Matemática Contemporânea, Vol 41, 61-74
https://doi.org/10.21711/231766362012/rmc415

Real Determinant Line Bundles

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March 8, 2013

Abstract

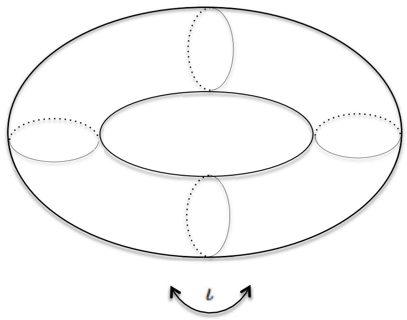
This article is an expanded version of the talk given by Ch. O. at the Second Latin Congress on "Symmetries in Geometry and Physics" in Curitiba, Brazil in December 2010. In this version we explain the topological and gauge-theoretical aspects of our paper "*Abelian Yang-Mills theory on Real tori and Theta divisors of Klein surfaces*" [15].

1 Introduction

In an ongoing research project we intend to develop a version of Seiberg-Witten theory and gauge theoretic Gromov-Witten theory in the presence of Real structures. The main problem is that the corresponding virtual fundamental class will be a homology class with coefficients in a local system (i.e., a locally constant sheaf) which, in order to construct $\mathbb{Z}$-valued invariants, must be determined explicitly.

Example: Let (C, ι) be a Klein surface, i.e., a Riemann surface C endowed with an anti-holomorphic involution ι. Put $g := h^0(\omega_C)$ and suppose $C^\iota \neq \emptyset$.

2000 *AMS Subject Classification.* 14H55, 14H60, 14H40, 14H51, 14P25, 14N10.
Key Words and Phrases. Real determinant line bundles, Klein surfaces, real Theta divisors, Yang-Mills connections
*Partially supported by the ANR grant ANR-10-BLAN-0118 (MNGNK)



For every $d \in \mathbb{N}$ we obtain induced Real structures $\hat{\iota} : \mathrm{Pic}^d(C) \to \mathrm{Pic}^d(C)$, and $\tilde{\iota} : C^{(d)} \to C^{(d)}$ on the complex manifolds $\mathrm{Pic}^d(C)$, and $C^{(d)}$.

In the example illustrated in the picture above, we have $(C^{(2)})^{\tilde{\iota}} \simeq T^2 \coprod (\mathbb{P}^2_{\mathbb{R}} \# \mathbb{P}^2_{\mathbb{R}})$. The fundamental class of the manifold $(C^{(d)})^{\tilde{\iota}}$ is an element

$$[(C^{(d)})^{\tilde{\iota}}] \in H_d((C^{(d)})^{\tilde{\iota}}, \mathcal{O}_{(C^{(d)})^{\tilde{\iota}}}) ,$$

where $\mathcal{O}_{(C^{(d)})^{\tilde{\iota}}}$ denotes its orientation sheaf. To understand this coefficient system we use the cartesian diagram:

$$\begin{array}{ccc} C \times C^{(d)} & \xrightarrow{\tilde{q}} & C^{(d)} \\ \Big\downarrow{\scriptstyle \mathrm{id} \times \rho} & & \Big\downarrow{\scriptstyle \rho} \\ C \times \mathrm{Pic}^d(C) & \xrightarrow{q} & \mathrm{Pic}^d(C) \end{array}$$

One has the following effective divisors:

1. *the universal divisor* $\Delta := \{(x, \delta) \in C \times C^{(d)} |\ x \in \delta\}$ on the product $C \times C^{(d)}$;

2. the ample divisor $D_{x_0} := \{\delta \in C^{(d)} |\ x_0 \in \delta\}$ on $C^{(d)}$, where $x_0 \in C$ is a fixed point.

We denote by $\mathcal{O}(\Delta)$, respectively $\mathcal{O}(D_{x_0})$ the corresponding line bundles on the manifolds $C \times C^{(d)}$, $C^{(d)}$ respectively. On the other hand, on $C \times \mathrm{Pic}^d(C)$ we consider the Poincaré line bundle with respect to x_0, denoted $\mathcal{P}_{x_0}$, which is characterized up to isomorphism by the properties:

$$\mathcal{P}_{x_0}|_{C \times \{[\mathcal{L}]\}} \simeq \mathcal{L} \ \ \forall [\mathcal{L}] \in \mathrm{Pic}^d(C) \ , \ \mathcal{P}_{x_0}|_{\{x_0\} \times \mathrm{Pic}^d(C)} \simeq \mathcal{O}_{\mathrm{Pic}^d(C)} \ .$$

We have a universal short exact sequence on $C \times C^{(d)}$:

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(\Delta) \longrightarrow \mathcal{O}_\Delta(\Delta) \longrightarrow 0$$

Using the well known facts [1]

i) $T_{C^{(d)}} = \tilde{q}_* \mathcal{O}_\Delta(\Delta)$,

ii) $\mathcal{O}(\Delta) = (1 \times \rho)^* \mathcal{P}_{x_0} \otimes \tilde{q}^* \mathcal{O}(D_{x_0})$,

and taking into account that determinant line bundles commute with base change [10], we obtain:

$$\det T_{C^{(d)}} = \det(\tilde{q}_* \mathcal{O}_\Delta(\Delta)) = \det(\tilde{q}_! \mathcal{O}_\Delta(\Delta)) = \det(\tilde{q}_! \mathcal{O}(\Delta)) \otimes \det(\tilde{q}_! \mathcal{O})^\vee \simeq$$

$$\simeq \det(\tilde{q}_![(1 \times \rho)^* \mathcal{P}_{x_0} \otimes \tilde{q}^* \mathcal{O}(D_{x_0})]) \simeq \rho^*[\det(q_! \mathcal{P}_{x_0})] \otimes \mathcal{O}((d+1-g)D_{x_0}) \tag{1}$$

When $x_0 \in C^\iota$ all bundles in these formulae are naturally Real holomorphic line bundles, and the isomorphisms in (1) are isomorphisms of Real line bundles. In particular we see that the orientation line bundle $\det T_{(C^{(d)})^{\tilde{\iota}}}$ of the manifold $(C^{(d)})^{\tilde{\iota}}$ – which can be identified with the fixed point locus of $\det T_{C^{(d)}}$ – is determined by the numerical data g, d and the Real holomorphic line bundle $\det(q_! \mathcal{P}_{x_0})$ on $(\mathrm{Pic}^d(C), \hat{\iota})$. Therefore, in order to determine the orientation sheaf $\mathcal{o}_{(C^{(d)})^{\tilde{\iota}}}$, we have to understand this Real determinant line bundle. To this end consider the Real isomorphism $\varphi_{x_0} : \mathrm{Pic}^0(C) \longrightarrow \mathrm{Pic}^d(C)$ given by

$$\varphi_{x_0}([\mathcal{L}]) = [\mathcal{L} \otimes \mathcal{O}_C(dx_0)] \ .$$

Proposition 1. *Let* $\Theta := \{[\mathcal{L}] \in \mathrm{Pic}^{g-1}(C) |\ h^0(\mathcal{L}) > 0\}$ *be the geometric theta divisor on* $\mathrm{Pic}^{g-1}(C)$. *There exists a canonical isomorphism of Real holomorphic line bundles*

$$\varphi_{x_0}^*[\det(q_! \mathcal{P}_{x_0})] \simeq \mathcal{O}_{\mathrm{Pic}^0(C)}(\Theta - [\mathcal{O}_C((g-1)x_0)]) \ .$$

Remark:

i) The isomorphism type of the underlying differentiable *complex* line bundle of the right hand side can be computed using the Grothendieck-Riemann-Roch theorem for proper morphisms, and the result is independent of x_0. On the other hand, we will see that the isomorphism type of the *Real* structure of this line bundle does depend on x_0, more precisely on the connected component of x_0 in C^ι. This striking result shows that there cannot exist any Grothendieck-Riemann-Roch-type theorem to compute this Real isomorphism type.

ii) When one considers more generally quot-schemes $\mathrm{Quot}^d_{\mathcal{E}}$ instead of the symmetric power $C^{(d)} = \mathrm{Quot}^d_{\mathcal{O}}$, one has to deal with virtual fundamental classes in order to define in a coherent way Real gauge theoretical Gromov-Witten invariants [14].

Therefore, our aim is now to determine $\mathcal{O}_{\mathrm{Pic}^0(C)}(\Theta - [\mathcal{O}_C((g-1)x_0)])$ as a Real holomorphic line bundle on the Real torus $(\mathrm{Pic}^0(C), \hat{\iota})$.

2 Real line bundles

Let (X, τ) be a Real space, i.e. a CW space endowed with an involution. A Real line bundle (in the sense of Atiyah [2]) over (X, τ) is a pair $(L, \tilde{\tau})$, where $L \to X$ is a complex line bundle, and $\tilde{\tau} : L \to L$ is a fibrewise anti-linear isomorphism over τ such that $\tilde{\tau}^2 = \mathrm{id}_L$. The isomorphism classes of Real line bundles over (X, τ) form a group, which is naturally isomorphic to the Grothendieck cohomology group [8]

$$H^1_{\mathbb{Z}_2}(X, \underline{S}^1(1)) = H^2_{\mathbb{Z}_2}(X, \underline{\mathbb{Z}}(1)) \simeq [(X, \tau), (\mathbb{P}^\infty(\mathbb{C}), {}^-)]^{\mathbb{Z}_2} ,$$

where $\underline{S}^1(1)$ denotes the $\mathbb{Z}_2$-sheaf of germs of continuous S^1-valued functions on X endowed with the $\mathbb{Z}_2$-action defined by the involution induced by τ and conjugation on S^1, and $\underline{\mathbb{Z}}(1)$ denotes the $\mathbb{Z}_2$-sheaf with fibre $\mathbb{Z}$ endowed with the $\mathbb{Z}_2$-action defined by the involution induced by τ and $-\mathrm{id}_{\mathbb{Z}}$ (see [8] for the cohomology theory of equivariant sheaves) .

Proposition 2. *Suppose that $X^\tau \neq \emptyset$. There exists a canonical exact sequence*

$$0 \to H^1_{\mathbb{Z}_2}(H^1(X,\mathbb{Z})(1)) \to H^2_{\mathbb{Z}_2}(X,\underline{\mathbb{Z}}(1)) \xrightarrow{c_1} H^2(X,\mathbb{Z})^{-\tau^*} \xrightarrow{\mathcal{O}} H^2_{\mathbb{Z}_2}(H^1(X,\mathbb{Z})(1)) .$$

We also have a natural restriction map

$$H^1_{\mathbb{Z}_2}(X,\underline{S}^1(1)) = H^2_{\mathbb{Z}_2}(X,\underline{\mathbb{Z}}(1)) \longrightarrow H^2(X^\tau,\underline{\mathbb{Z}}(1)) = H^1(X^\tau,\mathbb{Z}_2) ,$$

which maps the isomorphism class $[L,\tilde{\tau}]$ of a Real line bundle on (X,τ) onto $w_1(L^{\tilde{\tau}})$. Here $L^{\tilde{\tau}}$ is regarded as a real line bundle on the fixed point locus X^τ.

Example: For a Klein surface (C,ι) one has

$$H^2_{\mathbb{Z}_2}(C,\underline{\mathbb{Z}}(1)) \simeq H^2(C,\mathbb{Z}) \times_{\mathbb{Z}_2} H^1(C^\iota,\mathbb{Z}_2) ,$$

the isomorphism being given by $[L,\tilde{\iota}] \mapsto (c_1(L), w_1(L^{\tilde{\iota}}))$. The fact that the pair on the right belongs to the fibre product $H^2(C,\mathbb{Z})\times_{\mathbb{Z}_2} H^1(C^\iota,\mathbb{Z}_2)$ follows from the identity

$$\langle c_1(L),[C]\rangle \equiv \langle w_1(L^{\tilde{\iota}}),[C^\iota]\rangle \pmod 2 .$$

3 Gauge theoretical relevance

Let (X,τ) be a compact Real Riemannian manifold (i.e., τ is an isometric involution of X) and let L be a Hermitian line bundle on X. We denote by $\mathcal{T}(L)$ the moduli space of Yang-Mills connections, and put

$$\mathcal{T}_X := \coprod_{c\in H^2(X,\mathbb{Z})} \mathcal{T}(L_c) ,$$

where L_c is a Hermitian line bundle on X with $c_1(L_c) = c$. Note that $\mathcal{T}(L_c)$ depends only on c (i.e., is independent of the choice of L_c) up to canonical isomorphism. The manifold $\mathcal{T}_X$ comes with a natural Real structure $\hat{\tau} : \mathcal{T}_X \to \mathcal{T}_X$ defined by pull-back of Yang-Mills connections.

Proposition 3. *Suppose $X^\tau \neq \emptyset$.*

1. *The following conditions are equivalent:*

   (a) $\mathcal{T}(L_c)^{\hat{\tau}} \neq \emptyset$,

   (b) $\tau^*(c) = -c$, $\mathfrak{o}(c) = 0$,

   (c) L_c *admits Real structures with respect to the fixed involution* τ *on* X.

2. *There exists a natural morphism*

$$F : \mathcal{T}_X^{\hat{\tau}} \longrightarrow H^2_{\mathbb{Z}_2}(X, \underline{\mathbb{Z}}(1)) \ ,$$

which induces an isomorphism $f : \pi_0(\mathcal{T}_X^{\hat{\tau}}) \longrightarrow H^2_{\mathbb{Z}_2}(X, \underline{\mathbb{Z}}(1))$.

Problem: Compute the group $H^2_{\mathbb{Z}_2}(X, \underline{\mathbb{Z}}(1))$ classifying Real line bundles on a given Real 4-manifold (X, τ). Note that this is taken care of by Proposition 2 when the first Betti number of X vanishes.

4 Abelian Yang-Mills connections on a torus

Let V be an Euclidian vector space, $\Lambda \subset V$ a maximal lattice, and $T := V/\Lambda$ the associated flat torus. Recall that we have canonical isomorphisms $H^k(T, \mathbb{Z}) = \mathrm{Alt}^k(\Lambda, \mathbb{Z})$. Fix $u \in H^2(T, \mathbb{Z})$, denote by the same symbol the corresponding anti-symmetric bilinear form $u : \Lambda \times \Lambda \to \mathbb{Z}$, and by $u_{\mathbb{R}}$ its $\mathbb{R}$-linear extension, which can also be regarded as a harmonic 2-form on T. Using the notations of the previous section we have

$$\mathcal{T}(L_u) = \left\{ [A] \in {}^{\mathcal{A}(L_u)}\!/_{\mathcal{G}} \;\middle|\; \frac{i}{2\pi} F_A = u_{\mathbb{R}} \right\} .$$

Definition 4. *A* u*-character on* Λ *is a map* $\alpha : \Lambda \to S^1$ *such that*

$$\alpha(\lambda + \lambda') = \alpha(\lambda)\alpha(\lambda')e^{\pi i u(\lambda, \lambda')} \quad \forall \lambda, \lambda' \in \Lambda \ .$$

Note that the set $\mathrm{Hom}_u(\Lambda, S^1)$ of u-characters on Λ has a natural $\mathrm{Hom}(\Lambda, S^1)$-torsor structure.

Associated to any element $\lambda \in \Lambda$ one has a *standard loop* $c_\lambda : [0, 1] \to V/\Lambda$ given by $c_\lambda(t) := [\lambda t]$.

Proposition 5. *Taking holonomy along standard loops induces a bijection*

$$h : \mathcal{T}(L_u) \to \mathrm{Hom}_u(\Lambda, S^1)\ ,$$

which maps a Yang-Mills class $[A] \in \mathcal{T}(L_u)$ *to the* u*-character* $\lambda \mapsto \bar{h}^A_{c_\lambda}$.

Idea: The holonomy along the boundary of a 2-simplex is determined by the curvature form. We apply this principle to the image in T of a 2-simplex $[0, \lambda, \lambda'] \subset V$, and see that the holonomy of a Yang-Mills connection along the standard loops defines a u-character. Letting u vary in $\mathrm{Alt}^2(\Lambda, \mathbb{Z})$ we obtain

Theorem 6. *(Appel-Humbert theorem for Abelian Yang-Mills connections). The holonomy map defines a canonical isomorphism*

$$h : \mathcal{T}_T \longrightarrow \coprod_{u \in \mathrm{Alt}^2(\Lambda,\mathbb{Z})} \mathrm{Hom}_u(\Lambda, S^1)\ .$$

Remark: The classical Appel-Humbert theorem, describing holomorphic line bundles on complex tori, follows from this result and the Kobayashi-Hitchin correspondence [12]: if J is a compatible complex structure on V, we define a Hermitian form H_u on the complex vector space (V, J) by

$$H_u(v, w) := u_{\mathbb{R}}(v, Jw) + i u_{\mathbb{R}}(v, w)\ .$$

Clearly one has $\mathrm{im} H_u = u_{\mathbb{R}}$. We obtain a commutative diagram

$$\begin{array}{ccc}
\displaystyle\coprod_{\substack{H \text{ Hermitian} \\ \mathrm{im}H(\Lambda\times\Lambda)\subset\mathbb{Z}}} \mathrm{Hom}_{\mathrm{im}H}(\Lambda, S^1) & = & \displaystyle\coprod_{\substack{u\in\mathrm{Alt}^2(\Lambda,\mathbb{Z}) \\ J^*(u_{\mathbb{R}})=u_{\mathbb{R}}}} \mathrm{Hom}_u(\Lambda, S^1) \\
\downarrow{\scriptstyle AH} & & \downarrow{\scriptstyle h^{-1}} \\
\mathrm{Pic}(T, J) & \xleftarrow{\ KH\ } & \displaystyle\coprod_{u\in\mathrm{NS}(T,J)} \mathcal{T}(L_u)
\end{array}\ ,$$

where the map KH is the Kobayashi-Hitchin correspondence between equivalence classes of HE connections and isomorphism classes of polystable holomorphic bundles. We recall that a Hermitian connection on a differentiable Hermitian bundle over a compact Kähler manifold is Hermitian-Einstein if and only if it is Yang-Mills and its curvature has type (1,1).

Remark: Theorem 6 gives an interesting geometric interpretation of the classical Appel-Humbert data (H, α) as curvature, respectively holonomy of Hermitian-Einstein connections.

Problem 2: Generalize Theorem 6 to higher harmonic Deligne cohomology, i.e., higher Abelian gerbes with harmonic curvature. For the definition of these spaces we refer to [6].

5 Real line bundles on Real tori

Let V be an Euclidian vector space, $\tau : V \to V$ a linear isometric involution, and $\Lambda \subset V$ a τ-invariant maximal lattice. We denote by the same symbol τ the induced involution on the torus $T := V/\Lambda$, and by $\hat{\tau} : \mathcal{T}_T \to \mathcal{T}_T$ the induced involution on the total moduli space $\mathcal{T}_T$ of Yang-Mills connections on Hermitian line bundles over T. Using the notations and the results of the previous section we see that $\hat{\tau}$ acts – via the AH description given by Theorem 6 – by the formula

$$\hat{\tau}(u, \alpha) = (-\tau^* u, \overline{\tau^* \alpha}) \ ,$$

which shows that a fixed point (u, α) of $\hat{\tau}$ must satisfy the equations

1. $\alpha_{|\Lambda^\tau} \in \mathrm{Hom}(\Lambda^\tau, \{\pm 1\})$,
2. $\alpha(\lambda + \tau\lambda) = e^{\pi i u(\lambda, \tau\lambda)} \ \forall \lambda \in \Lambda$.

Therefore, restricting the map h of Theorem 6 to the fixed point locus of $\hat{\tau}$ we get a map

$$h^\tau : \mathcal{T}_T^{\hat{\tau}} \longrightarrow \mathrm{Alt}^2(\Lambda, \mathbb{Z})^{-\tau^*} \times_{\mathrm{Hom}((\mathrm{id}+\tau)\Lambda, \mathbb{Z}_2)} \mathrm{Hom}(\Lambda^\tau, \mathbb{Z}_2)$$

given by $h^\tau(u,\alpha) := (u, \alpha_{|\Lambda^\tau})$. Note that Proposition 3 yields a natural bijection $f : \pi_0(\mathcal{T}_T^{\hat\tau}) \to H^2_{\mathbb{Z}_2}(T, \underline{\mathbb{Z}}(1))$; clearly h^τ factorizes through f. Note also that one has canonical identifications

$$\mathrm{Alt}^2(\Lambda,\mathbb{Z})^{-\tau^*} = H^2(T,\mathbb{Z})^{-\tau^*}\ ,\ \mathrm{Hom}(\Lambda^\tau,\mathbb{Z}_2) = H^1(T_0^\tau,\mathbb{Z}_2)\ ,$$

where $T_0^\tau = V^\tau/\Lambda^\tau$ is the connected component of 0 in the fixed point locus T^τ.

Theorem 7. *Let (T,τ) be a Real torus with $T^\tau \neq \emptyset$. The mixed characteristic class (c_1, w_1) induces an isomorphism*

$$cw : H^2_{\mathbb{Z}_2}(T,\underline{\mathbb{Z}}(1)) \longrightarrow H^2(T,\mathbb{Z})^{-\tau^*} \times_{H^1(T,\mathbb{Z}_2)} H^1(T_0^\tau,\mathbb{Z}_2)$$

given by $cw([L,\tilde\tau]) = (c_1(L), w_1(L^{\tilde\tau}\big|_{T_0^\tau}))$.

Remark: The essential facts used in the proof are:

1. The second component α of an Appel-Humbert datum describes the holonomy of the corresponding Yang-Mills connection,
2. The connected component decomposition of the fixed point locus T^τ has the form

$$T^\tau = \coprod_{[\mu]\in\frac{1}{2}\Lambda^{-\tau}/\frac{1}{2}(\mathrm{id}-\tau)\Lambda} T^\tau_{[\mu]}\ ,$$

   where $T^\tau_{[\mu]} := T^\tau_0 + [\mu]$; the Stiefel-Whitney class $w_1(L^{\tilde\tau}) \in H^1(T^\tau,\mathbb{Z}_2)$ is determined by the Stiefel-Whitney class $w_1(L^{\tilde\tau}\big|_{T_0^\tau})$ of the restriction of $L^{\tilde\tau}$ to T_0^τ. This follows from the difference formula

$$w_1(L^{\tilde\tau}\big|_{T^\tau_{[\mu]}}) - w_1(L^{\tilde\tau}\big|_{T_0^\tau}) = u(2\mu,\cdot)\ (\mathrm{mod}\ 2)\ .$$

Problem 3: Silhol [16] constructed moduli spaces of Real Abelian varieties (A,τ) endowed with a compatible principal polarization $c \in H^2(A,\mathbb{Z})^{-\tau^*}$. For certain questions it is more natural to have moduli spaces of Real principally polarized Real Abelian varieties, i.e., triples (A,τ,cw), where $cw \in H^2_{\mathbb{Z}_2}(A,\underline{\mathbb{Z}}(1))$ is a class defining a principal polarization. These moduli spaces will be finite covers of Silhol's spaces.

6 Determinant bundles of Klein surfaces

Let (C, ι) be a Klein surface, $C^\iota = \coprod_{i=1}^n C_i$ the connected component decomposition of the fixed point locus C^ι, and let $x_0 \in C^\iota$. Our goal is to determine the topological type of the Real line bundle

$$\mathcal{L}_{x_0} := \mathcal{O}_{\mathrm{Pic}^0(C)}(\Theta - [\mathcal{O}_C((g-1)x_0)]) \ .$$

Taking into account the results explained in the previous section, this topological type is determined by the mixed characteristic class

$$cw(\mathcal{L}_{x_0}) \in \mathrm{Alt}^2(H^1(X,\mathbb{Z}),\mathbb{Z})^{(\iota^*)^*} \times_{\mathrm{Hom}((\mathrm{id}-\iota^*)H^1(C,\mathbb{Z}),\mathbb{Z}_2)} \mathrm{Hom}(H^1(C,\mathbb{Z})^{-\iota^*},\mathbb{Z}_2) \ .$$

The Grothendieck-Riemann-Roch theorem allows us to identify the first component of the pair $cw(\mathcal{L}_{x_0})$ [1]; the result is $c_1(\mathcal{L}_{x_0}) = u_C$, where

$$u_C \in H^2(\mathrm{Pic}^0(C),\mathbb{Z}) = \mathrm{Alt}^2(H^1(C,\mathbb{Z}),\mathbb{Z})$$

is the cup form:

$$u_C : H^1(C,\mathbb{Z}) \times H^1(C,\mathbb{Z}) \to \mathbb{Z} \ , \ (\lambda,\lambda') \mapsto \langle \lambda \cup \lambda', [C] \rangle$$

Therefore, it suffices to determine explicitly the Stiefel-Whitney class

$$w_1(\mathcal{L}_{x_0}^{\tilde{\iota}}\big|_{\mathrm{Pic}^0(C)_0^{\hat{\iota}}}) : H^1(C,\mathbb{Z})^{-\iota^*} \longrightarrow \mathbb{Z} \ . \qquad (2)$$

Using topological arguments [5], one can show that $H^1(C,\mathbb{Z})^{-\iota^*}$ is generated by the subgroup $(\mathrm{id}-\iota^*)H^1(C,\mathbb{Z})$ and the classes $[C_1]^\vee, \dots, [C_n]^\vee$, where $[C_i]^\vee$ denotes the Poincaré dual of the 1-homology class $[C_i]$ defined by an arbitrary orientation of the circle C_i. Since the restriction of $w_1(\mathcal{L}_{x_0}^{\tilde{\iota}}\big|_{\mathrm{Pic}^0(C)_0^{\hat{\iota}}})$ to $(\mathrm{id}-\iota^*)H^1(C,\mathbb{Z})$ is known (it is determined by the Chern class u_C, see section 5) we conclude that it suffices to compute the values of (2) on the classes $[C_i]^\vee$.

We will proceed in two steps:

1. compute $cw(\mathcal{L}_{[\kappa]})$, where $\mathcal{L}_{[\kappa]}$ is *a symmetric theta line bundle*,
2. compare $cw(\mathcal{L}_{x_0})$ to $cw(\mathcal{L}_{[\kappa]})$.

(1) We recall that a theta characteristic is square root of $[\omega_C]$. Denote by $\theta(C)$ the set of theta characteristics of C, i.e.,

$$\theta(C) := \{[\kappa] \in \mathrm{Pic}^{g-1}(C)|\ \kappa^{\otimes 2} \simeq \omega_C\}\ .$$

The cardinality of this set is 2^{2g}, and for every $[\kappa] \in \theta(C)$ we have an associated Mumford theta form $q_{[\kappa]} : \mathrm{Pic}^0(C)_2 \longrightarrow \mathbb{Z}_2$, defined on the 2-torsion subgroup $\mathrm{Pic}^0(C)_2 \subset \mathrm{Pic}^0(C)$ by

$$q_{[\kappa]}([\eta]) := h^0(\eta \otimes \kappa) - h^0(\kappa) \ (\mathrm{mod}\ 2)$$

(see [1]). Using the natural identification

$$\mathrm{Pic}^0(C)_2 = {\frac{1}{2}H^1(C,\mathbb{Z})}\Big/{H^1(C,\mathbb{Z})} = H_1(C,\mathbb{Z}_2)\ ,$$

we obtain a form $q_{[\kappa]} : H_1(C,\mathbb{Z}_2) \to \mathbb{Z}_2$ satisfying the *Riemann-Mumford* relations:

$$q_{[\kappa]}(a+b) = q_a + q_a + a \cdot b \tag{3}$$

The main idea is to use the translation by a theta characteristic instead of translation by $[\mathcal{O}_C((g-1)x_0)]$ to identify Pic^{g-1} with $\mathrm{Pic}^0(C)$. More precisely, we define

$$\mathcal{L}_{[\kappa]} := \mathcal{O}_{\mathrm{Pic}^0(C)}(\Theta - [\kappa])\ .$$

Theorem 8. *Let $[\kappa] \in \theta(C)$. Then the Appel-Humbert data of the holomorphic line bundle $\mathcal{L}_{[\kappa]}$ is $(u_C, \alpha_{[\kappa]})$, where $\alpha_{[\kappa]} : H^1(C,\mathbb{Z}) \to S^1$ is defined by*

$$\alpha_{[\kappa]}(\lambda) := (-1)^{q_{[\kappa]}(\overline{\lambda\cap[C]})}\ .$$

Here $\lambda \cap [C] \in H_1(C,\mathbb{Z})$ is the Poincaré dual of λ and $\overline{\lambda \cap [C]})$ denotes its image in $H_1(C,\mathbb{Z}_2)$.

Idea of proof: Since $\mathcal{L}_{[\kappa]}$ is *symmetric* in the sense that $(-1)^*(\mathcal{L}_{[\kappa]}) \simeq \mathcal{L}_{[\kappa]}$, it follows that

$$\alpha_{[\kappa]}(\lambda) = (-1)^{\mathrm{mult}_{[\frac{1}{2}\lambda]}(\Theta-[\kappa]) - \mathrm{mult}_{[0]}(\Theta-[\kappa])}$$

(see [4]). Now we use *Riemann's singularity theorem*, which states

$$\mathrm{mult}_{[\mathcal{L}]}\Theta = h^0(\mathcal{L}) \ .$$

Note that the u_C-character $\alpha_{[\kappa]}$ given by Theorem 8 involves the algebraic geometric data $q_{[\kappa]}$. Therefore, we have to make an additional step, which will give a purely topological interpretation of the values $q_{[\kappa]}([C_i]_2)$ when $\hat{\iota}[\kappa] = [\kappa]$.

Proposition 9. *Suppose* $[\kappa] \in \theta(C)^{\hat{\iota}}$. *Then*

$$q_{[\kappa]}([C_i]_2) = \langle w_1(\kappa^{\tilde{\iota}}), [C_i]_2\rangle + 1 \ .$$

Idea of proof: We use the diagram

$$\begin{array}{ccc} \theta(C) & \xrightarrow{\ q\ } & \\ \downarrow \xi & & Q(H_1(C,\mathbb{Z}_2),\cdot), \\ \mathrm{Spin}(C) & \xrightarrow{\ \omega\ } & \end{array}$$

where Spin(C) denotes the set of isomorphism classes of Spin-structures on C, $Q(H_1(C,\mathbb{Z}_2),\cdot)$ is the set of maps $H_1(C,\mathbb{Z}_2) \to \mathbb{Z}_2$ satisfying the Riemann-Mumford relations (3), q is the assignment $[\kappa] \mapsto q_{[\kappa]}$ given by the Mumford theta form, ξ is the bijection defined by Atiyah [3], and ω is a bijection defined by Johnson [9] in purely topological terms. We know by Mumford that q is a morphism of $H^1(C,\mathbb{Z}_2)$-torsors, according to Atiyah [3] ξ is a bijection, by Johnson [9] ω is a bijection, and according to Libgober [11] the diagram commutes. Combining these results, and using a direct geometrical argument, we obtain the following formula:

$$\omega_{\xi_\kappa}([C_i]_2) = \langle w_1(\kappa^{\tilde{\iota}}), [C_i]_2\rangle + 1 \qquad (4)$$

This completes step (1); for step (2) we use the fact that every component of $\mathrm{Pic}^{g-1}(C)^{\hat{\iota}}$ contains 2^g Real theta characteristics. Concluding, we get the following explicit formula which, combined with the results of section 6, completes the computation of the topological type of the Real line bundles $\mathcal{L}_{x_0}$.

Proposition 10.

$$w_1(\mathcal{L}^{\tilde{\iota}}_{x_0}\big|_{\mathrm{Pic}^0(C)_0})[C_i]_2^\vee = \begin{cases} 1 & \textit{when} \quad x_0 \notin C_i \\ g \pmod 2 & \textit{when} \quad x_0 \in C_i \end{cases}.$$

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