{R},\tau); (x,t) \in \Omega\}.$$

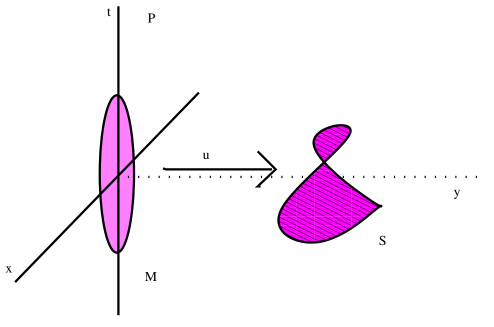

Figure 3: Horizontal graph.

Denote by N the unit normal vector to $\Sigma(f)$ such that $g(N, E_2) > 0$ and by H the mean curvature of $\Sigma(f)$ with respect to N. The mean curvature equation for horizontal graphs is given in the following lemma.

Lemma 4.3. Suppose the H is the mean curvature function of a horizontal graph $\Sigma(f)$. Then, the function f satisfies the equation

$$\frac{2HW^3}{f^2} = (f^2+f_t^2)f_{xx} - 2(f_x f_t - 2\tau f)f_{xt} + ((1+4\tau^2)+f_x^2)f_{tt} + f(1+f_x^2) + 2\tau f_x f_t,$$

where $W = \sqrt{f^2 + f_t^2 + f^2\left(f_x + \dfrac{2\tau f_t}{f}\right)^2}$.

5 Horizontal graphs

In this section, we deal with non-existence of horizontal graphs. First, we will use a family of minimal surfaces invariant by hyperbolic isometries to show that there is no entire horizontal graph. After that, we will consider horizontal minimal graphs defined over domains bounded by Jordan curves. Using a family of minimal surfaces invariant by parabolic isometries, we prove that for some Jordan curves such a horizontal graph cannot exist.

We recall that the space $\mathbb{H}^2$ in Euclidean coordinates is given by

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2; y > 0\}$$

endowed with the metric

$$d\sigma^2 = \lambda^2(dx^2 + dy^2)$$

where $\lambda = y^{-1}$. Changing for polar coordinates having pole at the point $(x_1, 0)$, that is

$$\begin{cases} (x - x_1) &=& e^{\varphi}\cos\theta, \\ y &=& e^{\varphi}\sin\theta, \end{cases}$$

where $-\infty < \varphi < +\infty$ and $0 < \theta < \pi$, the metric is given by

$$\frac{1}{\sin^2(\theta)}\left(d\varphi^2 + d\theta^2\right).$$

In this polar coordinates system, $\gamma(x_1)$ is given by $\{\theta = \pi/2\}$. Recall we have denoted by $D_1(x_1, x_2)$ the unbounded domain whose boundary is $\beta(x_1, x_2) \cup \Lambda(x_1, x_2)$ and $D_2(x_1, x_2) = \mathbb{H}^2 - (D_1(x_1, x_2) \cup \beta(x_1, x_2))$.

For each $x_1 > 0$, we denote by $\Omega_\theta(x_1)$ the region in $\mathbb{H}^2$ having boundary $\gamma_\theta(x_1) \cup \Gamma(x_1)$, where $\Gamma(x_1) = \{(x, 0); x > x_1\}$.

Now, we give the geometric description of a family of complete minimal vertical graphs which are invariant by one parameter family of hyperbolic isometries in $\widetilde{PSL}_{(2,\mathbb{H}^2)}(\mathbb{R}, \tau)$. This family was studied in [4], where the second author construct families of mean curvature surfaces invariant by one parameter group of isometries. For completeness, we describe a family which we will use to prove the non-existence of entire minimal horizontal graphs.

Proposition 5.1. *Let d be a real number. Consider the functions*

$$u_d^+(\theta) = \int_0^\theta \frac{d\sqrt{1+4\tau^2\cos^2(\nu)}}{\sqrt{1-d^2\sin^2(\nu)}}d\nu - 2\tau\theta, \quad 0 < \theta < \arcsin d^{-1} \tag{5.1}$$

and

$$u_d^-(\theta) = -\int_0^\theta \frac{d\sqrt{1+4\tau^2\cos^2(\nu)}}{\sqrt{1-d^2\sin^2(\nu)}}d\nu - 2\tau\theta, \quad 0 < \theta < \arcsin d^{-1}. \tag{5.2}$$

The vertical graphs of $u_d^+(\theta)$ and $u_d^-(\theta)$ are vertical minimal graphs which are invariant by hyperbolic translations along $\gamma(x_1)$ (recall that the polar coordinates is centered at $(x_1,0)$). Furthermore:

1. *If $d>1$, $u_d^+(\theta)$ and $u_d^-(\theta)$ give rise to a complete minimal surface $S_d(x_1)$ whose vertical projection on $\mathbb{H}^2$ is $\Omega_{\arcsin(1/d)}(x_1)$.*

   Defining

   $$h(d) := \int_0^{\arcsin(d^{-1})} \frac{d\sqrt{1+4\tau^2\cos^2(\nu)}}{\sqrt{1-d^2\sin^2(\nu)}}d\nu - 2\tau\arcsin(d^{-1}),$$

   we have

   $$\lim_{d\to\infty} h(d) = \frac{\pi}{2}\sqrt{1+4\tau^2}, \quad \lim_{d\to 1^+} h(d) = +\infty.$$

   Moreover the surface $S_d(x_1)$ is symmetric with respect to the slice $\{\mathbb{H}^2\times h(d)\}$ The asymptotic boundary of $S_d(x_1)$ is the union of $\{(x,0,t); t=0 \quad \text{or} \quad t=2h(d), x\geq x_1\}$ and the vertical segment joining the end points of these arcs.

2. *If $d=1$, $u_1^+(\theta)$ $(u_1^-(\theta))$ gives rise to a complete minimal vertical graph $S_1^+(x_1)$ (S_1^-) defined over $\Omega_{\pi/2}(x_1)$. The function $u_1^+(\theta)$ $(u_1^-(\theta))$ goes to 0 (0) on $\Gamma(x_1)$ and goes to $+\infty$ $(-\infty)$ on $\gamma(x_1)$. The asymptotic boundary of this surface is given by $\Gamma(x_1)$ and the vertical semi-lines in $\{(x,0,t); t\geq 0\}$ $(\{(x,0,t); t\leq 0\})$ leaving from the end points of $\Gamma(x_1)$. These surfaces are called Scherk type surfaces.*

3. *Finally, if $0<d<1$, the vertical graphs of u_d^+ and u_d^- are entire minimal vertical graphs.*

Proof. 1. The functions $u_d^+(\theta)$ and $u_d^-(\theta)$ are defined for $0 < \theta < \arcsin d^{-1}$. We have $u_d^-(\arcsin d^{-1}) = u_d^+(\arcsin d^{-1}) - 2h(d) - 4\tau \arcsin d^{-1}$, and over the equidistant curve $\gamma_{\arcsin 1/d}(x_1)$ the graphs of $u_d^+(\theta)$ and $u_d^-(\theta)$ have vertical tangent planes. So, after a vertical translation by $-2h(d)-4\tau \arcsin d^{-1}$ of the graph of $u_d^-(\theta)$, the union of these two graphs generates a minimal surface, which is denoted by $S_d(x_1)$ and is symmetric with respect to the plane $\mathbb{H}^2 \times h(d)$.

Now let us study the behavior of $h(d)$ when d goes to $+\infty$. Making a change of variables $v = d \sin \nu - 1$, we obtain

$$\begin{aligned} h(d) &= \int_{-1}^{0} \frac{\sqrt{d^2(1+4\tau^2) - 4\tau^2(v+1)^2}}{\sqrt{d^2(1-(v+1)^2) - (v+1)^2(1-(v+1)^2)}} dv \\ &\quad -2\tau \arcsin\left(\frac{1}{d}\right). \end{aligned}$$

Then, when d goes to $+\infty$, we have

$$\lim_{d \to \infty} h(d) = \frac{\pi}{2}\sqrt{1+4\tau^2}.$$

When $d > 1$ goes to 1, we have that $\lim_{d \to 1^+} h(d) = +\infty$. In fact, since $d > 1$,

$$\begin{aligned} h(d) &= \int_0^{\arcsin(\frac{1}{d})} \frac{d\sqrt{1+4\tau^2\cos^2(\nu)}}{\sqrt{1-d^2\sin^2(\nu)}} d\nu - 2\tau \arcsin\left(\frac{1}{d}\right) \\ &\geq \int_0^{\arcsin(\frac{1}{d})} \frac{1}{\sqrt{1-d^2\sin^2(\nu)}} d\nu - 2\tau \arcsin\left(\frac{1}{d}\right) \\ & \boxed{v = d\sin\theta} \\ &= \int_0^{1} \frac{1}{\sqrt{d^2-v^2}\sqrt{1-v^2}} dv - 2\tau \arcsin\left(\frac{1}{d}\right), \end{aligned}$$

then, when $d \to 1^+$, we have

$$\begin{aligned}\lim_{d\to 1^+} h(d) &= \int_0^1 \frac{1}{\sqrt{d^2-v^2}\sqrt{1-v^2}}dv - 2\tau \arcsin\left(\frac{1}{d}\right)\\ &= \int_0^1 \frac{1}{1-v^2}dv - 2\tau\arcsin\left(\frac{1}{d}\right)\\ &= -\frac{1}{2}\ln\left(\frac{1-v}{v+1}\right)\Big|_0^1 - 2\tau\arcsin\left(\frac{1}{d}\right)\\ &= +\infty.\end{aligned}$$

2. When $d = 1$,

$$u_1^+(\theta) = \int_0^\theta \frac{\sqrt{1+4\tau^2\cos^2(\nu)}}{\sqrt{1-\sin^2(\nu)}}d\nu - 2\tau\theta, \qquad 0 \le \theta < \frac{\pi}{2}.$$

   By continuity, $h(1) = \lim_{d\to 1^+} h(d) = +\infty$. Then, $u_1^+(\theta)$ takes boundary values 0 on the asymptotic boundary $\Gamma(x_1)$ and $u_1^+(\theta)$ goes to $+\infty$ when θ goes to $\pi/2$.

   Similarly, $u_1^-(\theta)$ takes boundary values 0 on the asymptotic boundary $\Gamma(x_1)$ goes to $-\infty$ when θ goes to $\pi/2$.

3. Finally, when $0 < d < 1$, u_d^+ and u_d^+ define entire minimal vertical graphs. ∎

Corollary 5.1. Following the notation on Proposition 5.1, given $d > 1$, there exists a minimal surface $M_d(0,x_1)$ having asymptotic boundary $\widetilde{R}_d(0,x_1)$, where $\widetilde{R}_d(0,x_1)$ is the rectangle given by the union $\{0 \le x \le x_1, y = 0, t = 0 \text{ or } t = 2h(d)\} \cup \{x = 0 \text{ or } x = x_1, y = 0, 0 \le t \le 2h(d)\}$.

This surface is continuous up to its boundary. Furthermore,

$$\lim_{d\to\infty} h(d) = \frac{\pi}{2}\sqrt{1+4\tau^2}, \quad \lim_{d\to 1^+} h(d) = +\infty.$$

The projection of $M_d(0,x_1)$ over $\mathbb{H}^2 \times \{t = 0\}$ is the region $D_{1,\theta}(0,x_1)$.

In the case $d = 1$, the Scherk type surface $M_1^+(0,x_1)$ takes zero value on $\Lambda(0,x_1)$ and $+\infty$ on the geodesic $\beta(0,x_1)$ and its projection on $\mathbb{H}^2$ is

$D_1(0, x_1)$. Similarly, the Scherk type surface $M_1^-(0, x_1)$ takes zero value on $\Lambda(0, x_1)$ and $-\infty$ on the geodesic $\beta(0, x_1)$ and its projection on $\mathbb{H}^2$ is $D_1(0, x_1)$.

Proof. The minimal surfaces $M_d(0, x_1)$, $M_1^+(0, x_1)$ and $M_1^-(0, x_1)$ are the image of the complete surface $S_d(x_1)$, $S_1^+(x_1)$ and $S_1^-(x_1)$, respectively, described in Lemma 5.1 by the isometry $L(x_1)$. Lemmas 3.5 and 3.4 guaranty the asymptotic behavior.

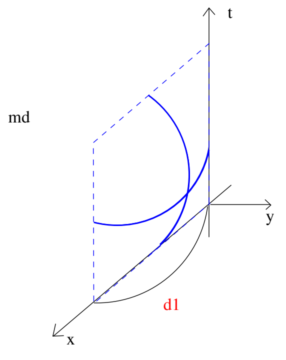

Figure 4: The surface $M_d(0, x_1)$, $d > 1$ and its projection $D_{1,\theta}(0, x_1)$.

$\blacksquare$

The geometric behavior of the family of minimal surfaces $M_d(0, x_1)$, $d > 1$ allow us to prove the following result.

Theorem 5.2. *There is no horizontal entire minimal graphs in* $\widetilde{PSL}_{(2,\mathbb{H}^2)}(\mathbb{R}, \tau)$.

Proof. We argue by contradiction. Assume that there is an entire horizontal minimal graph Σ. By the definition of horizontal graphs, we have that

$$\partial_\infty \Sigma \cap \partial_\infty M_d(0, x_1) = \emptyset,$$

for all $d > 1$. We fix $d > 1$ and $x_1 > 0$, then either $\Sigma \cap M_d(0, x_1) = \emptyset$ or $\Sigma \cap M_d(0, x_1) \neq \emptyset$.

Let $\gamma(x_1/2) = \{(x_1/2, y, 0) \in \widetilde{PSL}_{(2,\mathbb{H}^2)}(\mathbb{R},\tau); y > 0\}$ be a complete geodesic. Let $T_\xi(x,y,t) := \xi(x,y,0) + (x_1/2, 0, t)$ be a hyperbolic isometry of $\widetilde{PSL}_{(2,\mathbb{H}^2)}(\mathbb{R},\tau)$ induced by hyperbolic translations along $\gamma(x_1/2)$ of factor ξ.

If $\Sigma \cap M_d(0,x_1) = \emptyset$, we apply the isometry T_ξ, for $\xi > 1$. Observe that $\pi \circ T_\xi(M_d(0,x_1)) \subset \pi \circ T_{\xi'}(M_d(0,x_1))$, for all $\xi \leq \xi'$. Moreover, when ξ goes to $+\infty$, the projection $\pi \circ T_\xi(M_d(0,x_1))$ goes to $\mathbb{H}^2$. So, there exists a ξ_0 such that $T_{\xi_0}(M_d(0,x_1))$ has a first contact point with Σ, that is, if $\xi < \xi_0$, $\Sigma \cap T_\xi(M_d(0,x_1)) = \emptyset$ and $\Sigma \cap T_{\xi_0}(M_d(0,x_1)) \neq \emptyset$. Then at all points on the intersection $\Sigma \cap T_{\xi_0}(M_d(0,x_1))$ the surfaces Σ and $T_{\xi_0}(M_d(0,x_1))$ are tangent. Applying the maximum principle, we would conclude that $\Sigma = T_{\xi_0}(M_d(0,x_1))$. But this contradicts the fact that they don't have the same asymptotic boundary.

Similarly, if $\Sigma \cap M_d(0,x_1) \neq \emptyset$, we apply the isometry T_ξ, for $\xi < 1$. Observe that when ξ goes to 0, the projection $\pi \circ T_\xi(M_d(0,x_1))$ degenerates at the point $(x_1/2, 0, 0)$. So, there exists a ξ_0 such that $T_{\xi_0}(M_d(0,x_1))$ has a last contact point with Σ, that is, if $\xi < \xi_0$, $\Sigma \cap T_\xi(M_d(0,x_1)) = \emptyset$ and $\Sigma \cap T_{\xi_0}(M_d(0,x_1)) \neq \emptyset$. Then, at all points on the intersection $\Sigma \cap T_{\xi_0}(M_d(0,x_1))$ the surfaces Σ and $T_{\xi_0}(M_d(0,x_1))$ are tangent. Again, we obtain a contradiction by the maximum principle. ■

Now we focus our attention in the family of vertical minimal graphs which are invariant by parabolic isometries, such graphs are vertical graphs of functions [4, Lemma 4.2]

$$v_d(x,y) = v_d(y) = \sqrt{1+4\tau^2}\,\arcsin(dy), \quad d \in \mathbb{R},$$

v_d is defined over $\{(x,y,0) \in \widetilde{PSL}_{(2,\mathbb{H}^2)}(\mathbb{R},\tau); 0 < y < d^{-1}\}$. The graph of v_d has vertical tangent plane at $(x, d^{-1}, \sqrt{1+4\tau^2}\pi/2)$, for all $x \in \mathbb{R}$. Fix $d \neq 0$, after a rotation by π around the geodesic $\gamma(x_1)$, we obtain a complete minimal surface A_d. In the next lemma, we summarize the properties of A_d which we will use later.

Lemma 5.3. Let A_d the family of complete minimal surfaces invariant by parabolic isometry described above. Then for each $d \neq 0$, the surface A_d has the following properties

1. When d goes to $+\infty$, A_d goes to the slices $\{t = 0\} \cup \{t = \sqrt{1 + 4\tau^2}\pi\}$.

2. When d goes to 0, A_d goes to the slab

$$\{(x, 0, t), 0 \leq t \leq \sqrt{1 + 4\tau^2}\pi\}.$$

3. The asymptotic boundary of A_d are the lines $\{(x, 0, 0)\} \cup \{(x, 0, \sqrt{1 + 4\tau^2}\pi)\}$ and vertical segments joining their end points.

Proof. The proof follows from the analysis of the function $\arcsin(dy)$, taking into account that the surface A_d is invariant by parabolic isometries. ∎

Let Ω be a bounded domain in $\{(x, 0, t); x, t \in \mathbb{R}\}$ having smooth boundary. For each $x \in \mathbb{R}$, let

$$C(x) := \{t \in \mathbb{R}; (x, 0, t) \in \partial\Omega\}$$

and

$$l(x) := \sup_{C(x)} \{|t_i - t_j|\}.$$

We define the *width* of $\partial\Omega$ with respect to the t-axes by

$$w(x) = \sup_{x \in \mathbb{R}} l(x).$$

Now we will use the family of minimal surfaces A_d to prove that there is no horizontal minimal graph defined over a bounded domain whose boundary has width less than $\sqrt{1 + 4\tau^2}\pi$.

Theorem 5.4. *Let Ω be a domain in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ bounded by a Jordan curve $\partial\Omega \subset \partial_\infty \mathbb{H}^2 \times \mathbb{R}$ having width less than $\sqrt{1 + 4\tau^2}\pi$. Then there is no horizontal minimal graph defined in Ω continuous up to its asymptotic boundary.*

Proof. Suppose that there exists such a horizontal minimal graph Σ. After a vertical translation, we can assume that Ω is inside the simply connected region bounded by the horizontal planes $\{t=\sqrt{1+4\tau^2}\pi\}\cup\{t=0\}$. We fix $d>0$ large enough, such that A_d does not intersect the surface Σ. It is possible since when $d\to+\infty$ the surface A_d goes to the horizontal planes $\{t=\sqrt{1+4\tau^2}\pi\}\cup\{t=0\}$. Moreover, when d goes to 0, the surface A_d goes to $\{(x,0,t)\in\widetilde{PSL}_{(2,\mathbb{H}^2)}(\mathbb{R},\tau); 0<t<\sqrt{1+4\tau^2}\pi\}$. So, letting d decreasing to 0, it would occur a first contact point between A_d and Σ. And by the maximum principle, we would conclude that these surfaces are equal. The contradiction consists in the fact that these surfaces do not have the same asymptotic boundary, so, they can not be the same. ∎

6 Vertical graphs

In $\mathbb{R}^3$, the Berstein's theorem states that an entire minimal graph is a plane. In [3, Theorem 4], the authors proved that given a rectifiable Jordan curve in the asymptotic boundary of $\mathbb{D}^2\times\mathbb{R}$ there is a unique entire minimal graph having this Jordan curve as asymptotic boundary. We want to extend this theorem for $\widetilde{PSL}_2(\mathbb{R},\tau)$. Before this, we have to fix some notation.

We recall some notations about the Scherk's surfaces described in Lemma 5.1 and Corollary 5.1. We have denoted $S_1^+(x_1)$ $(S_1^-(x_1))$ the graphs over the unbounded domain $\Omega_{\pi/2}(x_1)$ whose boundary values are $+\infty$ $(-\infty)$ over $\gamma(x_1)$ and zero (zero) over $\Gamma(x_1)$. The image of $S_1^+(x_1)$ $(S_1^-(x_1))$ by the isometry L_{x_1} is denoted by $M_1^+(0,x_1)$ $(M_1^-(0,x_1))$, the projection over $\mathbb{H}^2$ of $M_1^+(0,x_1)$ $(M_1^-(0,x_1))$ is $D_1(0,x_1)$. After a parabolic isometry on $\widetilde{PSL}_{(2,\mathbb{H}^2)}(\mathbb{R},\tau)$, we ca assume that the projection of the Scherk surface $M_1^+(0,x_1)$ is a domain $D_1(x,\widetilde{x})$, where $\widetilde{x}-x=x_1$. The image of $M_1^+(0,x_1)$ by this parabolic isometry is denoted by $M_1^+(x,\widetilde{x})$. Since $x_1>0$ is arbitrary, we can choose $x,\widetilde{x}$, $x<\widetilde{x}$ arbitrarily. Similarly, we define $M_1^-(x,\widetilde{x})$.

Theorem 6.1. *Consider the cylinder model for $\widetilde{PSL}_2(\mathbb{R},\tau)$. Let $f:S^1\to$*

$\mathbb{R}$ *be a smooth function. There exist a unique function* $u : \mathbb{D}^2 \to \mathbb{R}$ *having boundary value* f *and whose vertical graph is a minimal surface.*

Proof. We start extending f smoothly to $\mathbb{D}^2$. For instance, let $\overline{u} : \mathbb{D}^2 \to \mathbb{R}$ the function having boundary value f and whose graph in the product space $\mathbb{D}^2 \times \mathbb{R}$ is minimal, the existence of such $\overline{u}$ is proved in [3, Theorem 4].

Now let Ω_n be the disc centered at the origin having hyperbolic radius n. Let $u_n : \Omega_n \to \mathbb{R}$ be the unique function whose vertical graph is minimal and $u_n(x, y) = \overline{u}(x, y)$ for all (x, y) in $\partial\Omega_n$. The existence of such u_n is given in [8].

We have to prove that the sequence $\{u_n\}$ converges to a function $u : \mathbb{D}^2 \to \mathbb{R}$ having boundary value f. First we prove that the sequence $\{u_n\}$ is uniformly bounded. For this, let J be the isometry given in Lemma 3.2. Fix $x > 1$, we define $\Sigma^+ := J^{-1}(M_1^+(x, x))$ and $\Sigma^- := J^{-1}(M_1^-(x, x))$. Let R_π be the elliptic isometry of $\widetilde{PSL}_{(2,\mathbb{D}^2)}(\mathbb{R}, \tau)$ generated by the rotation around the origin on $\mathbb{D}^2$. Set $C_0 := \max_{z\in S^1} f(z)$ and $C_1 := \min_{z\in S^1}(f(z))$. We observe that $\{u_n|_{\pi(\Sigma^+)}\}$ is uniformly bounded above by $C_0 + \Sigma^+$ by maximum principle, the maximum principle can be applied since the minimal equation can be extended to the asymptotic boundary. Also, $\{u_n|_{\pi(\Sigma^+)}\}$ is uniformly bounded bellow by $C_1 + \Sigma^-$ by the maximum principle. The same reasoning can be applied to $\{u_n|_{\pi(R_\pi(\Sigma^+))}\}$. So, we conclude that the sequence $\{u_n\}$ is uniformly bounded.

Now we prove that u has boundary value f. Fix $p \in S^1$, given $\varepsilon > 0$, by the continuity of f, there exists $\delta_1 > 0$ such that for all $q \in S^1$ such that $|p - q|_{S^1} < \delta_1$, where $|.|_{S^1}$ denotes the metric on S^1, then

$$|f(q) - f(p)| < \varepsilon/2.$$

On the other hand, let $\pi_3; \widetilde{PSL}_2(\mathbb{R}, \tau) \to \mathbb{R}$ be the projection on the third factor, that is $\pi_3(x, y, t) = t$. Define $(x_0, 0, t_0) := J(p, f(p) + 3\varepsilon/4)$, where J is the isometry define on Lemma 3.2. Then by continuity of J^{-1}, there exists $\delta_2 > 0$, such that if $|x - x_0| < \delta_2$, then

$$|\pi_3(J^{-1}(x_0, 0, t_0)) - \pi_3(J^{-1}(x, 0, t_0))| < \varepsilon/4.$$

Then,

$$\pi_3(J^{-1}(x,0,t_0)) > f(p) + \varepsilon/2, \tag{6.1}$$

for all $(x,0,t_0)$ such that $|x-x_0| < \delta_2$.

Let $Q_\nu(x,y,t) := (x,y,t+\nu)$ be the vertical translation. Taking δ_1 smaller, if necessary, inequality (6.1) guaranties that the surface

$$J^{-1}(Q_{t_0}(M_1^+(x_0-\delta_2, x_0+\delta_2)))$$

is above u near $(p, f(p))$.

Similar argument prove that if $(x_1,0,t_1) := J(p, f(p-3\varepsilon/4))$ the surface

$$J^{-1}(Q_{t_1}(M_1^-(x_1-\delta_2, x_1+\delta_2)))$$

is bellow u near $(p, f(p))$. The arbitrariness of $p \in S^1$ and $\varepsilon > 0$, ensures that u takes boundary value f.

Assume that there are u_1 and u_2 two smooth functions having prescribed boundary values f and whose graphs are minimal. We can translate vertically the graph of u_1 upward such that the translated graph of u_1 has no intersection with the graph of u_2. Then, we translate the new graph of u_1 downward, the maximum principle assures that the first contact point occur at the boundary, and the conclusion is that $u_2 \leq u_1$, for all points in $\mathbb{D}^2$. Similarly, translating downward the graph of u_1, we conclude that $u_2 \geq u_1$, for all points in $\mathbb{D}^2$. So $u_1 \equiv u_2$. ■

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Abigail Folha
Instituto de Matemática e Es–tatística - Departamento de Geometria, Universidade Fede–ral Fluminense,
Brazil
abigailfolha@vm.uff.br

Carlos Penãfiel
Instituto de Matemática e Es–tatística, Universidade Fede–ral de Rio de Janeiro,
Brazil
penafiel@im.ufrj.br