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# Integer index of $p$-broom-like graphs

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**Abstract**

A $p$-broom-like graph is obtained by identifying each vertex of a $p$-clique with the root of a copy of a rooted broom. We prove that the class of $p$-broom-like graphs with $n$ vertices is total and strictly ordered by the index (the largest eigenvalue of the adjacency matrix). Moreover, for $p$-broom-like maximal graphs (higher index) we obtain integrality conditions, both for the index as for the graph itself, proving the existence of an integral $p$-broom-like graph for each $p$-clique with $p \geq 5$.

## 1 Introduction

Given a connected simple undirected graph $G$, with $n$ vertices, its adjacency matrix $A(G)$ is a square matrix of order $n$, whose entries are

$$a_{ij} = \begin{cases} 1, & \text{if } ij \text{ is an edge of } G; \\ 0, & \text{otherwise.} \end{cases}$$

The eigenvalues of this matrix are also called eigenvalues of the graph $G$. As $A(G)$ is symmetric, its eigenvalues are all real and, therefore,

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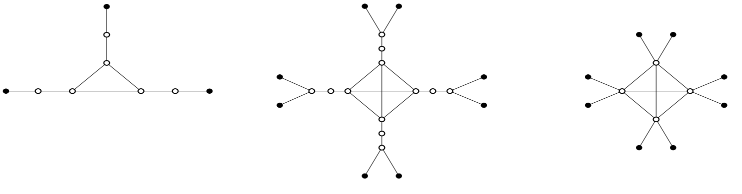
can be displayed in non-increasing order $\lambda_1(G) \geq \lambda_2(G) \geq \ldots \geq \lambda_n(G)$. In fact, the first inequality is strict, since Perron-Frobenius Theorem [4] guarantees that the largest eigenvalue has multiplicity 1. We refer to it by *index* of the graph, denoted simply by $\lambda(G)$. In this paper, we study the index of $p$-broom-like graphs, which are a variation of the graph defined by Tyomkyna and Uzzellem in [7]. Initially, let us define the rooted broom $B(a;r)$: this is the tree with $a+r$ vertices, obtained by hanging $a \geq 1$ edges in the vertex $v_1$ of the path $P_r = v_1 \ldots v_r$ and considering its root in the vertex $v_r$ of $P_r$. If $r = 1$, the rooted broom $B(a;r)$ coincides with the star $S_{a+1}$ rooted in its central vertex.

A $p$-broom-like graph $K_p \sqcap B(a;r)$ is a graph with $p(a+r)$ vertices, obtained by *hierarchical product* [1] of the complete graph $K_p$ by the rooted broom $B(a;r)$, where $p \geq 3$, $a \geq 1$ e $r \geq 1$, that is, identifying each vertex of the $p$-clique with the root of a copy of $B(a;r)$. Some examples of $p$-broom-like graphs are shown in Figure 1.



(a) $K_3 \sqcap B(1;2)$ (b) $K_4 \sqcap B(2;3)$ (c) $K_4 \sqcap B(2;1)$

Figure 1: *p-broom-like graphs*

As observed by Stevanović [6], "*one of the basic and hardest problems of spectral graph theory is to reconstruct a graph from its spectrum alone. This problem has been solved for a few well defined families of spectra* (...)" In the first section we discuss this problem in the class of $p$-broom-like graphs with $n$ vertices, proving that this class is complete and strictly ordered by the index. Therefore, there are no coespectral pairs of non-isomorphic $p$-broom-like graphs.

Another problem, also considered quite difficult, is to build families of integral graphs, i.e., graphs whose eigenvalues are all integers. Stevanović [6], as Cvetković and Simić [3] indicate a growing interest in this area. In the second section we study the integrality of maximal $p$-broom-like graphs (i.e., having largest index), obtained when $r = 1$. We proved that, if $p = 3, 4$, there are no integral graphs in this class. On the other hand, for each $p \geq 5$, we exhibit integral graphs $K_p \sqcap B(a; 1)$.

## 2 Ordering the class of $p$-broom-like graphs

In the following theorem we prove that the class of $p$-broom-like graphs with $n$ vertices is totally ordered. As the number of vertices is $n = p\cdot(a+r)$, we are considering the parameters $p$ and $c = a + r$ constants. More specifically, we show that there is a total and strict ordering , given by the index of these graphs:

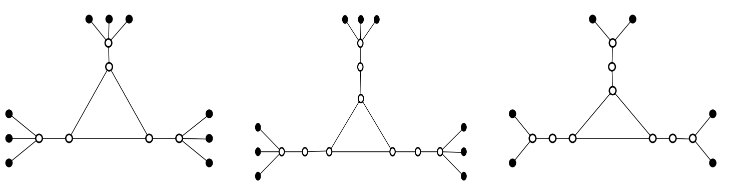
$$\lambda(K_p \sqcap B(c-1; 1)) > \lambda(K_p \sqcap B(c-2; 2)) > \ldots > \lambda(K_p \sqcap B(1; c-1)) \ .$$

**Theorem 2.1.** Let $p \geq 3$ and $c \geq 2$ be integers. Then:

$$\lambda(K_p \sqcap B(a-1; c-a+1)) < \lambda(K_p \sqcap B(a; c-a)) \ ,$$

for all $2 \leq a \leq c-1$.

Proof. The proof of this theorem is divided in two parts. In the first part, we consider $r = c - a \geq 2$ and we use the method of subdividing an edge of an *internal path* of the graph [5], illustrated in Figure 2.



(a) $\lambda \approx 2,65544$ (b) $\lambda \approx 2,54926$ (c) $\lambda \approx 2,51658$

Figure 2: *Subdividing an edge in three internal paths of* $K_3 \sqcap B(3; 2)$

As $r = c - a \geq 2$, $K_p \sqcap B(a;r)$ has $p$ internal paths, each one given by the edges of the path $P_r$ contained in the rooted broom $B(a;r)$. By a result from Hoffman and Smith [5] (see also in [8], Lemma 2.5) it follows that, subdividing an edge of each of these paths, the index of the graph obtained, namely $K_p \sqcap B(a;r+1)$, will be strictly smaller than the index of the original graph. Since $K_p \sqcap B(a-1;r+1)$ is a proper subgraph of $K_p \sqcap B(a;r+1)$ and they are both connected, it follows that $\lambda(K_p \sqcap B(a-1;r+1)) < \lambda(K_p \sqcap B(a;r+1))$. Therefore, we prove in this case that $\lambda(K_p \sqcap B(a-1;c-a+1)) < \lambda(K_p \sqcap B(a;c-a))$, for all $2 \leq a \leq c-2$.

Now, in the second part of the proof, we considere $r = c - a = 1$, and we compare the indices of $K_p \sqcap B(a;1)$ and $K_p \sqcap B(a;2)$ computing their characteristic polynomial. The characteristic polynomial of $K_p \sqcap B(a;1)$ can be calculated using Theorem 2.2.5 of [2], obtaining

$$p_{K_p \sqcap B(a;1)} = x^{p(a-1)}\ [x^2 - (p-1)x - a]\ [x^2 + x - a]^{p-1}\ . \tag{1}$$

Therefore, the index $\gamma_1$ of $K_p \sqcap B(a;1)$ is the largest root of $p_1(x) = x^2 - (p-1)x - a$. On the other hand, from Theorem 2.2 of [1], we have that the characteristic polynomial $q(x)$ of $K_p \sqcap B(a-1;2)$ is given by the formula

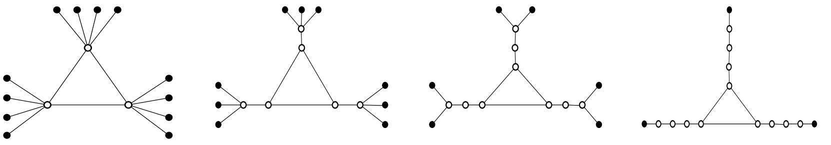
$$q(x) = [p_{S_a}(x)]^p\ \ p_{K_p}\left(\frac{p_{S_{a+1}}(x)}{p_{S_a}(x)}\right)\ .$$

As $p_{S_a}(x) = x^{a-2}(x^2 - a + 1)$ and $p_{K_p} = (x+1)^{p-1}(x-p+1)$, we deduce that

$$q(x) = x^{p(a-2)}\ [x(x^2-a) + x^2 - a + 1]^{p-1}\ [x(x^2-a) - (p-1)(x^2-a+1)]\ .$$

Since the index $\gamma_2$ of $K_p \sqcap B(a-1;2)$ is a root of $q(x)$ with multiplicity 1, it will be the largest root of $p_2(x) = x(x^2-a) - (p-1)(x^2-a+1)$. As we want to prove that $\gamma_2 < \gamma_1$, it is sufficient to check that $p_1(\gamma_2) < 0$. As $\gamma_2 > 0$, we verify that $\gamma_2\ p_1(\gamma_2) < 0$, thereby concluding the proof. □

In Figure 3 we exemplify this theorem ordering the 3-broom-like graphs with 15 vertices.



(a) $\lambda \approx 3,23607$ (b) $\lambda \approx 2,65544$ (c) $\lambda \approx 2,51658$ (d) $\lambda \approx 2,49889$

Figure 3: *Order of* 3*-broom-like graphs with* 15 *vertices*

# 3 Integral $K_p \sqcap B(a;1)$ graphs

From the previous theorem, we have that the maximal $p$-broom-like graphs are given by $K_p \sqcap B(a;1)$. In the following proposition we give necessary and sufficient conditions for the integrality of index of $K_p \sqcap B(a;1)$ and also for the integrality of the other eigenvalues of the graph. We prove here that, for each $p \geq 3$, there is an infinite family of $p$-broom-like graphs having integer index, but not necessarily integral. Indeed, the condition of integrality of these graphs is more difficult to achieve, since the parameter $a$ must satisfy simultaneously two nonlinear diophantine equations. Corollaries 3.1 and 3.2 illustrate this difficulty.

**Proposition 3.1.** Given integers $p \geq 3$ and $a \geq 1$, $K_p \sqcap B(a;1)$ has integer index $\lambda$ if and only if there is a positive integer $q$ such that

$$a = q(p-1+q) \ .$$

In this case, $\lambda = p-1+q$ and $K_p \sqcap B(a;1)$ is integral if and only if there is a positive integer $s$ such that

$$q(p-1+q) = s(s+1) \ .$$

Proof. From (1) we have that the distinct nonzero eigenvalues of $K_p \sqcap B(a;1)$ are:

$$\lambda = \frac{p-1+\sqrt{(p-1)^2+4a}}{2}, \quad \frac{-1+\sqrt{1+4a}}{2}, \quad \frac{p-1-\sqrt{(p-1)^2+4a}}{2} \quad \text{and} \quad \frac{-1-\sqrt{1+4a}}{2}.$$

Thus, $\lambda$ is integer if and only if $(p-1)^2 + 4a$ is a perfect square. The remaining eigenvalues are integers if, furthermore, $1 + 4a$ is also a perfect square. We will verify firstly that $(p-1)^2 + 4a$ is a perfect square if and only if there is a positive integer $q$ such that $a = q(p - 1 + q)$, calculating the index $\lambda$. We do this dividing the analysis in two cases: when $p$ is odd and when $p$ is even.

If $p = 2t + 1$ $(t \geq 1)$, then $(p-1)^2 + 4a = 4(t^2 + a)$ is a perfect square if and only if $t^2 + a = x^2$, for some positive integer $x$. Hence, $a = (x + t)(x - t)$. Considering $q = x - t$, we have $a = q(q + 2t)$ and so, $a = q(q + p - 1)$. It follows that $(p-1)^2 + 4a = 4x^2 = 4[q + \frac{p-1}{2}]^2$ and $\lambda = \frac{p-1}{2} + [q + \frac{p-1}{2}] = q + p - 1$.

If $p = 2t$ $(t \geq 2)$, then $(p-1)^2 + 4a = 4(t^2 - t + a) + 1$ is odd. Thus, it is a perfect square if and only if there is a positive integer $x$ such that $4(t^2 - t + a) + 1 = (2x + 1)^2$, ie $t^2 - t + a = x^2 + x$. Hence, $a = (x + t)(x - t + 1)$. In this case, consider $q = x - t + 1$, which enables us to write $a = q(q + 2t - 1) = q(q + p - 1)$. As $(p-1)^2 + 4a = (2x + 1)^2 = (2q + p - 1)^2$, we have $\lambda = q + p - 1$, as we wanted.

To finish the proof, it remains to verify that $1 + 4a$ is a perfect square if and only if there is a positive integer $s$ such that $a = s(s + 1)$. Indeed, $1 + 4a = x^2$, for some odd positive $x$, if and only if $a = (\frac{x-1}{2})(\frac{x+1}{2})$, since $x - 1$ and $x + 1$ are both even. As $\frac{x+1}{2} = \frac{x-1}{2} + 1$, the assertion is proved. $\square$

From this proposition we see that, if $p = a$, $(p-1)^2 + 4a = (p+1)^2$. Then, $K_p \sqcap B(p; 1)$ has integer index, namely, $\lambda = p$. But this graph is not always integral. Indeed, it will be integral, if we have $p = s(s+1)$, for some positive integer $s \geq 2$. So, $p$ must be even.

**Corollary 3.1.** Given an integer $p \geq 3$, $K_p \sqcap B(p; 1)$ has integer index $\lambda = p$. It is integral if and only if $p$ is even and is given by the product of two consecutive positive integers.

In what follows, we apply Proposition 3.1 in order to obtain infinite families of integral graphs $K_p \sqcap B(a; 1)$, for which $p \neq a$. Clearly, we will consider the case where they have integer index: $a = q(q + p - 1)$, where $q \geq 1$ is integer. First, however, we observe:

**Corollary 3.2.** For all $a \geq 1$, the graphs $K_3 \sqcap B(a;1)$ and $K_4 \sqcap B(a;1)$ are not integral.

In the following corollary we prove, for each $p \geq 5$, the existence of an integral graph $K_p \sqcap B(a;1)$.

**Corollary 3.3.** For each pair of $p$ and $a$ below, we have that $K_p \sqcap B(a;1)$ is integral:

(i) $p = 2t + 1$ and $a = t^2(t^2 - 1)$, for $t \geq 2$ integer. In this case, $\lambda = t(t + 1)$.

(ii) $p = 2t$ and $a = \frac{(t-2)(t-1)}{2} \cdot \frac{t(t+1)}{2}$, for $t \geq 3$ integer. In this case, $\lambda = \frac{t(t+1)}{2}$.

(iii) $p = 6t - 1$ and $a = 4t(4t - 1)$, for $t \geq 1$ integer. In this case, $\lambda = 8t - 2$.

(iv) $p = 6t + 3$ and $a = 4t(4t + 1)$, for $t \geq 1$ integer. In this case, $\lambda = 8t + 2$.

# 4 Conclusion

As noted before, the problems of characterizing, in a specific class, which graphs are determined, up to isomorphism, by its eigenvalues and graphs which are integral are considered in the literature as difficult issues. In the class of $p$-broom-like graphs, we have obtained positive answers to both problems. Regarding the first question, the answer was obtained by ordering the indices of the graphs in this class. Note that, although it is possible to calculate the index of a graph from its adjacency matrix, the same is not true when we know only structural properties of the graph. Actually, for $r = 1$, we calculate the index of $K_p \sqcap B(a;1)$ in function of $a$, but we do not know an expression for the index of $K_p \sqcap B(a;r)$ (valid for any $a$ and $r$). We managed however, to get an order of $p$-broom-like graphs, described in Theorem 2.1.

Regarding the second proposal, we exhibit infinite families of $K_p \sqcap B(a;1)$ integral graphs. Note that to obtain an integral $p$-broom-like graph corresponds to a solution of a system of non linear Diophantine equations, and we prove that there are no solutions of this system if $p = 3, 4$. Considering the case where $p \geq 5$ is odd, we get more than one distinct family of such graphs. We observed that the same occurs if $p$ is even, although we have not determined in this case, another infinite family, parameterized by a single parameter.

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