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AVD-total-colouring of complete equipartite graphs

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Abstract

An *AVD-total-colouring* of a simple graph G is a mapping $\phi : V(G) \cup E(G) \to \mathcal{C}$, with $\mathcal{C}$ a set of colours, such that: (i) for each adjacent or incident elements $x, y \in V(G) \cup E(G)$, $\phi(x) \neq \phi(y)$; (ii) and for each pair of adjacent vertices $x, y \in V(G)$, sets $\{\phi(x)\} \cup \{\phi(xv) : xv \in E(G)\}$ and $\{\phi(y)\} \cup \{\phi(yv) : yv \in E(G)\}$ are distincts. The *AVD-total-chromatic number*, $\chi''_a(G)$, is the smallest number of colours for which G admits an AVD-total-colouring. In 2005, Zhang et al. conjectured that $\chi''_a(G) \leq \Delta(G)+3$ for any simple graph G. In this article this conjecture is verified for complete equipartite graphs G and it is also shown that $\chi''_a(G) = \Delta(G)+2$, if G has even order.

1 Introduction

Let $G := (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The cardinality of $V(G)$ is the *order* of G. We denote an edge $e \in E(G)$ by uv when u and v are its ends. An *element* of G is a

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vertex or an edge of G. As usual, we denote by $d(v)$ the *degree* of a vertex $v \in V(G)$, and by $\Delta(G)$ the *maximum degree* of G.

Let $S := V(G) \cup E(G)$ and let $\mathcal{C}$ be a set of colours. A *total-colouring* of G is a mapping $\phi : S \to \mathcal{C}$, such that for each adjacent or incident elements $x, y \in S$, we have $\phi(x) \neq \phi(y)$. If $|\mathcal{C}| = k$, then mapping ϕ is called a *k-total-colouring* of G. If $S = V(G)$, then ϕ is a *vertex-colouring* of G, and if $S = E(G)$, then ϕ is an *edge-colouring* of G.

The *chromatic number* of G, $\chi(G)$, is the smallest number of colours for which G admits a vertex-colouring. Similarly, we define the *chromatic index* of G, $\chi'(G)$, as the smallest number of colours for which G admits an edge-colouring; and the *total-chromatic number* of G, $\chi''(G)$, as the smallest number of colours for which G admits a total-colouring.

Let ϕ be a total-colouring of G and let $C(u) := \{\phi(u)\} \cup \{\phi(uv) : uv \in E(G)\}$ be the set of colours that *occurs* in a vertex $u \in V(G)$. Two vertices u and v are *distinguishable* when $C(u) \neq C(v)$. If this property is true for every pair of adjacent vertices, then ϕ is an *adjacent-vertex-distinguishing-total-colouring (AVD-total-colouring)*. The *AVD-total-chromatic number*, $\chi''_a(G)$, is the smallest number of colours for which G admits an AVD-total-colouring. If ϕ uses k colours, then it is called a *k-AVD-total-colouring*.

A *vertex-distinguishing-proper-edge-colouring* is an edge-colouring of G that requires $C(u) \neq C(v)$ for each $u, v \in V(G)$. This colouring was first examined by Burris and Schelp [3], and further investigated by many others, including Bazgan et al. [2] and Balister et al. [1]. The motivation for studying vertex-distinguishing-proper-edge-colourings came from irregular networks. In these networks, it is necessary to associate positive integer weights to the edges in such a way that the sum of weights of the edges incident with each vertex form a set of distinct numbers [2]. Zhang et al. [9] considered edge-colourings in which only adjacent vertices were distinguishable. After that, around 2005, they studied the problem of distinguishable vertices in the context of total-colourings [10], giving rise to AVD-total-colourings. In their seminal article, Zhang et al. determined the AVD-total-chromatic number for some classes of graphs and, based on their results, the authors posed the following conjecture:

Conjecture 1.1 (**AVD-total-colouring conjecture**)**.** If G is a simple graph, then $\chi''_a(G) \leq \Delta(G) + 3$.

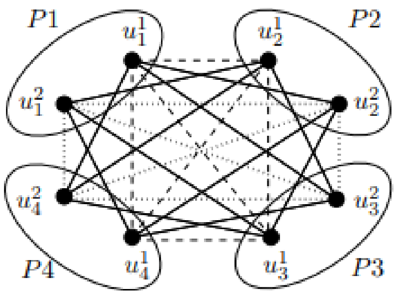
This conjecture has been verified for some classes of graphs, which include complete graphs, complete bipartite graphs, trees [10], hypercubes [4], graphs with $\Delta(G) = 3$ [6], outerplanar graphs [8], indifference graphs [7], and Halin graphs [5]. In this work, we consider the class of complete equipartite graphs. We prove that the AVD-total-colouring conjecture holds for this class and we determine the AVD-total-chromatic number for complete equipartite graphs of even order.

2 Main results

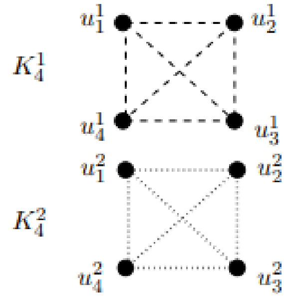
A subset of $V(G) \cup E(G)$ is *independent* if its elements are pairwise nonadjacent and nonincident. For positive integers r and n, a *complete equipartite graph*, $K_{r(n)}$, is a simple graph whose vertex set can be partitioned into r independent sets *(parts)* of cardinality n, where any two vertices that belong to different parts are joined by an edge. In this note, we verify the AVD-total-colouring conjecture for complete equipartite graphs. We consider graphs $K_{r(n)}$ with $r \geq 2$ and $n \geq 2$ since the results when $r < 2$ or $n < 2$ are known [10]. Moreover, we also determine the AVD-total-chromatic number for even order complete equipartite graphs.

A *canonical labelling* of $K_{r(n)}$ is a labelling of the vertices of $K_{r(n)}$, such that for each part j, $1 \leq j \leq r$, each vertex in the part receives a distinct label u^i_j, where $1 \leq i \leq n$. For $r \geq 2$, we define the *canonical decomposition* $[\mathcal{K}, \mathcal{B}]$ of $K_{r(n)}$ as the union of edge-disjoint subgraphs. This decomposition is described in the following.

Let $K_{r(n)}$ be a complete equipartite graph endowed with canonical labelling. Considering $G[S]$ denotes the subgraph induced by set $S \subseteq V(G)$, note that subgraphs $K^i_r := G[\{u^i_1, \ldots, u^i_r\}]$, $1 \leq i \leq n$, are isomorphic to the complete graph K_r. Thus, $K_{r(n)}$ has n disjoint copies of K_r as induced subgraphs. Figure 1 illustrates $K_{4(2)}$ endowed with canonical labelling and two induced subgraphs K_4^1 and K_4^2 isomorphic to K_4.



(a) $K_{4(2)}$ endowed with canonical labelling. The four parts of $K_{4(2)}$ are identified by $P1$, $P2$, $P3$, and $P4$.



(b) Two disjoint induced subgraphs of $K_{4(2)}$ isomorphic to K_4.

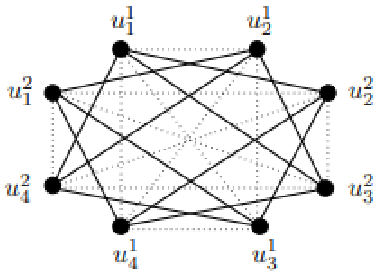
Figure 1: $K_{4(2)}$ and its induced subgraphs K_4^1 and K_4^2 that are isomorphic to K_4.

The subgraph induced by edges joining vertices of K_r^i to vertices of K_r^j, is a bipartite graph, denoted by $B_{ij} = G[V(K_r^i), V(K_r^j)]$, $1 \leq i < j \leq n$. Moreover, B_{ij} is an $(r-1)$-regular graph. In fact, edges $u_x^i u_x^j$ $(1 \leq x \leq r)$ do not exist since vertices u_x^i and u_x^j are in the same part of $K_{r(n)}$. Figure 2 illustrates a $K_{4(2)}$ endowed with canonical labelling and its unique bipartite subgraph, B_{12}, induced by the edges joining vertices from K_4^1 to vertices of K_4^2.

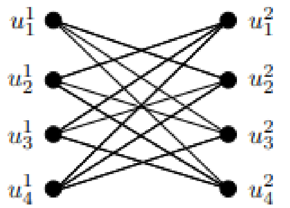
Using the above notation, we define the canonical decomposition $[\mathcal{K}, \mathcal{B}]$ of $K_{r(n)}$ as:

$$\mathcal{K} := \bigcup_{1 \leq i \leq n} K_r^i, \qquad \text{and} \qquad \mathcal{B} := \bigcup_{1 \leq i < j \leq n} B_{ij}.$$

The previous definition implies that $K_{r(n)} \cong (\mathcal{K} \cup \mathcal{B})$. Also, note that $\mathcal{K}$ is a disconnected graph composed by exactly n components K_r^i, each one isomorphic to a complete graph K_r.



(a) $K_{4(2)}$ endowed with canonical labelling. The bold edges induce the bipartite subgraph B_{12}.



(b) The bipartite subgraph B_{12}. Note that edges $u_j^1 u_j^2$, $1 \leq j \leq 4$, do not exist.

Figure 2: $K_{4(2)}$ endowed with canonical labelling and its induced bipartite subgraph B_{12}.

Let G_R be the underlying simple graph obtained from $[\mathcal{K}, \mathcal{B}]$ by shrinking each K_r^i into a vertex v_i. Graph G_R is called the *representative graph* of $K_{r(n)}$ since the previous decomposition can be represented by G_R in the following way: each vertex $v_i \in V(G_R)$ represents a component $K_r^i \subseteq \mathcal{K}$ and each edge $v_i v_j \in E(G_R)$ represents a bipartite graph $B_{ij} \subseteq \mathcal{B}$. Note that $G_R \cong K_n$. For example, observe that the representative graph of $K_{4(2)}$ is the complete graph K_2. Figure 3 illustrates the canonical decomposition of $K_{4(3)}$ and its representative graph.

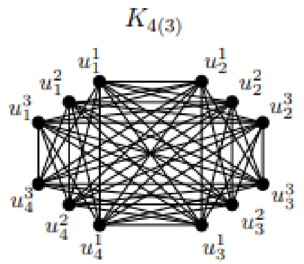
Now, we are ready to establish our main result.

Theorem 2.1. Let $G := K_{r(n)}$ be a complete equipartite graph with $r \geq 2$ and $n \geq 2$. If G has even order, then $\chi''_a(G) = \Delta(G) + 2$; otherwise, $\chi''_a(G) \leq \Delta(G) + 3$.

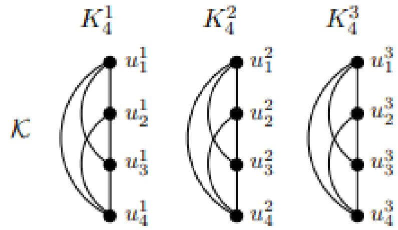
Proof. (Sketch)

Initially, note that $\chi''_a(G) \geq \Delta(G) + 2$ since G has two adjacent vertices of maximum degree. Therefore, to prove Theorem 2.1, it is enough to build a $(\Delta(G) + 2)$-AVD-total-colouring for G of even order and a $(\Delta(G) + 3)$-AVD-total-colouring for G of odd order.

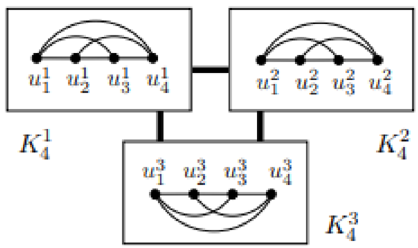
In order to build the required colouring, we decompose $K_{r(n)}$ into the



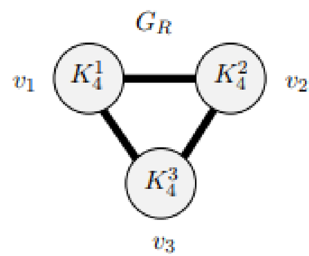
(a) $K_{4(3)}$ endowed with canonical labelling.



(b) The subgraph $\mathcal{K}$ composed by 3 componentes isomorphic to K_4.



(c) A scheme showing the canonical decomposition of $K_{4(3)}$. Each square represents a component $K_r^i \subseteq \mathcal{K}$. Thick lines joining squares represent the edges of bipartite graphs B_{12}, B_{13}, and B_{23}.



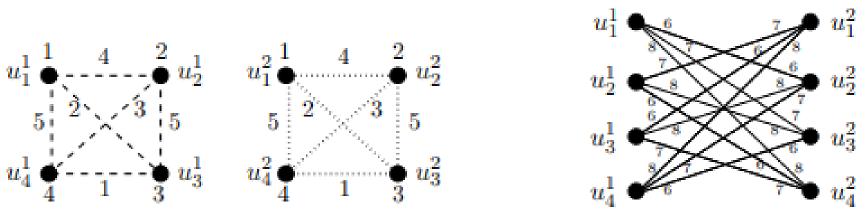
(d) The representative graph G_R of $K_{4(3)}$. Note that $G_R \cong K_3$.

Figure 3: Canonical decomposition of $K_{4(3)}$ and its representative graph G_R.

canonical decomposition $[\mathcal{K}, \mathcal{B}]$ and consider four cases depending on the parity of n and r. In each case, using the representative graph G_R, we assign suitable edge-colourings to subgraph $\mathcal{B}$ and an AVD-total-colouring to subgraph $\mathcal{K}$ in such a way that the result is an AVD-total-colouring of $K_{r(n)}$.

As an illustration, for the case n and r even, the components of subgraph $\mathcal{K}$ receive an AVD-total-colouring with $r+1$ colours, while subgraph $\mathcal{B}$ receives an edge-colouring with $(n-1)(r-1)$ new colours. The result is a $(\Delta(K_{r(n)})+2)$-AVD-total-colouring of $K_{r(n)}$. For example, using the canonical decomposition $[\mathcal{K}, \mathcal{B}]$ of $K_{4(2)}$, Figure 4(a) shows a 5-AVD-total-colouring of subgraph $\mathcal{K}$ and Figure 4(b) shows a 3-edge-colouring of subgraph $\mathcal{B}$. The result is an 8-AVD-total-colouring of $K_{4(2)}$.

■



(a) A 5-AVD-total-colouring of subgraph $\mathcal{K}$ using colours $1, 2, 3, 4, 5$.

(b) The subgraph $\mathcal{B}$ endowed with a 3-edge-colouring using colours $6, 7, 8$.

Figure 4: A canonical decomposition of $K_{4(2)}$ showing its 8-AVD-total colouring.

3 Concluding Remarks

According to Theorem 2.1, the AVD-total-colouring conjecture holds for complete equipartite graphs. Although the conjecture holds for $K_{r(n)}$ of odd order, the AVD-total-chromatic number is not determined for this case. Nevertheless, we have obtained $(\Delta(K_{r(n)})+2)$-AVD-total-colourings for some equipartite graphs of odd order. Based on these findings, we pose the following conjecture.

Conjecture 3.1. If $K_{r(n)}$ has odd order, then $\chi''_a(K_{r(n)}) = \Delta(K_{r(n)}) + 2$.

It is well known that the restriction of an AVD-total-colouring to a proper subgraph H of G is not necessarily an AVD-total-colouring of H. However, we observe that the restriction of our AVD-total-colouring of $K_{r(n)}$, with r and n even, to certain subgraphs of $K_{r(n)}$ is an AVD-total colouring for these subgraphs. Therefore, an extension of this work could be the study of the conditions under which the restriction of AVD-total-colourings of complete equipartite graphs to their proper subgraphs results in AVD-total colourings for these subgraphs.

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