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The asymptotic Plateau problem for convex hypersurfaces of constant curvature in Hyperbolic space

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Abstract

We show that for a very general class of curvature functions defined in the positive cone, the problem of finding a complete strictly locally convex hypersurface in $\mathbb{H}^{n+1}$ satisfying $f(\kappa) = \sigma \in (0,1)$ with a prescribed asymptotic boundary Γ at infinity has at least one smooth solution with uniformly bounded hyperbolic principal curvatures. Moreover if Γ is (Euclidean) starshaped, the solution is unique and also (Euclidean) starshaped while if Γ is mean convex the solution is unique. We also show via a strong duality theorem that analogous results hold in De Sitter space. A novel feature of our approach is a "global interior curvature estimate".

1 Introduction

The asymptotic Plateau problem for complete strictly locally convex hypersurfaces of constant Gauss curvature was initiated by Labourie [9] in $\mathbb{H}^3$ and by Rosenberg-Spruck [11] in $\mathbb{H}^{n+1}$ and subsequently extended to more general curvature functions in [5], [6], [7], [12]. In this survey

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paper (which is based on joint work with Bo Guan and Ling Xiao), we will sketch the complete solution (Theorem 1.3) to the asymptotic Plateau problem for locally strictly convex hypersurfaces of constant curvature for essentially arbitrary "elliptic curvature functions". A novel feature of our recent work [8] is the derivation of a "global interior curvature bound" (Theorem 1.2) that besides yielding optimal existence allows us to infer that the convex solutions are starshaped for sharshaped asymptotic boundary (Theorem 1.4) and unique for mean convex asymptotic boundary (Theorem 1.5).

Given $\Gamma \subset \partial_\infty \mathbb{H}^{n+1}$ and a smooth symmetric function f of n variables, the *asymptotic Plateau problem* is to find a complete locally strictly convex hypersurface Σ in $\mathbb{H}^{n+1}$ satisfying

$$f(\kappa[\Sigma]) = \sigma \tag{1}$$

$$\partial\Sigma = \Gamma \tag{2}$$

where $\kappa[\Sigma] = (\kappa_1, \dots, \kappa_n)$ denotes the induced (positive) hyperbolic principal curvatures of Σ and $\sigma \in (0,1)$ is a constant.

The function f is to satisfy the standard structure conditions [2] in the positive cone

$$K = K_n^+ := \big\{\lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0\big\}, \tag{3}$$

$$f > 0 \ \text{ in } K, \tag{4}$$

$$f = 0 \ \text{ on } \partial K, \tag{5}$$

$$f_i(\lambda) \equiv \frac{\partial f(\lambda)}{\partial \lambda_i} > 0 \ \text{ in } K, \ \ 1 \leq i \leq n, \tag{6}$$

$$f \text{ is a concave function in } K. \tag{7}$$

In addition, we assume that f is normalized

$$f(1, \dots, 1) = 1 \tag{8}$$

and satisfies

$$f \text{ is homogeneous of degree one.} \tag{9}$$

By contrast we will drop the following more technical assumption of [5], [6], [7], [12]:

$$\lim_{R\to+\infty} f(\lambda_1, \cdots, \lambda_{n-1}, \lambda_n + R) \geq 1 + \varepsilon_0 \quad \text{uniformly in } B_{\delta_0}(\mathbf{1}) \tag{10}$$

for some fixed $\varepsilon_0 > 0$ and $\delta_0 > 0$, where $B_{\delta_0}(\mathbf{1})$ is the ball of radius δ_0 centered at $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$. This technical condition is the main assumption used in the proof of boundary estimates.

We will use the half-space model

$$\mathbb{H}^{n+1} = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$$

equipped with the hyperbolic metric

$$ds^2 = \frac{\sum_{i=1}^{n+1} dx_i^2}{x_{n+1}^2}. \tag{11}$$

Thus $\partial_\infty \mathbb{H}^{n+1}$ is naturally identified with $\mathbb{R}^n = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ and (2) may be understood in the Euclidean sense. For convenience we say Σ has compact asymptotic boundary if $\partial\Sigma \subset \partial_\infty \mathbb{H}^{n+1}$ is compact with respect to the Euclidean metric in $\mathbb{R}^n$.

In this paper all hypersurfaces in $\mathbb{H}^{n+1}$ we consider are assumed to be connected and orientable. If Σ is a complete hypersurface in $\mathbb{H}^{n+1}$ with compact asymptotic boundary at infinity, then the normal vector field of Σ is chosen to be the one pointing to the unique unbounded region in $\mathbb{R}^{n+1}_+ \setminus \Sigma$, and the (both hyperbolic and Euclidean) principal curvatures of Σ are calculated with respect to this normal vector field.

As in our earlier work we will take $\Gamma = \partial\Omega$ where $\Omega \subset \mathbb{R}^n$ is a smooth domain and seek Σ as the graph of a function $u(x)$ over Ω, i.e.

$$\Sigma = \{(x, x_{n+1}) : x \in \Omega,\ x_{n+1} = u(x)\}.$$

Then the coordinate vector fields and upper unit normal are given by

$$X_i = e_i + u_i e_{n+1},\ \mathbf{n} = u\nu = u\frac{(-u_i e_i + e_{n+1})}{w},$$

where $w = \sqrt{1+|\nabla u|^2}$ and ν is the Euclidean upward unit normal to Σ. The first fundamental form g_{ij} is then given by

$$g_{ij} = \langle X_i, X_j \rangle = \frac{1}{u^2}(\delta_{ij} + u_i u_j) = \frac{g^e_{ij}}{u^2}\ . \tag{12}$$

To compute the second fundamental form h_{ij} we use

$$\Gamma^k_{ij} = \frac{1}{x_{n+1}}\,(-\delta_{jk}\delta_{in+1} - \delta_{ik}\delta_{jn+1} + \delta_{ij}\delta_{kn+1}) \tag{13}$$

to obtain

$$\nabla_{X_i} X_j = (\frac{\delta_{ij}}{x_{n+1}} + u_{ij} - \frac{u_i u_j}{x_{n+1}})e_{n+1} - \frac{u_j e_i + u_i e_j}{x_{n+1}}\ . \tag{14}$$

Then

$$\begin{aligned} h_{ij} &= \langle \nabla_{X_i} X_j, u\nu \rangle = \frac{1}{uw}(\frac{\delta_{ij}}{u} + u_{ij} - \frac{u_i u_j}{u} + 2\frac{u_i u_j}{u}) \\ &= \frac{1}{u^2 w}(\delta_{ij} + u_i u_j + u u_{ij}) = \frac{h^e_{ij}}{u} + \frac{\nu^{n+1}}{u^2} g^e_{ij}. \end{aligned} \tag{15}$$

The hyperbolic principal curvatures κ_i of Σ are the roots of the characteristic equation

$$\det(h_{ij} - \kappa g_{ij}) = u^{-n}\det(h^e_{ij} - \frac{1}{u}(\kappa - \frac{1}{w})g^e_{ij}) = 0.$$

Therefore,

$$\kappa_i = u\kappa^e_i + \nu^{n+1}. \tag{16}$$

The relations (15) and (16) are easily seen to hold for parametric hypersurfaces.

One important consequence of (16) is the following result of [5].

Theorem 1.1. Let Σ be a complete locally strictly convex C^2 hypersurface in $\mathbb{H}^{n+1}$ with compact asymptotic boundary at infinity. Then Σ is the (vertical) graph of a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $u > 0$ in Ω and $u = 0$ on $\partial\Omega$, for some domain $\Omega \subset \mathbb{R}^n$:

$$\Sigma = \left\{(x, u(x)) \in \mathbb{R}^{n+1}_+ : x \in \Omega\right\}$$

such that

$$\{\delta_{ij} + u_i u_j + u u_{ij}\} > 0 \ \ in\ \Omega. \tag{17}$$

That is, the function $u^2 + |x|^2$ is strictly (Euclidean) convex.

According to Theorem 1.1, our assumption that Σ is a graph is completely general for locally strictly convex hypersurfaces, that is, the asymptotic boundary Γ must be the boundary of some bounded domain Ω in $\mathbb{R}^n$.

In our recent paper [8], we discovered a new phenomenon of "convexity arising from infinity" that forces the principal curvatures of solutions to the asymptotic problem to be uniformly bounded. This leads to substantial improvements of our earlier results for the convex cone K_n^+. The main new technical idea is a global curvature estimate for locally strictly convex solutions of (1), (2) *which is obtained from interior curvature estimates.* More precisely we have

Theorem 1.2 ([8]). Let $\Gamma = \partial\Omega \times \{0\} \subset \mathbb{R}^{n+1}$ where Ω is a bounded smooth domain in $\mathbb{R}^n$. Suppose $\sigma \in (0, 1)$ and that f satisfies conditions (3)- (4), (6)-(9). Let $\Sigma = \mathrm{graph}(u)$ be a smooth locally strictly convex graph in $\mathbb{H}^{n+1}$ satisfying $f(\kappa) = \sigma$, $\partial_\infty\Sigma = \Gamma$ and

$$\nu^{n+1} \geq 2a > 0 \ on\ \Sigma. \tag{18}$$

For $\mathbf{x} \in \Sigma$ let $\kappa_{\max}(\mathbf{x})$ be the largest principal curvature of Σ at $\mathbf{x}$. Then for $0 < b \leq \frac{a}{4}$,

$$\max_\Sigma u^b \frac{\kappa_{\max}}{\nu^{n+1} - a} \leq \frac{8}{a^{\frac{5}{2}}} (\sup_\Sigma u)^b. \tag{19}$$

In particular,

$$\kappa_{\max}(\mathbf{x}) \leq \frac{8}{a^{\frac{5}{2}}}. \tag{20}$$

To solve the asymptotic Plateau problem for the curvature function f, we apply the existence theorem of [7] to the curvature function $f^\theta := \theta K^{\frac{1}{n}} + (1-\theta)f$ which satisfies conditions (3)-(9) as well as (10). We then obtain a complete strictly locally convex solution $\Sigma^\theta = \text{graph}(u^\theta)$ in $\mathbb{H}^{n+1}$ satisfying (1)-(2) (with f replaced by f^θ) with bounded principal curvatures depending on θ. Using Theorem 1.2, we find that $u^\theta \in C^{0,1}(\overline{\Omega})$, $(u^\theta)^2 \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$, $u^\theta + u^\theta|D^2u^\theta| + |Du^\theta| \le C$ where C is independent of θ. We can now let θ tend to 0 to obtain the following existence theorem for $\Gamma = \partial\Omega$ satisfying a uniform exterior ball condition.

Theorem 1.3 ([8]). Let $\Gamma = \partial\Omega \times \{0\} \subset \mathbb{R}^{n+1}$ where Ω is a bounded domain in $\mathbb{R}^n$ satisfying a uniform exterior ball condition. Suppose $\sigma \in (0,1)$ and that f satisfies conditions (4)-(9) in K_n^+. Then there exists a complete locally strictly convex hypersurface $\Sigma = \text{graph}(u)$ in $\mathbb{H}^{n+1}$ satisfying (1)-(2) with uniformly bounded principal curvatures

$$\frac{1}{C} \le \kappa_i \le C \quad \text{on } \Sigma. \tag{21}$$

Furthermore, $u \in C^{0,1}(\overline{\Omega})$, $u^2 \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$, $u|D^2u| + |Du| \le C$ and

$$\sqrt{1+|Du|^2} = \frac{1}{\sigma} \quad \text{on } \partial\Omega \quad \text{if } \partial\Omega \in C^2. \tag{22}$$

Note that no uniqueness of solutions is asserted. In [7] we showed uniqueness if

$$\sum f_i > \sum \lambda_i^2 f_i \text{ in } K_n^+ \cap \{0 < f < 1\}.$$

In particular, uniqueness holds for the curvature quotients $f = (\frac{H_n}{H_l})^{\frac{1}{n-l}}$ with $l = n-1$ or $l = n-2$. Of course it is well-known that if Ω is strictly (Euclidean) starshaped about the origin, then there is uniqueness. However, much more can be said in this case.

Theorem 1.4 ([8]). Let $\Omega \in C^1$ be as in Theorem 1.3 and in addition be strictly (Euclidean) starshaped about the origin. Then the unique solution given in Theorem 1.3 is strictly (Euclidean) starshaped about the origin, i.e. $X \cdot \nu > 0$.

We also proved uniqueness in case Ω is mean convex.

Theorem 1.5 ([8])**.** Assume Ω is a $C^{2+\alpha}$ mean convex domain, that is, the Euclidean mean curvature $\mathcal{H}_{\partial\Omega} \geq 0$. Then the solution Σ of Theorem 1.3 is unique.

We end by describing an application of Theorem 1.3 to the existence of constant curvature spacelike hypersurfaces in de Sitter space. There is a natural asymptotic Plateau problem dual to (1), (2) for strictly spacelike hypersurfaces [12] which takes place in the steady state subspace $\mathcal{H}^{n+1} \subset dS_{n+1}$ of de Sitter space. Following Montiel [10], there is a halfspace model for $\mathcal{H}^{n+1}$ which can be identified with $\mathbb{R}^{n+1}_+$ endowed with the Lorentz metric

$$ds^2 = \frac{1}{x_{n+1}^2}(dx^2 - dx_{n+1}^2), \tag{23}$$

It is important to note that *the isometry from $\mathcal{H}^{n+1}$ to the halfspace model reverses the time orientation.* The dual asymptotic Plateau problem seeks to find a strictly spacelike hypersurface S satisfying

$$\begin{aligned} f(\kappa[S]) &= \sigma > 1 && (24)\\ \partial S &= \Gamma && (25)\end{aligned}$$

where $\kappa[S] = (\kappa_1, \ldots, \kappa_n)$ denote the principal curvatures of S in the induced de Sitter metric.

If S is a complete spacelike hypersurface in $\mathcal{H}^{n+1}$ with compact asymptotic boundary at infinity, then the normal vector field N of S is chosen to be the one pointing to the unique unbounded region in $\mathbb{R}^{n+1}_+ \setminus S$, and the de Sitter principal curvatures of S are calculated with respect to this normal vector field.

Because S is strictly spacelike, we are essentially forced to take $\Gamma = \partial V$ where $V \subset \mathbb{R}^n$ is a bounded domain and seek S as the graph of a "spacelike" function $v(x)$ over Ω, i.e.

$$S = \{(y, y_{n+1}) : y \in V,\ y_{n+1} = v(x)\}, \quad |\nabla v| < 1, \text{ in } \overline{V}. \tag{26}$$

In [12] we have computed the first and second fundamental forms of S with respect to the induced de Sitter metric. We use

$$X_i = e_i + v_i e_{n+1}, \quad N = v\nu = v\frac{v_i e_i + e_{n+1}}{w},$$

where $w = \sqrt{1-|\nabla v|^2}$ and ν is the normal vector field of S viewed as a Minkowski space $R^{n,1}$ graph. The first and second fundamental forms g_{ij} and h_{ij} are then given by

$$g_{ij} = \langle X_i, X_j\rangle_D = \frac{1}{v^2}(\delta_{ij} - v_i v_j) \tag{27}$$

and

$$\begin{aligned} h_{ij} = \langle \nabla_{X_i} X_j, v\nu\rangle_D &= \frac{1}{vw}\left(\frac{\delta_{ij}}{v} - v_{ij} + \frac{v_i v_j}{v} - 2\frac{v_i v_j}{v}\right) \\ &= \frac{1}{v^2 w}(\delta_{ij} - v_i v_j - v v_{ij}) \end{aligned} \tag{28}$$

Note that from (28), S is locally strictly convex if and only if

$$y^2 - v^2 \text{ is (Euclidean) locally strictly convex .} \tag{29}$$

There is a well known Gauss map duality for locally strictly convex hypersurfaces in dS_{n+1} For our purposes we will need a very concrete formulation of this duality [12]. Montiel [10] showed that if we use the upper halfspace representation for both $\mathcal{H}^{n+1}$ and $\mathbb{H}^{n+1}$, then the Gauss map N corresponds to the map

$$L : S \to \mathbb{H}^{n+1}$$

defined by

$$L((y, v(y))) = (y - v(y)\nabla v(y), v(y)\sqrt{1-|\nabla v|^2}), \quad y \in V. \tag{30}$$

We now identify the map L in terms of a hodograph map and its associated Legendre transform. Define the map $x = \nabla p(y) : V \subset \mathbb{R}^n \to \mathbb{R}^n$ by

$$x = \nabla p(y),\ y \in V \ \text{ where } p(y) = \tfrac{1}{2}(y^2 - v(y)^2). \tag{31}$$

Note that p is strictly convex in the Euclidean sense by (28) and hence the map x is globally one to one. Therefore $u(x) := v(y)\sqrt{1 - |\nabla v(y)|^2}$ is well defined in $\Omega := x(V)$. The associated Legendre transform is the function $q(x)$ defined in Ω by $p(y) + q(x) = x \cdot y$ or $q(x) = -p(y) + y \cdot \nabla p(y)$.

Theorem 1.6 ([12])**.** Let L be defined by (30) and let x be defined by (31). Then the image of S by L is the locally strictly convex graph (with respect to the induced hyperbolic metric)

$$\Sigma = \{(x, u(x)) \in \mathbb{R}^{n+1}_+ : u \in C^\infty(\overline{\Omega}), u(x) > 0\},$$

with principal curvatures $\kappa_i^* = \kappa_i^{-1}$. Here $\kappa_i > 0$, $i = 1, \ldots, n$ are the principal curvatures of S with respect to the induced de Sitter metric. Moreover the inverse map $L^{-1} : \Sigma \to S$ defined by

$$L^{-1}((x, u(x))) = (x + u(x)\nabla u(x), u(x)\sqrt{1 + |\nabla u(x)|^2}) \quad y \in \Omega$$

is the dual Legendre transform and hodograph map $y = \nabla q(x)$, $q(x) = \frac{1}{2}(x^2 + u(x)^2)$.

Note that when $\Sigma = \text{graph}(u)$ over Ω is a strictly locally convex solution of the asymptotic Plateau problem (1)-(2) in $\mathbb{H}^{n+1}$, then its Gauss image $S = \text{graph}(v)$ is a locally strictly convex spacelike graph also defined over Ω which solves the asymptotic Plateau problem $f^*(\kappa) = \frac{1}{\sigma} > 1$. We now define f^*.

Definition 1.1. Given a curvature function $f(\kappa)$ in the positve cone K_n^+, define the dual curvature function $f^*(\kappa)$ by

$$f^*(\kappa) := (f(\frac{1}{\kappa_1}, \ldots, \frac{1}{\kappa_n}))^{-1} \quad \kappa \in K_n^+ \tag{32}$$

Note that f^* may in fact be naturally defined in a cone $K \supseteq K_n^+$. For example if $f(\kappa) = \left(\frac{H_n}{H_l}\right)^{\frac{1}{n-l}}$, $n > l \geq 0$ defined in K_n^+, then

$$f^*(\kappa) = \left(H_{n-l}\right)^{\frac{1}{n-l}}$$

is in fact defined in the standard Garding cone $K = \Gamma_{n-l}$. Here H_k is the normalized kth elementary symmetric function.

Using the duality Theorem 1.6, we can transplant the existence Theorem 1.3 to $\mathcal{H}^{n+1}$.

Theorem 1.7 ([8])**.** Let $\Gamma = \partial\Omega \times \{0\} \subset \mathbb{R}^{n+1}$ where Ω is a bounded domain in $\mathbb{R}^n$ satisfying a uniform exterior ball condition. Suppose $\sigma > 1$ and that f satisfies conditions (4)-(9) in K_n^+. Then there exists a complete locally strictly convex spacelike graph $S = \mathrm{graph}(v)$ in $\mathcal{H}^{n+1}$ satisfying (24)-(25) for the curvature function $f^*(\kappa)$ with uniformly bounded principal curvatures

$$\frac{1}{C} \leq \kappa_i \leq C \text{ on } S. \tag{33}$$

Furthermore, $v \in C^{0,1}(\overline{\Omega})$, $v^2 \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$, $v|D^2 v| + |Dv| \leq C$ and

$$\sqrt{1 - |Dv|^2} = \frac{1}{\sigma} \quad \textit{on } \partial\Omega \quad \textit{if } \partial\Omega \in C^2. \tag{34}$$

Corollary 1.1 ([8])**.** Let $\Gamma = \partial\Omega \times \{0\} \subset \partial_\infty \mathcal{H}^{n+1}$ where Ω is a bounded domain in $\mathbb{R}^n$ satisfying a uniform exterior ball condition. Then there exists a complete locally strictly convex spacelike hypersurface S in $\mathcal{H}^{n+1}$ satisfying

$$(H_l)^{\frac{1}{l}} = \sigma > 1, \quad 1 \leq l \leq n$$

with $\partial S = \Gamma$ and having uniformly bounded principal curvatures

$$\frac{1}{C} \leq \kappa_i \leq C \text{ on } \Sigma. \tag{35}$$

Moreover, $S = \mathrm{graph}(v)$ with $v \in C^\infty(\Omega) \cap C^{0,1}(\bar{\Omega})$ and $v^2 \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$, $v|D^2 v| + |Dv| \leq C$. Further, if $l = 1$ or $l = 2$ (mean curvature and normalized scalar curvature) or if $\partial\Omega$ is mean convex, we have uniqueness among convex solutions and even among all solutions (convex or not) if Ω *is simple.*

The uniqueness part of Corollary 1.1 follows from the uniqueness Theorem 1.6 of [7] or Theorem 1.5 and a continuity and deformation argument like that used in [11]. Note that Montiel [10] proved existence for $H = \sigma > 1$ assuming $\partial\Omega$ is mean convex. Our result shows that for arbitrary Ω there is always a unique locally strictly convex solution. If Ω

is mean convex the solutions constructed by Montiel must agree with the ones we construct.

An outline of the paper is as follows. Sections 2-4 contain all the important identities and formulas needed in the proof of our main technical result Theorem 5.1 which is our “global interior curvature estimate”. This is proved in Section 5 and Theorem 1.2 follows immediately. For the reader’s convenience and completeness we have repeated some of the proofs which are contained in our earlier paper [7]. These preliminary results are interesting and important in themselves and will orient the reader to our point of view. They are also needed in Section 4 to prove Corollary 4.1 which shows that the condition $\nu^{n+1} \geq 2a$ is satisfied for $\partial\Omega$ satisfying a uniform exterior ball condition with an a of order σ. In Section 6, we prove the strict starshapedness of solutions in Theorem 1.3. In Section 7 we prove the uniqueness Theorem 1.5 making essential use of Theorem 1.2. Finally in Section 8, we sketch the proof from [12] of the duality stated in Theorem 1.6.

2 Formulas on hypersurfaces and some basic identities

In this section we recall some basic identities on a hypersurface that were derived in [7] by comparing the induced hyperbolic and Euclidean metrics.

Let Σ be a hypersurface in $\mathbb{H}^{n+1}$. We shall use g and ∇ to denote the induced hyperbolic metric and Levi-Civita connection on Σ, respectively. As Σ is also a submanifold of $\mathbb{R}^{n+1}$, we shall usually distinguish a geometric quantity with respect to the Euclidean metric by adding a ‘tilde’ over the corresponding hyperbolic quantity. For instance, $\tilde{g}$ denotes the induced metric on Σ from $\mathbb{R}^{n+1}$, and $\tilde{\nabla}$ is its Levi-Civita connection.

Let $\mathbf{x}$ be the position vector of Σ in $\mathbb{R}^{n+1}$ and set

$$u = \mathbf{x} \cdot \mathbf{e}$$

where $\mathbf{e}$ is the unit vector in the positive x_{n+1} direction in $\mathbb{R}^{n+1}$, and '$\cdot$' denotes the Euclidean inner product in $\mathbb{R}^{n+1}$. We refer u as the *height function* of Σ.

Throughout the paper we assume Σ is orientable and let $\mathbf{n}$ be a (global) unit normal vector field to Σ with respect to the hyperbolic metric. This also determines a unit normal ν to Σ with respect to the Euclidean metric by the relation

$$\nu = \frac{\mathbf{n}}{u}.$$

We denote $\nu^{n+1} = \mathbf{e} \cdot \nu$.

Let $(z_1, \ldots, z_n)$ be local coordinates and

$$\tau_i = \frac{\partial}{\partial z_i}, \quad i = 1, \ldots, n.$$

The hyperbolic and Euclidean metrics of Σ are given by

$$g_{ij} = \langle \tau_i, \tau_j \rangle, \quad \tilde{g}_{ij} = \tau_i \cdot \tau_j = u^2 g_{ij},$$

while the second fundamental forms are

$$\begin{aligned} h_{ij} &= \langle D_{\tau_i} \tau_j, \mathbf{n} \rangle = -\langle D_{\tau_i} \mathbf{n}, \tau_j \rangle, \\ \tilde{h}_{ij} &= \nu \cdot \tilde{D}_{\tau_i} \tau_j = -\tau_j \cdot \tilde{D}_{\tau_i} \nu, \end{aligned} \tag{1}$$

where D and $\tilde{D}$ denote the Levi-Civita connection of $\mathbb{H}^{n+1}$ and $\mathbb{R}^{n+1}$, respectively. The following relations are well known (see (15), (16)):

$$h_{ij} = \frac{1}{u} \tilde{h}_{ij} + \frac{\nu^{n+1}}{u^2} \tilde{g}_{ij} \tag{2}$$

and

$$\kappa_i = u \tilde{\kappa}_i + \nu^{n+1}, \quad i = 1, \cdots, n \tag{3}$$

where $\kappa_1, \cdots, \kappa_n$ and $\tilde{\kappa}_1, \cdots, \tilde{\kappa}_n$ are the hyperbolic and Euclidean principal curvatures, respectively. The Christoffel symbols are related by the formula

$$\Gamma_{ij}^k = \tilde{\Gamma}_{ij}^k - \frac{1}{u}(u_i \delta_{kj} + u_j \delta_{ik} - \tilde{g}^{kl} u_l \tilde{g}_{ij}). \tag{4}$$

It follows that for $v \in C^2(\Sigma)$

$$\nabla_{ij} v = v_{ij} - \Gamma^k_{ij} v_k = \tilde{\nabla}_{ij} v + \frac{1}{u}(u_i v_j + u_j v_i - \tilde{g}^{kl} u_l v_k \tilde{g}_{ij}) \tag{5}$$

where (and in sequel)

$$v_i = \frac{\partial v}{\partial z_i},\ v_{ij} = \frac{\partial^2 v}{\partial z_i z_j},\ \text{etc.}$$

In particular,

$$\nabla_{ij} u = \tilde{\nabla}_{ij} u + \frac{2u_i u_j}{u} - \frac{1}{u}\tilde{g}^{kl} u_k u_l \tilde{g}_{ij} \tag{6}$$

and

$$\nabla_{ij} \frac{1}{u} = -\frac{1}{u^2}\tilde{\nabla}_{ij} u + \frac{1}{u^3}\tilde{g}^{kl} u_k u_l \tilde{g}_{ij}. \tag{7}$$

Moreover,

$$\nabla_{ij} \frac{v}{u} = v \nabla_{ij} \frac{1}{u} + \frac{1}{u}\tilde{\nabla}_{ij} v - \frac{1}{u^2}\tilde{g}^{kl} u_k v_l \tilde{g}_{ij}. \tag{8}$$

In $\mathbb{R}^{n+1}$,

$$\begin{aligned} \tilde{g}^{kl} u_k u_l &= |\tilde{\nabla} u|^2 = 1 - (\nu^{n+1})^2 \\ \tilde{\nabla}_{ij} u &= \tilde{h}_{ij} \nu^{n+1}. \end{aligned} \tag{9}$$

Therefore, by (3) and (7),

$$\begin{aligned} \nabla_{ij} \frac{1}{u} &= -\frac{\nu^{n+1}}{u^2}\tilde{h}_{ij} + \frac{1}{u^3}(1 - (\nu^{n+1})^2)\tilde{g}_{ij} \\ &= \frac{1}{u}(g_{ij} - \nu^{n+1} h_{ij}). \end{aligned} \tag{10}$$

We note that (8) and (10) still hold for general local frames $\tau_1, \ldots, \tau_n$. In particular, if $\tau_1, \ldots, \tau_n$ are orthonormal in the hyperbolic metric, then $g_{ij} = \delta_{ij}$ and $\tilde{g}_{ij} = u^2 \delta_{ij}$.

We now consider equation (1) on Σ. Let $\mathcal{A}$ be the vector space of $n \times n$ matrices and

$$\mathcal{A}^+ = \{A = \{a_{ij}\} \in \mathcal{A} : \lambda(A) \in K_n^+\},$$

where $\lambda(A) = (\lambda_1, \ldots, \lambda_n)$ denotes the eigenvalues of A. Let F be the function defined by

$$F(A) = f(\lambda(A)),\quad A \in \mathcal{A}^+ \tag{11}$$

and denote

$$F^{ij}(A) = \frac{\partial F}{\partial a_{ij}}(A), \quad F^{ij,kl}(A) = \frac{\partial^2 F}{\partial a_{ij}\partial a_{kl}}(A). \tag{12}$$

Since $F(A)$ depends only on the eigenvalues of A, if A is symmetric then so is the matrix $\{F^{ij}(A)\}$. Moreover,

$$F^{ij}(A) = f_i \delta_{ij}$$

when A is diagonal, and

$$F^{ij}(A)a_{ij} = \sum f_i(\lambda(A))\lambda_i = F(A), \tag{13}$$

$$F^{ij}(A)a_{ik}a_{jk} = \sum f_i(\lambda(A))\lambda_i^2. \tag{14}$$

Equation (1) can therefore be rewritten in a local frame $\tau_1, \ldots, \tau_n$ in the form

$$F(A[\Sigma]) = \sigma \tag{15}$$

where $A[\Sigma] = \{g^{ik}h_{kj}\}$. Let $F^{ij} = F^{ij}(A[\Sigma])$, $F^{ij,kl} = F^{ij,kl}(A[\Sigma])$.

Lemma 2.1 ([7]). *Let Σ be a smooth hypersurface in $\mathbb{H}^{n+1}$ satisfying equation* (1)*. Then in a local orthonormal frame,*

$$F^{ij}\nabla_{ij}\frac{1}{u} = -\frac{\sigma\nu^{n+1}}{u} + \frac{1}{u}\sum f_i. \tag{16}$$

and

$$F^{ij}\nabla_{ij}\frac{\nu^{n+1}}{u} = \frac{\sigma}{u} - \frac{\nu^{n+1}}{u}\sum f_i\kappa_i^2. \tag{17}$$

Proof. The first identity follows immediately from (10), (13) and assumption (9). To prove (17) we recall the identities in $\mathbb{R}^{n+1}$

$$\begin{aligned} (\nu^{n+1})_i &= -\tilde{h}_{ij}\tilde{g}^{jk}u_k, \\ \tilde{\nabla}_{ij}\nu^{n+1} &= -\tilde{g}^{kl}(\nu^{n+1}\tilde{h}_{il}\tilde{h}_{kj} + u_l\tilde{\nabla}_k\tilde{h}_{ij}). \end{aligned} \tag{18}$$

By (2), (13), (14), and $\tilde{g}^{ik} = \delta_{jk}/u^2$ we see that

$$\begin{aligned} F^{ij}\tilde{g}^{kl}\tilde{h}_{il}\tilde{h}_{kj} &= \frac{1}{u^2}F^{ij}\tilde{h}_{ik}\tilde{h}_{kj} \\ &= F^{ij}(h_{ik}h_{kj} - 2\nu^{n+1}h_{ij} + (\nu^{n+1})^2\delta_{ij}) \\ &= f_i\kappa_i^2 - 2\nu^{n+1}\sigma + (\nu^{n+1})^2\sum f_i. \end{aligned} \tag{19}$$

As a hypersurface in $\mathbb{R}^{n+1}$, it follows from (3) that Σ satisfies

$$f(u\tilde{\kappa}_1 + \nu^{n+1}, \ldots, u\tilde{\kappa}_n + \nu^{n+1}) = \sigma,$$

or equivalently,

$$F(\{\tilde{g}^{ik}(u\tilde{h}_{kj} + \nu^{n+1}\tilde{g}_{kj})\}) = \sigma. \tag{20}$$

Differentiating equation (20) and using $\tilde{g}_{ik} = u^2\delta_{ik}$, $\tilde{g}^{ik} = \delta_{ik}/u^2$, we obtain

$$F^{ij}(u\tilde{\nabla}_k\tilde{h}_{ij} + u_k\tilde{h}_{ij} + (\nu^{n+1})_k u^2\delta_{ij}) = 0. \tag{21}$$

That is,

$$\begin{aligned} F^{ij}\tilde{\nabla}_k\tilde{h}_{ij} + (\nu^{n+1})_k u \sum F^{ii} &= -\frac{u_k}{u}F^{ij}\tilde{h}_{ij} \\ &= -u_k F^{ij}(h_{ij} - \nu^{n+1}\delta_{ij}) \\ &= -u_k\Big(\sigma - \nu^{n+1}\sum f_i\Big). \end{aligned} \tag{22}$$

Finally, combining (8), (16), (18), (19), (22), and the first identity in (9), we derive

$$\begin{aligned} F^{ij}\nabla_{ij}\frac{\nu^{n+1}}{u} &= \nu^{n+1}F^{ij}\nabla_{ij}\frac{1}{u} + \frac{|\tilde{\nabla}u|^2}{u}F^{ij}\tilde{h}_{ij} - \frac{\nu^{n+1}}{u^3}F^{ij}\tilde{h}_{ik}\tilde{h}_{kj} \\ &= \frac{\nu^{n+1}}{u}\Big(\sum f_i - \nu^{n+1}\sigma\Big) + \frac{|\tilde{\nabla}u|^2}{u}\Big(\sigma - \nu^{n+1}\sum f_i\Big) \\ &\quad - \frac{\nu^{n+1}}{u}\Big(f_i\kappa_i^2 - 2\nu^{n+1}\sigma + (\nu^{n+1})^2\sum f_i\Big) \\ &= \frac{\sigma}{u} - \frac{\nu^{n+1}}{u}\sum f_i\kappa_i^2. \end{aligned} \tag{23}$$

This proves (17). □

3 Height estimates and the asymptotic angle condition

In this section let Σ be a hypersurface in $\mathbb{H}^{n+1}$ with $\partial\Sigma \subset P(\varepsilon) := \{x_{n+1} = \varepsilon\}$ so Σ separates $\{x_{n+1} \geq \varepsilon\}$ into an inside (bounded) region and an outside (unbounded) one. Let Ω be the region in $\mathbb{R}^n \times \{0\}$ such that its vertical lift Ω^ε to $P(\varepsilon)$ is bounded by $\partial\Sigma$ (and $\mathbb{R}^n \setminus \Omega$ is connected and unbounded). It is allowable that Ω has several connected components. Suppose $\kappa[\Sigma] \in K$ and $f(\kappa) = \sigma \in (0,1)$ with respect to the outer normal.

Let $B_1 = B_R(a)$ be a ball of radius R centered at $a = (a', -\sigma R) \in \mathbb{R}^{n+1}$ where $\sigma \in (0,1)$ and $S_1 = \partial B_1 \cap \mathbb{H}^{n+1}$. Then $\kappa_i[S_1] = \sigma$ for all $1 \le i \le n$ with respect to its outward normal. Similarly, let $B_2 = B_R(b)$ be a ball of radius R centered at $b = (b', \sigma R) \in \mathbb{R}^{n+1}$ with $S_2 = \partial B_2 \cap \mathbb{H}^{n+1}$. Then $\kappa_i[S_2] = \sigma$ for all $1 \le i \le n$ with respect to its inward normal.

These so called equidistant spheres serve as useful barriers.

Lemma 3.1 ([7])**.**

$$\begin{aligned}
&(i)\ \ \Sigma \cap \{x_{n+1} < \varepsilon\} = \emptyset \\
&(ii)\ \textit{If}\ \partial\Sigma \subset B_1,\ \textit{then}\ \Sigma \subset B_1\ . \\
&(iii)\ \textit{If}\ B_1 \cap P(\varepsilon) \subset \Omega^\varepsilon,\ \textit{then}\ B_1 \cap \Sigma = \emptyset\ . \\
&(iv)\ \textit{If}\ \ B_2 \cap \Omega^\varepsilon = \emptyset,\ \textit{then}\ B_2 \cap \Sigma = \emptyset\ .
\end{aligned} \tag{1}$$

Proof. For (i) let $c = \min_{x\in\Sigma} x_{n+1}$ and suppose $0 < c < \varepsilon$. Then the horosphere $P(c)$ satisfies $f(\kappa) = 1$ with respect to the upward normal, lies below Σ and has an interior contact violating the maximum principle. Thus $c = \varepsilon$. For (ii),(iii), (iv) we perform homothetic dilations from $(a', 0)$ and $(b', 0)$ respectively which are hyperbolic isometries and use the maximum principle. For (ii), expand B_1 continuously until it contains Σ and then reverse the process. Since the curvatures of Σ and S_1 are calculated with respect to their outward normals and both hypersurfaces satisfy $f(\kappa) = \sigma$, there cannot be a first contact. For (iii) and (iv) we shrink B_1 and B_2 until they are respectively inside and outside Σ. When we expand B_1 there cannot be a first contact as above. Now shrink B_2 until it lies below $P(\varepsilon)$ and so is disjoint (outside) from Σ. Now reverse the process and suppose there is a first interior contact. Then the outward normal to Σ at this contact point is the inward normal to S_2. Since the curvatures of S_2 are calculated with respect to its inner normal and it satisfies $f(\kappa) = \sigma$, this contradicts the maximum principle. $\square$

Lemma 3.2 ([7]). *Suppose f satisfies (3), (7) and (8). Assume that $\partial\Sigma$ satisfies a uniform interior and/or exterior ball condition and let u denote the height function of Σ. Then for $\varepsilon > 0$ sufficiently small,*

$$-\frac{\varepsilon\sqrt{1-\sigma^2}}{r_2}-\frac{\varepsilon^2(1+\sigma)}{r_2^2}<\nu^{n+1}-\sigma<\frac{\varepsilon\sqrt{1-\sigma^2}}{r_1}+\frac{\varepsilon^2(1-\sigma)}{r_1^2} \quad \text{on } \partial\Sigma \quad (2)$$

where r_2 and r_1 are the maximal radii of exterior and interior spheres to $\partial\Omega$, respectively. In particular, $\nu^{n+1} \to \sigma$ on $\partial\Sigma$ as $\varepsilon \to 0$.

Proof. Assume first $r_2 < \infty$. Fix a point $x_0 \in \partial\Omega$ and let e_1 be the outward pointing unit normal to $\partial\Omega$ at x_0. Let B_1, B_2 be balls in $\mathbb{R}^{n+1}$ with centers $a_1 = (x_0 - r_1 e_1, -R_1\sigma$, $a_2 = (x_0 + r_2 e_1, R_2\sigma)$ and radii R_1, R_2 respectively satisfying

$$R_1^2 = r_1^2 + (R_1\sigma + \varepsilon)^2,\ R_2^2 = r_2^2 + (R_2\sigma - \varepsilon)^2\ . \tag{3}$$

Then $B_1 \cap P(\varepsilon)$ is an n-ball of radius r_1 internally tangent to $\partial\Omega^\varepsilon$ at x_0 while $B_2 \cap P(\varepsilon)$ is an n-ball of radius r_2 externally tangent to $\partial\Omega^\varepsilon$ at x_0. By Lemma 3.1 (iii) and (iv), $B_i \cap \Sigma = \emptyset$, $i = 1, 2$. Hence,

$$-\frac{u-\sigma R_2}{R_2} < \nu^{n+1} < \frac{u+\sigma R_1}{R_1} \text{ at } x_0\ .$$

That is,

$$-\frac{\varepsilon}{R_2} < \nu^{n+1} - \sigma < \frac{\varepsilon}{R_1} \text{ at } x_0\ . \tag{4}$$

From (3),

$$\frac{1}{R_1} = \frac{\sqrt{(1-\sigma^2)r_1^2+\varepsilon^2}-\varepsilon\sigma}{r_1^2+\varepsilon^2} < \frac{\sqrt{1-\sigma^2}}{r_1} + \frac{\varepsilon(1-\sigma)}{r_1^2}\ ,$$

and

$$\frac{1}{R_2} = \frac{\sqrt{(1-\sigma^2)r_2^2+\varepsilon^2}+\varepsilon\sigma}{r_2^2+\varepsilon^2} < \frac{\sqrt{1-\sigma^2}}{r_2} + \frac{\varepsilon(1+\sigma)}{r_2^2}\ ,$$

These estimates and (4) give (2), completing the proof of the lemma. □

4 The asymptotic angle maximum principle and gradient estimates

In this section we show that the upward unit normal of a solution tends to a fixed asymptotic angle on approach to the boundary. This implies a global gradient bound on solutions.

Theorem 4.1 ([7]). Let Σ be a smooth strictly locally convex hypersurface in $\mathbb{H}^{n+1}$ satisfying equation (1). Suppose Σ is globally a graph:

$$\Sigma = \{(x, u(x)) : x \in \Omega\}$$

where Ω is a domain in $\mathbb{R}^n \equiv \partial\mathbb{H}^{n+1}$. Then

$$F^{ij}\nabla_{ij}\frac{\sigma - \nu^{n+1}}{u} \geq \sigma(1-\sigma)\frac{(\sum f_i - 1)}{u} \geq 0 \tag{1}$$

and so,

$$\frac{\sigma - \nu^{n+1}}{u} \leq \sup_{\partial\Sigma}\frac{\sigma - \nu^{n+1}}{u} \quad \text{on } \Sigma. \tag{2}$$

Moreover, if $u = \epsilon > 0$ on $\partial\Omega$ (satisfying a uniform exterior ball condition), then there exists $\epsilon_0 > 0$ depending only on $\partial\Omega$, such that for all $\epsilon \leq \epsilon_0$,

$$\frac{\sigma - \nu^{n+1}}{u} \leq \frac{\sqrt{1-\sigma^2}}{r_2} + \frac{\varepsilon(1+\sigma)}{r_2^2} \quad \text{on } \Sigma \tag{3}$$

where r_2 is the maximal radius of exterior tangent spheres to $\partial\Omega$.

Proof. Set $\eta = \frac{\sigma - \nu^{n+1}}{u}$. By (16) and (17) we have

$$F^{ij}\nabla_{ij}\eta = \frac{\sigma}{u}\Big(\sum f_i - 1\Big) + \frac{\nu^{n+1}}{u}\Big(\sum f_i\kappa_i^2 - \sigma^2\Big).$$

On the other hand,

$$\sum \kappa_i^2 f_i \geq \frac{(\sum \kappa_i f_i)^2}{\sum f_i} = \frac{\sigma^2}{\sum f_i}.$$

Hence,

$$F^{ij}\nabla_{ij}\eta \geq \frac{\sigma}{u}\Big(\sum f_i - 1\Big)\Big(1 - \frac{\sigma\nu^{n+1}}{\sum f_i}\Big) \geq \frac{\sigma(1-\sigma)}{u}\Big(\sum f_i - 1\Big) \geq 0.$$

So (2) follows from the maximum principle, while (3) follows from (2) and the approximate asymptotic angle condition,

$$\eta \leq \frac{\sqrt{1-\sigma^2}}{r_2} + \frac{\varepsilon(1+\sigma)}{r_2^2} \quad \text{on } \partial\Sigma$$

which is proved in Lemma 3.2 □

Proposition 4.1 ([7])**.** Let Σ be a smooth strictly locally convex graph

$$\Sigma = \{(x, u(x)) : x \in \Omega\}$$

in $\mathbb{H}^{n+1}$ satisfying $u \geq \varepsilon$ in Ω, $u = \varepsilon$ on $\partial\Omega$. Then at an interior maximum of $\frac{u}{\nu^{n+1}}$ we have $\frac{u}{\nu^{n+1}} \leq \max_\Omega u$. Hence for ε small compared to σ,

$$\nu^{n+1} \geq \frac{u}{\max_\Omega u} \quad \text{in } \Omega \tag{4}$$

Proof. Let $h = \frac{u}{\nu^{n+1}} = uw$ and suppose that h assumes its maximum at an interior point x_0. Then at x_0,

$$\partial_i h = u_i w + u\frac{u_k u_{ki}}{w} = (\delta_{ki} + u_k u_i + u u_{ki})\frac{u_k}{w} = 0 \ \ \forall\, 1 \leq i \leq n.$$

Since Σ is strictly locally convex, this implies that $\nabla u = 0$ at x_0 so the proposition follows immediately from Theorem 4.1. □

Combining Theorem 4.1 and Proposition 4.1 gives

Corollary 4.1 ([8])**.** Let Σ be a smooth strictly locally convex graph

$$\Sigma = \{(x, u(x)) : x \in \Omega\}$$

in $\mathbb{H}^{n+1}$ satisfying $u \geq \varepsilon$ in Ω, $u = \varepsilon$ on $\partial\Omega$. Assume that $\partial\Omega$ satisfies a uniform exterior ball condition. Then for ε sufficiently small compared to σ

$$\nu^{n+1} \geq 2a := \frac{\sigma}{1 + M\max_\Omega u} \tag{5}$$

where $M = \frac{\sqrt{1-\sigma^2}}{r_2} + \frac{\varepsilon(1+\sigma)}{r_2^2}$.

Proof. By Theorem 4.1 we have $\nu^{n+1} \geq \sigma - Mu$ while by Proposition 4.1 we have $\nu^{n+1} \geq \frac{u}{\max_\Omega u}$. Hence if $u \leq \lambda\sigma$ we find $\nu^{n+1} \geq \sigma(1-\lambda M)$ while if $u \geq \lambda\sigma$ we find $\nu^{n+1} \geq \frac{\lambda\sigma}{\max_\Omega u}$. Choosing $\lambda = \frac{\max_\Omega u}{1+M\max_\Omega u}$ completes the proof. □

5 The global interior curvature estimate

In this section we prove an interior curvature estimate (see Theorem 5.1 below) for the largest principal curvature of locally strictly convex graphs satisfying $f(\kappa) = \sigma$. What is remarkable is that the bound obtained is independent of the "cutoff " function u^b which vanishes at $\partial\Omega$. Hence we can let b tend to zero to prove the global estimate Theorem 1.2.

Let Σ be a smooth strictly locally convex hypersurface in $\mathbb{H}^{n+1}$ satisfying $f(\kappa) = \sigma$ with $\partial\Sigma \subset \partial_\infty\mathbb{H}^{n+1}$. For a fixed point $\mathbf{x}_0 \in \Sigma$ we choose a local orthonormal frame $\tau_1, \ldots, \tau_n$ around $\mathbf{x}_0$ such that $h_{ij}(\mathbf{x}_0) = \kappa_i\delta_{ij}$. The calculations below are done at $\mathbf{x}_0$. For convenience we shall write $v_{ij} = \nabla_{ij}v$, $h_{ijk} = \nabla_k h_{ij}$, $h_{ijkl} = \nabla_{lk}h_{ij} = \nabla_l\nabla_k h_{ij}$, etc.

Since $\mathbb{H}^{n+1}$ has constant sectional curvature -1, by the Codazzi and Gauss equations we have $h_{ijk} = h_{ikj}$ and

$$\begin{aligned} h_{iijj} &= h_{jjii} + (h_{ii}h_{jj} - 1)(h_{ii} - h_{jj}) \\ &= h_{jjii} + (\kappa_i\kappa_j - 1)(\kappa_i - \kappa_j). \end{aligned} \tag{1}$$

Consequently for each fixed j,

$$F^{ii}h_{jjii} = F^{ii}h_{iijj} + (1+\kappa_j^2)\sum f_i\kappa_i - \kappa_j\sum f_i - \kappa_j\sum\kappa_i^2 f_i. \tag{2}$$

Theorem 5.1 ([8])**.** Let Σ be a smooth strictly locally convex graph in $\mathbb{H}^{n+1}$ satisfying $f(\kappa) = \sigma$, $\partial_\infty\Sigma \subset \partial_\infty\mathbb{H}^{n+1}$ and

$$\nu^{n+1} \geq 2a > 0 \text{ on } \Sigma. \tag{3}$$

For $\mathbf{x} \in \Sigma$ let $\kappa_{\max}(\mathbf{x})$ be the largest principal curvature of Σ at $\mathbf{x}$. Then for $0 < b \leq \frac{a}{4}$,

$$\max_{\Sigma} u^b \frac{\kappa_{\max}}{\nu^{n+1} - a} \leq \frac{8}{a^{\frac{5}{2}}} (\sup_{\Sigma} u)^b. \tag{4}$$

Proof. Let

$$M_0 = \max_{\mathbf{x}\in\Sigma} u^b \frac{\kappa_{\max}(x)}{\nu^{n+1} - a}. \tag{5}$$

Then $M_0 > 0$ is attained at an interior point $\mathbf{x}_0 \in \Sigma$. Let $\tau_1, \ldots, \tau_n$ be a local orthonormal frame around $\mathbf{x}_0$ such that $h_{ij}(\mathbf{x}_0) = \kappa_i \delta_{ij}$, where $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of Σ at $\mathbf{x}_0$. We may assume $\kappa_1 = \kappa_{\max}(\mathbf{x}_0)$. Thus, at $\mathbf{x}_0$, $u^b \frac{h_{11}}{\nu^{n+1}-a}$ has a local maximum and so

$$\frac{h_{11i}}{h_{11}} + b\frac{u_i}{u} - \frac{\nabla_i \nu^{n+1}}{\nu^{n+1} - a} = 0, \tag{6}$$

$$\frac{h_{11ii}}{h_{11}} + b\frac{u_{ii}}{u} - \frac{\nabla_{ii} \nu^{n+1}}{\nu^{n+1} - a} - (b + b^2)\frac{u_i^2}{u^2} + 2b\frac{u_i}{u}\frac{\nabla_i \nu^{n+1}}{\nu^{n+1} - a} \leq 0. \tag{7}$$

Using (4), we find after differentiating the equation $F(h_{ij}) = \sigma$ twice that

Lemma 5.1. *At* $\mathbf{x}_0$*,*

$$F^{ii} h_{11ii} = -F^{ij,rs} h_{ij1} h_{rs1} + \sigma(1 + \kappa_1^2) - \kappa_1 (\sum f_i + \sum \kappa_i^2 f_i). \tag{8}$$

By Lemma 2.1 we immediately derive

Lemma 5.2 ([7])**.** *Let* Σ *be a smooth hypersurface in* $\mathbb{H}^{n+1}$ *satisfying* $f(\kappa) = \sigma$*. Then in a local orthonormal frame,*

$$\begin{aligned} F^{ij} \nabla_{ij} \nu^{n+1} &= \frac{2}{u} F^{ij} \nabla_i u \nabla_j \nu^{n+1} + \sigma(1 + (\nu^{n+1})^2) \\ &\quad - \nu^{n+1} \Big(\sum f_i + \sum f_i \kappa_i^2 \Big), \\ F^{ij} \frac{\nabla_{ij} u}{u} &= 2 \sum f_i \frac{u_i^2}{u^2} + \sigma \nu^{n+1} - \sum f_i\ . \end{aligned} \tag{9}$$

Using Lemma 5.1 and Lemma 5.2 we find from (7)

$$\begin{aligned} 0 \geq & - F^{ij,rs} h_{ij1} h_{rs1} + \sigma\Big(1 + \kappa_1^2 - \frac{1 + (\nu^{n+1})^2}{\nu^{n+1} - a}\kappa_1\Big) \\ & + \frac{a\kappa_1}{\nu^{n+1} - a}\Big(\sum f_i + \sum \kappa_i^2 f_i\Big) - b\kappa_1 \sum f_i \\ & + (b - b^2)\kappa_1 \sum f_i \frac{u_i^2}{u^2} - \frac{(2 - 2b)\kappa_1}{\nu^{n+1} - a} F^{ij} \frac{u_i}{u} \nabla_j \nu^{n+1}. \end{aligned} \tag{10}$$

Next we use an inequality due to Andrews [1] and Gerhardt [4] which states

$$- F^{ij,kl} h_{ij1} h_{kl,1} \geq \sum_{i \neq j} \frac{f_i - f_j}{\kappa_j - \kappa_i} h_{ij1}^2 \geq 2 \sum_{i \geq 2} \frac{f_i - f_1}{\kappa_1 - \kappa_i} h_{i11}^2. \tag{11}$$

Recall that (see (18))

$$\nabla_i \nu^{n+1} = \frac{u_i}{u}(\nu^{n+1} - \kappa_i).$$

Then at $\mathbf{x}_0$, we obtain from (6)

$$h_{11i} = \kappa_1 \frac{u_i}{u}\Big(\frac{\nu^{n+1} - \kappa_i}{\nu^{n+1} - a} - b\Big). \tag{12}$$

Inserting this into (11) we derive

$$- F^{ij,kl} h_{ij1} h_{kl,1} \geq 2{\kappa_1}^2 \sum_{i \geq 2} \frac{f_i - f_1}{\kappa_1 - \kappa_i} \frac{u_i^2}{u^2}\Big(\frac{\kappa_i - \nu^{n+1}}{\nu^{n+1} - a} + b\Big)^2. \tag{13}$$

Note that we may write

$$\sum f_i + \sum \kappa_i^2 f_i = (1 - (\nu^{n+1})^2) \sum f_i + \sum (\kappa_i - \nu^{n+1})^2 f_i + 2\sigma\nu^{n+1}. \tag{14}$$

Combining (11), (13) and (14) gives at $\mathbf{x}_0$

$$\begin{aligned} 0 \geq\ & \sigma\Big(1 + \kappa_1^2 - \frac{1 + (\nu^{n+1})^2}{\nu^{n+1} - a}\kappa_1\Big) - b\kappa_1 \sum f_i \\ & + (b - b^2) \sum f_i \frac{u_i^2}{u^2} + \frac{a\kappa_1}{2(\nu^{n+1} - a)}\Big(\sum f_i + \sum \kappa_i^2 f_i\Big) \\ & + \frac{a\kappa_1}{2(\nu^{n+1} - a)}\Big((1 - (\nu^{n+1})^2) \sum f_i + \sum (\kappa_i - \nu^{n+1})^2 f_i + 2\sigma\nu^{n+1}\Big) \\ & + (2 - 2b)\kappa_1 \sum f_i \frac{u_i^2}{u^2}\Big(\frac{\kappa_i - \nu^{n+1}}{\nu^{n+1} - a}\Big) + 2{\kappa_1}^2 \sum_{i \geq 2} \frac{f_i - f_1}{\kappa_1 - \kappa_i} \frac{u_i^2}{u^2}\Big(\frac{\kappa_i - \nu^{n+1}}{\nu^{n+1} - a} + b\Big)^2. \end{aligned} \tag{15}$$

Note that (assuming $\kappa_1 \geq \frac{2}{a}$ and $b \leq \frac{a}{4}$) all the terms of (15) are positive except possibly the ones in the sum involving $(\kappa_i - \nu^{n+1})$ and only if $\kappa_i < \nu^{n+1}$.

Therefore define

$$J = \{i : \kappa_i - \nu^{n+1} < 0,\ f_i < \theta^{-1} f_1\},$$
$$L = \{i : \kappa_i - \nu^{n+1} < 0,\ f_i \geq \theta^{-1} f_1\},$$

where $\theta \in (0,1)$ is to be chosen later. Since $\sum u_i^2/u^2 = |\tilde{\nabla} u|^2 = 1 - (\nu^{n+1})^2 \leq 1$ and $\kappa_1 f_1 \leq \sigma$, we have

$$\sum_{i\in J} (\kappa_i - \nu^{n+1}) f_i \frac{u_i^2}{u^2} \geq -\frac{f_1}{\theta} \geq -\frac{\sigma}{\theta\kappa_1}. \tag{16}$$

Finally,

$$\begin{aligned}
&2\kappa_1^2 \sum_{i\in L} \frac{f_i - f_1}{\kappa_1 - \kappa_i} \frac{u_i^2}{u^2} \Big(\frac{\kappa_i - \nu^{n+1}}{\nu^{n+1} - a} + b\Big)^2 + (2-2b)\kappa_1 \sum_{i\in L} f_i \frac{u_i^2}{u^2} \frac{(\kappa_i - \nu^{n+1})}{\nu^{n+1} - a} \\
&\quad\geq 2(1-\theta)\kappa_1 \sum_{i\in L} f_i \frac{u_i^2}{u^2} \Big(\frac{\kappa_i - \nu^{n+1}}{\nu^{n+1} - a}\Big)^2 + 2(1 + b(1-2\theta))\kappa_1 \sum_{i\in L} f_i \frac{u_i^2}{u^2} \frac{(\kappa_i - \nu^{n+1})}{\nu^{n+1} - a} \\
&\quad\geq \frac{2\kappa_1}{(\nu^{n+1} - a)^2} \sum_{i\in L} f_i \frac{u_i^2}{u^2} (\kappa_i^2 - (a + \nu^{n+1})\kappa_i + a\nu^{n+1}) \\
&\quad\quad - \frac{2\theta}{a} \frac{\kappa_1}{\nu^{n+1} - a} \sum_{i\in L} f_i (\kappa_i - \nu^{n+1})^2 + 2b(1-2\theta)\kappa_1 \sum_{i\in L} f_i \frac{u_i^2}{u^2} \frac{(\kappa_i - \nu^{n+1})}{\nu^{n+1} - a} \\
&\quad\geq -\frac{6\sigma}{a}\kappa_1 - \frac{2b\kappa_1}{\nu^{n+1} - a}(1 - (\nu^{n+1})^2) \sum f_i - \frac{2\theta}{a} \frac{\kappa_1}{\nu^{n+1} - a} \sum_{i\in L} f_i (\kappa_i - \nu^{n+1})^2.
\end{aligned} \tag{17}$$

In deriving (17) we have used that $\kappa_i f_i \leq \sigma$ for each i and that $\nu^{n+1} \geq 2a$. We now fix $\theta = \frac{a^2}{4}$ and $0 < b \leq \frac{a}{4}$. From (16) and (17) we see that the right hand side of (15) at $\mathbf{x}_0$ is strictly greater than

$$\sigma\Big(1 + \kappa_1^2 - \frac{8}{a}\kappa_1 - \frac{8}{a^3}\Big)\ . \tag{18}$$

Then (18) is strictly positive if for example $\kappa_1 \geq \frac{8}{a^{\frac{3}{2}}}$. Therefore $\kappa_1 \leq \frac{8}{a^{\frac{3}{2}}}$ at $\mathbf{x}_0$, completing the proof of Theorem 5.1. $\square$

6 Strict Euclidean starshapedness for convex solutions

In this section we give the proof (taken from [[8]]) of Theorem 1.4 by direct construction in Theorem 6.1 below of a strictly starshaped locally strictly convex solution with boundary in the horosphere $\{x_{n+1} = \varepsilon\}$. Then by compactness and uniqueness, we can pass to the limit as ε tends to zero. We use the continuity method by deforming from the horosphere solution $u \equiv \varepsilon$ for $\sigma = 1$. Under this deformation we will show that the property of being strictly sharshaped, i.e. $X \cdot \nu > 0$, persists as long as a solution exists. This property is intertwined with the demonstration that the full linearized operator has trivial kernel.

Suppose Σ is locally represented as the graph of a function $u \in C^2(\Omega)$, $u > 0$, in a domain $\Omega \subset \mathbb{R}^n$:

$$\Sigma = \{(x, u(x)) \in \mathbb{R}^{n+1} :\ x \in \Omega\}.$$

oriented by the upward (Euclidean) unit normal vector field ν to Σ:

$$\nu = \Big(\frac{-Du}{w}, \frac{1}{w}\Big), \quad w = \sqrt{1 + |Du|^2}.$$

The Euclidean metric and second fundamental form of Σ are given respectively by

$$g^e_{ij} = \delta_{ij} + u_i u_j, \;\; h^e_{ij} = \frac{u_{ij}}{w}.$$

According to [3], the Euclidean principal curvatures $\kappa^e[\Sigma]$ are the eigenvalues of the symmetric matrix $A^e[u] = \{a^e_{ij}\}$:

$$a^e_{ij} := \frac{1}{w} \gamma^{ik} u_{kl} \gamma^{lj}, \tag{1}$$

where

$$\gamma^{ij} = \delta_{ij} - \frac{u_i u_j}{w(1 + w)}. \tag{2}$$

Note that the matrix $\{\gamma^{ij}\}$ is invertible with inverse

$$\gamma_{ij} = \delta_{ij} + \frac{u_i u_j}{1 + w} \tag{3}$$

which is the square root of $\{g^e_{ij}\}$, i.e., $\gamma_{ik}\gamma_{kj} = g^e_{ij}$. By (16) the hyperbolic principal curvatures $\kappa[u]$ of Σ are the eigenvalues of the matrix $A[u] = \{a_{ij}[u]\}$:

$$a_{ij}[u] := \frac{1}{w}\Big(\delta_{ij} + u\gamma^{ik}u_{kl}\gamma^{lj}\Big). \tag{4}$$

Let $\mathcal{S}$ be the vector space of $n \times n$ symmetric matrices and

$$\mathcal{S}_+ = \{A \in \mathcal{S} : \lambda(A) \in K_n^+\},$$

where $\lambda(A) = (\lambda_1, \ldots, \lambda_n)$ denotes the eigenvalues of A. Define a function F by

$$F(A) = f(\lambda(A)), \;\; A \in \mathcal{S}_+. \tag{5}$$

We denote

$$F^{ij}(A) = \frac{\partial F}{\partial a_{ij}}(A), \;\; F^{ij,kl}(A) = \frac{\partial^2 F}{\partial a_{ij}\partial a_{kl}}(A). \tag{6}$$

The matrix $\{F^{ij}(A)\}$, which is symmetric, has eigenvalues $f_1, \ldots, f_n$, and therefore is positive definite for $A \in \mathcal{S}_+$ if f satisfies (6), while (7) implies that F is concave for $A \in \mathcal{S}_+$ (see [2]), that is

$$F^{ij,kl}(A)\xi_{ij}\xi_{kl} \leq 0, \;\; \forall \, \{\xi_{ij}\} \in \mathcal{S}, \; A \in \mathcal{S}_+. \tag{7}$$

We have

$$F^{ij}(A)a_{ij} = \sum f_i(\lambda(A))\lambda_i, \tag{8}$$

$$F^{ij}(A)a_{ik}a_{jk} = \sum f_i(\lambda(A))\lambda_i^2. \tag{9}$$

Problem (1)-(2) reduces to the Dirichlet problem for a fully nonlinear second order equation which we shall write in the form

$$G(D^2u, Du, u) = \sigma, \;\; u > 0 \;\; \text{in } \Omega \subset \mathbb{R}^n \tag{10}$$

with the boundary condition

$$u = 0 \;\; \text{on } \partial\Omega. \tag{11}$$

The function G in equation (10) is determined by

$$G(D^2u, Du, u) = F(A[u]) \tag{12}$$

where $A[u] = \{a_{ij}[u]\}$ is given by (4). Let

$$\mathcal{L} = G^{st}\partial_s\partial_t + G^s\partial_s + G_u \tag{13}$$

be the linearized operator of G at u, where

$$G^{st} = \frac{\partial G}{\partial u_{st}},\ G^s = \frac{\partial G}{\partial u_s},\ G_u = \frac{\partial G}{\partial u}. \tag{14}$$

We shall not need the exact formula for G^s but note that

$$\begin{aligned} G^{st} &= \frac{u}{w}F^{ij}\gamma^{is}\gamma^{jt} \\ G^{st}u_{st} &= uG_u = G - \frac{1}{w}\sum F^{ii} \end{aligned} \tag{15}$$

and

$$G^{pq,st} := \frac{\partial^2 G}{\partial u_{pq}\partial u_{st}} = \frac{u^2}{w^2}F^{ij,kl}\gamma^{is}\gamma^{tj}\gamma^{kp}\gamma^{ql} \tag{16}$$

where $F^{ij} = F^{ij}(A[u])$, etc. It follows that, under condition (6), equation (10) is elliptic for u if $A[u] \in \mathcal{S}_+$, while (7) implies that $G(D^2u, Du, u)$ is concave with respect to D^2u.

Since $X \cdot \nu = \frac{u - \sum x_k u_k}{w}$, the following lemma is important.

Lemma 6.1. $\mathcal{L}(u - \sum x_k u_k) = 0$.

Proof. Write $\mathcal{L} = L + G_u$. Since horizontal translation is an isometry, $\mathcal{L}(u_k) = 0$. Then

$$\mathcal{L}(\sum x_k u_k) = L(\sum x_k u_k) + G_u \sum x_k u_k$$

$$= \sum [x_k L(u_k) + u_k L(x_k) + 2G^{ij}\delta_{ki}u_{kj}] + G_u \sum x_k u_k$$

$$= \sum x_k \mathcal{L} u_k + \sum u_k G^k + 2G^{ij}u_{ij} = G^{ij}u_{ij} + G^k u_k + G_u u = \mathcal{L}u\ ,$$

since $G^{ij}u_{ij} = uG_u$.

Lemma 6.2. *Suppose $\mathcal{L}\phi = 0$ in Ω, $\phi = 0$ on $\partial\Omega$ and there exists $w > 0$ in $\overline{\Omega}$ satisfying $\mathcal{L}w = 0$. Then $\phi \equiv 0$.*

Proof. Set $h = \frac{\phi}{w}$. Then a simple computation shows that h satisfies $Lh + 2G^{ij}\frac{w_i}{w}h_j = 0$ in Ω, $h = 0$ on $\partial\Omega$. The lemma now follows by the maximum principle. $\square$

Consider for $0 \leq t \leq 1$, the family of Dirichlet problems

$$\begin{aligned} G(D^2u^t, Du^t, u^t) = \sigma^t := t\sigma + (1-t) \qquad &\text{in } \Omega, \\ u^t = \varepsilon \qquad &\text{on } \partial\Omega, \\ u^0 \equiv \varepsilon. & \end{aligned} \tag{17}$$

Theorem 6.1. Let Ω be a strictly starshaped $C^{2+\alpha}$ domain. Then the Dirichlet problem

$$\begin{aligned} G(D^2u, Du, u) = \sigma \;\; &\text{in } \Omega, \\ u = \varepsilon \;\; &\text{on } \partial\Omega, \end{aligned} \tag{18}$$

has a smooth solution.

For Ω a $C^{2+\alpha}$ strictly starshaped domain, we find (starting from $u^0 \equiv \varepsilon$) a smooth family of solutions u^t, $0 \leq t \leq 2t_0$ by the implicit function theorem since $G_u|_{u^0} \equiv 0$ implies $\mathcal{L}^0$ initially has trivial kernel. By elliptic regularity it is now well understood that if we can find uniform estimates in C^2 for $0 < t_0 \leq t \leq 1$ and if $\mathcal{L}^t$ has trivial kernel, then the set of t for which we can solve (17) is both open and closed. By Lemma 3.1, Lemma 3.2 and Theorem 1.2. $||u^t||_{C^2}$ is uniformly bounded independent of t, by a constant depending only on σ and the exterior ball condition satisfied by Ω. Hence to solve the Dirichlet problem (17) for $t = 1$ it remains only to show that $\mathcal{L}^t$ has trivial kernel. Note that for t sufficiently small, $w^t := u^t - \sum x_k u^t_k > 0$ in $\overline{\Omega}$ and $\mathcal{L}^t w^t = 0$ by Lemma 6.1. Moreover for $\mathbf{n}$ the exterior unit normal to $\partial\Omega$, $w^t = \varepsilon - \sum x_k u_k = \varepsilon + |\nabla u^t|\, x \cdot \mathbf{n} > \varepsilon$ on $\partial\Omega$ since $\partial\Omega$ is strictly starshaped. Since for t sufficiently small, $w^t > 0$ and $w^t > \varepsilon$ on $\partial\Omega$, the maximum principle implies $w^t > 0$ on $\overline{\Omega}$ as long as w^t exists. Hence by Lemma 6.2, $\mathcal{L}^t$ has trivial kernel and Theorem 6.1 is proven. $\square$

7 Uniqueness for mean convex Ω

In this section we give the proof (taken from [[8]]) of Theorem 1.5 . The main step is to show there is always a solution $\Sigma_2 = \text{graph}(u)$ of the asymptotic probem (1)-(2) in Ω with $G_u < 0$ and moreover $u \le v$ for any other solution $\Sigma_1 = \text{graph}(v)$. Then we show that Σ_2 is the unique solution. The proof we give is slightly circuitous in order to avoid delicate issues of boundary regularity caused by the degeneracy of the problem at the asymptotic boundary.

Proposition 7.1. Suppose that the Euclidean mean curvature $\mathcal{H}_{\partial D} \ge 0$. Then for any smooth solution $\Sigma = \text{graph}(u)$ of (1) over D with $u = \varepsilon > 0$ on ∂D, we have $G_u < 0$ in D. Consequently the linearized operator $\mathcal{L}$ satisfies the maximum principle and so has trivial kernel.

Proof. Let $\eta = \frac{\sigma - \nu^{n+1}}{u}$. Then by (15), $G_u \le \eta$ so we need to show $\eta < 0$ in D. According to Theorem 4.1, η cannot have an interior maximum. Suppose the maximum of η is achieved at $0 \in \partial D$ and choose coordinates so that the x_n direction is the interior unit normal to ∂D at 0. Then at 0,

$$\eta_n = \frac{u_n u_{nn}}{uw^3} - \eta \frac{u_n}{u} < 0 \text{ or equivalently } \tfrac{u_{nn}}{w^3} < \eta. \tag{1}$$

On the other hand by the concavity of $f(\kappa)$, the hyperbolic mean curvature $H(\Sigma) \ge \sigma$. Equivalently,

$$\frac{1}{w}(\delta_{ij} - \frac{u_i u_j}{w^2})u_{ij} \ge n\eta \tag{2}$$

Restricting (2) to ∂D implies (since $\sum_{\alpha<n} u_{\alpha\alpha} = -u_n(n-1)\mathcal{H}_{\partial D}$)

$$\frac{u_{nn}}{w^3} - \frac{u_n}{w}(n-1)\mathcal{H}_{\partial D} \ge n\eta \tag{3}$$

Combining (1) and (3) yields $\eta(0) < -\frac{u_n}{w}\mathcal{H}_{\partial D} \le 0$. This gives $\sup_{\partial D} \eta < 0$ so $\eta < 0$ in D. $\square$

Proposition 7.2. Let $f(\kappa)$ satisfy (3)-(9) and also (10). Let $D \in C^{2+\alpha}$ be as in Proposition 7.1. Then for $s \in (0,1)$ and ε sufficiently small, there is a continuous family of solutions $\Sigma^t = \mathrm{graph}(u^t)$, $0 \le t \le 1$ of (1) over D with $u^t = \varepsilon > 0$ on ∂D and $f(\kappa) = ts + (1-t)$ on Σ^t such that u^t tends uniformly to ε in $\overline{D}$ as t tends to 0. Moreover $G_u < 0$ on D for each solution u^t.

Proof. Consider for $0 \le t \le 1$ the family of Dirichlet problems

$$\begin{aligned} G(D^2u^t, Du^t, u^t) = s(t) := ts + (1-t) \qquad &\text{in } \Omega, \\ u^t = \varepsilon \qquad &\text{on } \partial\Omega, \\ u^0 \equiv \varepsilon. \qquad & \end{aligned} \tag{4}$$

For D a $C^{2+\alpha}$ domain, we find (starting from $u^0 \equiv \varepsilon$) a smooth family of solutions u^t, $0 \le t \le 2t_0$ by the implicit function theorem since $G_u|_{u^0} \equiv 0$. By Propositon 7.1, the linearized operator (at a solution u^t) satisfies the maximum principle, i.e. $G_u < 0$, and so has trivial kernel. Hence the set of t for which (4) is solvable is open. By elliptic regularity it is now well understood that if we can find uniform estimates in C^2 for $0 < t_0 \le t \le 1$ then we can solve (1). In [5] we obtained such estimates $u^t + |Du^t| + u^t|D^2u^t| \le C$ where C depends on D, s and the uniformity of constants in (10). Hence the Proposition follows. □

Corollary 7.1. Let $f(\kappa)$ satisfy (3)-(9) and let D be as in Proposition 7.2. Then for any $\sigma \in (0,1)$ there exist a solution u of the asymptotic problem (1)-(2) in D with bounded principal curvatures and $G_u < 0$.

Proof. Given $f(\kappa)$ satisfying (3)-(9), let $f^\theta := (1-\theta)f + \theta K^{\frac{1}{n}}$. Then f^θ satisfies (3)-(9) and also (10). We can apply Proposition 7.2 with $s = \sigma$ and obtain a solution of the approximate problem $f^\theta = \sigma$ with $u = \varepsilon$ on ∂D. Letting $\varepsilon \to 0$ yields a solution u^θ of the asymptotic problem for $f^\theta = \sigma$. By Theorem 1.2 the principal curvatures of $\Sigma^\theta = \mathrm{graph}(u^\theta)$ are uniformly bounded by a constant C depending only on D and σ. Hence as $\theta \to 0$ we obtain by passing to a subsequence the desired solution u. □

Proposition 7.3. Let Ω be a $C^{2+\alpha}$ domain with $\mathcal{H}_{\partial\Omega} \geq 0$. Let $\Sigma_v =$ graph(v) be a solution of the asymptotic problem (1)-(2) where $f(\kappa)$ satisfy (3)-(9). Then $v \geq u$ where u is a solution of the asymptotic problem satisfying $G_u < 0$. The inequality is strict unless $v \equiv u$.

Proof. By Theorem 1.2, $f^\theta \leq (1-\theta)\sigma + C\theta$ on Σ_v. Let Γ_ε be the inward parallel hypersurface to $\Gamma = \partial\Omega$ at distance ε and let D_ε be the domain bounded by Γ^ε. Then for ε sufficiently small, $\Gamma^\varepsilon \in C^{2+\alpha}$ and $\mathcal{H}_{\Gamma^\varepsilon} \geq 0$. Moreover, $v \geq 2\lambda\varepsilon$ in D_ε for a uniform constant $\lambda > 0$ independent of ε. Now let $s = (1-\theta)\sigma + C\theta$ and let u^t be the continuous family of solutions (θ fixed) given in Proposition 7.2 with ε replaced by $\lambda\varepsilon$ with θ so small that $\frac{\sigma}{2} < s < \frac{1+\sigma}{2}$. Then for t close to 0, $u^t < v$ in D and by the maximum principle this property must continue until $t = 1$. As $\varepsilon \to 0$ we obtain $v \geq u^\theta$ in Ω. Finally as $\theta \to 0$ we obtain $v \geq u$ in Ω. □

We now prove Theorem 1.5. Suppose that $\Sigma_1 =$ graph(v) and $\Sigma_2 =$ graph(u) are two distinct solutions of the asymptotic problem (1)-(2) with $G_u < 0$ in Ω. By Proposition 7.3 either $v > u$ or $v \equiv u$ in Ω. Suppose for contradiction that $\max_\Omega(v-u) = v(x_0)-u(x_0) > 0$. Set $w^t := tv+(1-t)u$. We claim that in a small neighborhood of x_0, graph(w^t) is locally strictly convex, that is, $\frac{(w^t)^2+|x-x_0|^2}{2}$ is strictly Euclidean convex. At x_0, $\nabla u = \nabla v$ and $D^2 v \leq D^2 u$. A simple computation shows

$$w^t w^t_{ij} - tvv_{ij} - (1-t)uu_{ij} = t(1-t)(v-u)(u_{ij} - v_{ij}) \geq 0 \ \text{ at } x_0 \ .$$

Hence at x_0, $w^t w^t_{ij} + w^t_i w^t_j + \delta_{ij} \geq t(vv_{ij} + v_i v_j + \delta_{ij}) + (1-t)(uu_{ij} + u_i u_j + \delta_{ij}) > 0$ and the claim follows.

Note that $\frac{d}{dt}G(D^2 w^t, Dw^t, w^t)(x_0) = \mathcal{L}^t w(x_0)$ where $w(x_0) = (v - u)(x_0) > 0$. Evaluating at $t = 0$ gives

$$\frac{d}{dt}G(D^2 w^t, Dw^t, w^t)(x_0)|_{t=0} = G^{ij} w_{ij}(x_0) + G_u w(x_0) < 0$$

since $(G^{ij}) > 0$, $(w_{ij})(x_0) < 0$, $\nabla w(x_0) = 0$, $G_u < 0$ and $w(x_0) > 0$. Hence for $t > 0$ small enough, $\varphi(t) := G(D^2 w^t, Dw^t, w^t)(x_0) < \sigma$. In particular there is a $t_0 \in (0,1]$ such that

$$\varphi(t_0) = \sigma,\ \varphi(t) < \sigma \ \text{ on } (0, t_0)\ .$$

Using the integral form of the mean value theorem, we may write

$$0 = \varphi(t_0) - \varphi(0) = [a^{ij} w_{ij} + a^s w_s + c(x)w](x_0) := Lw(x_0) + c(x_0)w(x_0)\ ,$$

where

$$a^{ij} = \int_0^{t_0} G^{ij}|_{w^t} dt,\ a^s = \int_0^{t_0} G^s|_{w^t} dt,\ c(x) = \int_0^{t_0} G_u|_{w^t} dt\ .$$

Since graph(w^t) is hyperbolic locally strictly convex in a small neighborhood of x_0, the operator $L = a^{ij}\frac{\partial^2}{\partial x_i \partial x_j} + a^s \frac{\partial}{\partial x_s}$ is elliptic in this neighborhood. Suppose for the moment that also $c(x_0) < 0$. Then $Lw(x_0) = -c(x_0)w(x_0) > 0$ and w has a strict interior maximum at x_0 contradicting the maximum principle.

We show $c(x_0) < 0$ to complete the proof of Theorem 1.5. According to (15), $w^t G_u|_{w^t}(x_0) = \varphi(t) - \nu_u^{n+1}(x_0) < \sigma - \nu_u^{n+1}(x_0) < 0$ on $(0, t_0)$. Hence $c(x_0) = \int_0^{t_0} G_u|_{w^t}(x_0) dt < 0$.

8 Hyperbolic-de Sitter space duality and the asymptotic Plateau problem

In this section we sketch the proof of the duality theorem [12].

Theorem 8.1. Let L be defined by (30) and let x be defined by (31). Then the image of S by L is the locally strictly convex graph (with respect to the induced hyperbolic metric)

$$\Sigma = \{(x, u(x)) \in \mathbb{R}^{n+1}_+ : u \in C^\infty(\overline{\Omega}), u(x) > 0\},$$

with principal curvatures $\kappa_i^* = \kappa_i^{-1}$. Here $\kappa_i > 0$, $i = 1, \ldots, n$ are the principal curvatures of S with respect to the induced de Sitter metric. Moreover the inverse map $L^{-1} : \Sigma \to S$ defined by

$$L^{-1}((x, u(x))) = (x + u(x)\nabla u(x), u(x)\sqrt{1 + |\nabla u(x)|^2}) \ \ y \in \Omega$$

is the dual Legendre transform and hodograph map $y = \nabla q(x)$.

Lemma 8.1. *The Legendre transform $q(x)$ is given by*

$$q(x) = \frac{1}{2}(x^2 + u(x)^2) \quad \textit{where } u(x) := v(y)\sqrt{1 - |\nabla v(y)|^2}\ .$$

Moreover, $\sqrt{1 + |\nabla u(x)|^2} = (1 - |\nabla v(y)|^2)^{-\frac{1}{2}}$ *and* $v(y) = u(x)\sqrt{1 + |\nabla u(x)|^2}$. *Therefore* $y = \nabla q(x)$, $(q_{ij}(x)) = (p_{ij}(y))^{-1}$ *and the inverse map* L^{-1} *of* L *is given by* $L^{-1}(x, u(x)) = (y, v(y))$.

Proof. We calculate

$$\begin{gathered} p(y) + q(x) = \tfrac{1}{2}(y^2 - v(y)^2) + \tfrac{1}{2}(x^2 + u(x)^2) \\ = \tfrac{1}{2}(y^2 - v(y)^2) + \tfrac{1}{2}(y^2 - 2v(y)y \cdot \nabla v(y) + v^2|\nabla v|^2) + \tfrac{1}{2}(v^2(1 - |\nabla v(y)|^2) \\ = y^2 - v(y)y \cdot \nabla v(y) = y \cdot x\ , \end{gathered}$$

as required. It is then standard that $y = \nabla q(x)$ and $(q_{ij}(x)) = (p_{ij}(y))^{-1}$. Then $y = \nabla q(x) = x + u\nabla u(x)$ and $x = y - v(y)\nabla v(y)$ implies $v\nabla v = u\nabla u$ so $v^2|\nabla v|^2 = u^2|\nabla u|^2 = v^2(1 - |\nabla v|^2)|\nabla u|^2$ and so $|\nabla u(x)|^2 = \frac{|\nabla v(y)|^2}{1 - |\nabla v(y)|^2}$. Therefore,

$$\sqrt{1 + |\nabla u(x)|^2} = (1 - |\nabla v|^2)^{-\frac{1}{2}} \quad \text{and } v(y) = u(x)\sqrt{1 + |\nabla u(x)|^2}\ . \qquad \square$$

Proof of Theorem 1.6: By Lemma 8.1, it remains only to show that the principal curvatures of Σ are κ_i^{-1} . The principal curvatures of S, Σ are respectively the eigenvalues of the matrices

$$A[v] = (\gamma^{ij})(h_{ij})(\gamma^{ij}), \quad A[u] = (\gamma^{*ij}))(h^*_{ij})(\gamma^{*ij})\ ,$$

where

$$g_{ij} = \frac{\delta_{ij} - v_i v_j}{v^2}, \quad (\gamma^{ij}) = (g_{ij})^{-\frac{1}{2}}, \quad h_{ij} = \frac{\delta_{ij} - v v_{ij} - v_i v_j}{v^2\sqrt{1 - |\nabla v|^2}}\ ,$$

$$g^*_{ij} = \frac{\delta_{ij} + u_i u_j}{u^2}, \quad (\gamma^{*ij}) = (g^*_{ij})^{-\frac{1}{2}}, \quad h^*_{ij} = \frac{\delta_{ij} + u u_{ij} + u_i u_j}{u^2\sqrt{1 + |\nabla u|^2}}\ .$$

By Lemma 8.1,

$$h^*_{ij} = \frac{q_{ij}}{u^2\sqrt{1+|\nabla u|^2}} = \frac{v^2\sqrt{1-|\nabla v|^2}}{v^2u^2} q_{ij} = \frac{1}{u^2v^2}(h_{ij})^{-1} \ ,$$

$$g^*_{ij} = \frac{\delta_{ij}+u_iu_j}{u^2} = \frac{\delta_{ij}+\frac{v_iv_j}{1-|\nabla v|^2}}{u^2} = \frac{g^{ij}}{u^2v^2}, \quad (\gamma^{*ij}) = uv(\gamma^{ij})^{-1} \ ,$$

and therefore $A[u] = (A[v])^{-1}$ completing the proof.

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