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Quantum Sheaf Cohomology, a précis

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Abstract

We present a brief introduction to quantum sheaf cohomology, a generalization of quantum cohomology based on the physics of the (0,2) nonlinear sigma model.

Throughout, we will consider X to be a Kähler manifold of complex dimension n. In addition, we will consider $\mathcal{E} \to X$ to be a holomorphic Hermitian vector bundle of rank k satisfying $c_i(\mathcal{E}) = c_i(X)$, $i = 1, 2$, where these conditions are to be understood in the Chow ring. The matching of the first Chern class implies that $\det \mathcal{E}^\vee \simeq \omega_X$, but not canonically. In order to consistently normalize correlation functions across different instanton sectors, a specific isomorphism ψ is fixed as part of the initial data[1].

As matching of second Chern characters of $\mathcal{E}$ and X is the usual Green-Schwarz anomaly cancellation condition (implied by our conditions), we will a bundle satisfying the Chern class conditions *omalous*[2]. One may

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[1]When $\mathcal{E} = T_X$, the quantum sheaf cohomology is the ordinary quantum cohomology – such a choice is not required since the class of a point fixes the isomorphism canonically.

consider the first Chern class condition to be an analogue of the usual condition for existence of the B-model, though in this context it guarantees that both the classical and quantum algebras are Frobenius. A bundle satisfying these conditions may be obtained by, for example, selecting a deformation of the tangent bundle of X.

1 Quantum Cohomology

1.1 Ordinary cohomology

We now give some elementary facts about the cohomology of X, stated in a way that will facilitate our point of view on quantum sheaf cohomology. Since X is Kähler, there is a Hodge decomposition on $H^\bullet(X, \mathbb{C})$,

$$H^\bullet(X, \mathbb{C}) \simeq \bigoplus_{p,q} H^p(X, \textstyle\bigwedge^q T_X^\vee).$$

By a slight abuse of language, we will refer to elements of the sheaf cohomology groups $H^p(X, \bigwedge^q T_X^\vee)$ as (p, q)-forms – clearly $H^\bullet(X, \mathbb{C})$ possesses a basis consisting of such forms. The antisymmetric cup product

$$H^p(X, \textstyle\bigwedge^q T_X^\vee) \wedge H^{p'}(X, \bigwedge^{q'} T_X^\vee) \to H^{p+p'}(X, \bigwedge^{q+q'} T_X^\vee)$$

furnishes this vector space with the structure of a bigraded $\mathbb{C}$-algebra. Finally, integration of forms induces a trace on the algebra; in terms of a basis element ω,

$$\mathrm{tr}(\omega) = \begin{cases} \displaystyle\int_X \omega & \omega \in H^n(X, \bigwedge^n T_X^\vee) \\ 0 & \text{otherwise.} \end{cases}$$

The pairing $(\alpha, \beta) \mapsto \mathrm{tr}(\alpha \smile \beta)$ induced by this trace is a non-degenerate bilinear form satisfying $(\alpha, \beta \smile \gamma) = (\alpha \smile \beta, \gamma)$, so that $H^\bullet(X, \mathbb{C})$ is a bigraded Frobenius algebra.

[2]That is, *not anomalous* – this delightful terminology is due to Ron Donagi. One should also consider the choice of isomorphism as part of the omality condition.

1.2 Physics

The relationship between ordinary cohomology and quantum cohomology may be elucidated by appealing to physics – in particular to a topologically-twisted (2,2) nonlinear sigma model of maps $\mathbb{P}^1 \to X$. Of the many intriguing aspects of this quantum field theory, we will be most interested in its algebra of massless supersymmetric operators[3]. Using elementary physics arguments, one identifies a basis for the set of such operators that may be set into one-to-one correspondence with (p, q)-forms on X.

The (2,2) supersymmetry of the model forces the product of two massless supersymmetric operators to be massless and supersymmetric. The particular form of the product is obtained by considering three-point correlation functions in the quantum field theory: the quantum product of two massless operators $\mathcal{O}_1$ and $\mathcal{O}_2$ is defined to be the unique operator $(\mathcal{O}_1 * \mathcal{O}_2)$ such that for all massless operators $\mathcal{O}_3$,

$$\langle \mathbb{1}(\mathcal{O}_1 * \mathcal{O}_2)\mathcal{O}_3 \rangle = \langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3 \rangle \,. \tag{1.1}$$

Here, $\mathbb{1}$ denotes the operator corresponding to $1 \in H^0(X, \bigwedge^0 T_X^\vee)$. Such a correlation function is computed using the instanton expansion

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3 \rangle = \sum_{\beta \in H_2(X,\mathbb{Z})} \langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3 \rangle_\beta \, q^\beta. \tag{1.2}$$

Although they have intrinsic meaning in physics, we will consider the expressions q^β to comprise a set of formal variables endowed with the structure of a monoid via the product $q^\alpha q^\beta = q^{\alpha+\beta}$. We denote by $\mathbb{C}[\![\underline{q}]\!]$ the ring of formal power series with complex coefficients in these variables – one sometimes insists on convergence or other properties, but such subtleties are beyond the scope of this review.

Mathematically, one defines the expression $\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3 \rangle_\beta$ as the Gromov-Witten invariant[4] $\langle I_{0,3,\beta} \rangle(\omega_1, \omega_2, \omega_3)$, where ω_i are the forms correspond-

[3]More precisely, it is the algebra of local, scalar, supersymmetric operators [Wit].

ing to the operator $\mathcal{O}_i$. Physically, one says that $\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle_\beta\, q^\beta$ denotes the contribution of instantons of degree β to the correlation function $\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle$. This expression is morally the integral of induced forms on some compactification $\overline{M(X,\beta)}$ of the moduli space of holomorphic maps $f:\mathbb{P}^1\to X$ of class $\beta = f_*[\mathbb{P}^1]$. We will write the induced forms schematically using maps

$$\zeta_\beta : H^p(X, \textstyle\bigwedge^q T_X^\vee) \to H^p\left(\overline{M(X,\beta)}, \textstyle\bigwedge^q T^\vee_{\overline{M(X,\beta)}}\right). \tag{1.3}$$

If ω_i are the forms corresponding to operators $\mathcal{O}_i$, modulo the subtleties of obstruction bundles we have that

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle_\beta = \int_{\overline{M(X,\beta)}} \zeta_\beta(\omega_1)\smile\zeta_\beta(\omega_2)\smile\zeta_\beta(\omega_3).$$

Depending on the compactification, there may be more than one such map – in the case of the stable maps compactification, pullbacks via evaluation maps play the rôle of ζ_β. For toric varieties, one often uses the Morrison-Plesser compactification [MRP] wherein – as indicated in equation (1.3) – one map suffices for each β.

The three-point correlation functions in equation (1.2) induce a non-degenerate bilinear pairing $(\omega_1,\omega_2) = \langle \mathbb{1}\mathcal{O}_1\mathcal{O}_2\rangle$ on the unital algebra $\bigoplus_{p,q} H^p(X,\bigwedge^q T_X^\vee)[\![\underline{q}]\!]$, leading to the following definition.

Definition 1.1. *The* quantum cohomology *of X is the Frobenius algebra*

$$QH^\bullet(X) := \bigoplus_{p,q} H^p(X,\textstyle\bigwedge^q T_X^\vee)[\![\underline{q}]\!],$$

with the product and bilinear pairing induced by (2,2) three-point functions.

Here, the (2,2) correlation functions are defined either via Gromov-Witten invariants or as correlation functions in the quantum field theory, depending on whether your tastes tend to the mathematical or to the physical.

[4] See equation 7.4 of [CK] for a precise definition of Gromov-Witten invariants.

Example 1.2. *The "classical sector" is the set of maps homotopic to a point, $\beta = 0$, and the moduli space of such maps is simply X itself. Thus, in this sector, the quantum product reduces to the cup product on forms; ordinary cohomology is the "classical limit" of quantum cohomology. This sector may be isolated by setting $\underline{q} = 0$. For example, the ordinary and quantum cohomology of $\mathbb{P}^n$ are respectively*

$$\begin{aligned} H^\bullet(\mathbb{P}^n, \mathbb{C}) &\simeq \frac{\mathbb{C}[H]}{\langle H^{n+1} \rangle}, \\ QH^\bullet(\mathbb{P}^n) &\simeq \frac{\mathbb{C}[H][\![q]\!]}{\langle H^{n+1} - q \rangle}. \end{aligned}$$

Here H denotes the hyperplane class. For $\mathbb{P}^n \times \mathbb{P}^m$, the equivalent expressions are

$$\begin{aligned} H^\bullet(\mathbb{P}^n \times \mathbb{P}^m, \mathbb{C}) &\simeq \frac{\mathbb{C}[H_1, H_2]}{\left\langle H_1^{n+1}, H_2^{m+1} \right\rangle}, \\ QH^\bullet(\mathbb{P}^n \times \mathbb{P}^m) &\simeq \frac{\mathbb{C}[H_1, H_2][\![q_1, q_2]\!]}{\left\langle H_1^{n+1} - q_1, H_2^{m+1} - q_2 \right\rangle}. \end{aligned} \tag{1.4}$$

2 Quantum Sheaf Cohomology

As in our study of the passage from ordinary cohomology to quantum cohomology, we first consider the "ordinary sheaf cohomology" – in particular that of an omalous bundle $\mathcal{E} \to X$. Here, by ordinary sheaf cohomology we mean polysection cohomology,

$$\bigoplus_{p,q} H^p(X, \textstyle\bigwedge^q \mathcal{E}^\vee). \tag{2.1}$$

Again by a slight abuse of language, we will refer to elements of $H^p(X, \bigwedge^q \mathcal{E}^\vee)$ as (p, q)-forms – clearly the vector space (2.1) possesses a basis consisting of such forms, and the cup product furnishes it with the structure of a bigraded $\mathbb{C}$-algebra.

The trace on this algebra is slightly more subtle and follows from the omality of $\mathcal{E}$. In particular, one uses the induced isomorphism $\psi : H^n(X, \bigwedge^k \mathcal{E}) \to H^n(X, \omega_X)$ to define, for a basis element ω,

$$\operatorname{tr}(\omega) = \begin{cases} \int_X \psi(\omega) & \omega \in H^n(X, \bigwedge^k \mathcal{E}^\vee) \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

The pairing $(\alpha, \beta) \mapsto \operatorname{tr}(\alpha \smile \beta)$ induced by this trace endows $\bigoplus_{p,q} H^p(X, \bigwedge^q \mathcal{E}^\vee)$ with the structure of a bigraded Frobenius algebra.

2.1 Physics

To understand the relationship between sheaf cohomology and quantum sheaf cohomology we again appeal to physics – in particular a topologically-twisted (0,2) nonlinear sigma model of maps $\mathbb{P}^1 \to X$. A recent physics review of this and related models may be found in [McO]. We will again be most interested in its algebra of massless supersymmetric operators[5]. The same elementary physics arguments used for the (2,2) theory identify a basis for this set of operators that may be placed into one-to-one correspondence with (p, q)-forms (that is, elements of (2.1)), and the quantum product of two massless operators is defined using three-point correlation functions of the (0,2) in analogy to equation (1.1). Unlike the (2,2) case, however, there is no mathematical definition of $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle_\beta$ in a (0,2) theory so the following definition is purely physical.

Definition 2.1. *The* quantum sheaf cohomology *of an omalous bundle $\mathcal{E} \to X$ is the Frobenius algebra*

$$QH^\bullet(X, \mathcal{E}) := \bigoplus_{p,q} H^p(X, \textstyle\bigwedge^q \mathcal{E}^\vee) \otimes \mathbb{C}[\![\underline{q}]\!]$$

[5] As explained in [ADE], we are actually interested in local, scalar, supersymmetric operators with vanishing holomorphic conformal weight, but for continuity we will refer to them as massless supersymmetric or simply massless.

with the product and bilinear pairing induced by (0,2) three-point functions.

As in the case of ordinary quantum cohomology, the classical limit of quantum sheaf cohomology is precisely the ordinary sheaf cohomology with the Frobenius structure induced by equation (2.2). Unlike the case in (2,2) theories, (0,2) supersymmetry is *not* enough to guarantee that the product of massless operators is massless: one needs to work harder to show that the algebra closes in the set of all operators.

2.2 Existence

The (modern) history of quantum sheaf cohomology begins with the observation in [ABS] of an analogue of $QH^\bullet(X)$ for $(0,2)$ theories. Therein, the quantum sheaf cohomology of a one-parameter family of deformations of the tangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$ was computed using a conjectured form of mirror symmetry for $(0,2)$ models. Their calculations were confirmed in a sheaf-cohomology-based computation by Katz and Sharpe [KS]. Inspired by these results, Adams, Distler, and Ernebjerg [ADE] gave a physics definition of quantum sheaf cohomology and found a physics proof of two sufficient conditions for its existence. We restate these conditions here as conjectures.

Conjecture 2.2. *Let U be a family of bundles, $\gamma\colon[0,1] \to U$ continuous, and $\gamma(t)$ omalous for all $t \in [0,1]$. Then for $\gamma(0) = \mathcal{E}$ and $\gamma(1) = \mathcal{E}'$, $QH^\bullet(X, \mathcal{E})$ exists iff $QH^\bullet(X, \mathcal{E}')$ exists.*

Conjecture 2.3. *If $\mathcal{E} \to X$ is omalous and rk $\mathcal{E} < 8$, then $QH^\bullet(X, \mathcal{E})$ exists.*

Since $QH^\bullet(X, T_X) = QH^\bullet(X)$, the former condition implies the existence of quantum sheaf cohomology for all omalous one-parameter families of tangent-bundle deformations. The latter is likely an artefact of the technique used in the physics proof – there are no known examples of omalous bundles of rank eight or higher for which the massless operators fail to close under the quantum product, and there are no physical reasons to expect such a bundle to exist.

2.3 Computation example

Although there is no definition for the invariants $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle_\beta$, a number of physics-inspired techniques exist to compute them when the omalous bundle is a deformation of the tangent bundle of a toric variety [KS, GK, MM1] or a complete intersection therein[MM2]. One of the advantages of using a toric variety X is that deformations of T_X are easily obtained by deforming the Euler exact sequence:

$$0 \longrightarrow \mathcal{O}_X^r \xrightarrow{E_0} \bigoplus_{\rho \in \Delta} \mathcal{O}_X(D_\rho) \longrightarrow T_X \longrightarrow 0.$$

Here, r is the rank of the Picard group, Δ denotes the set of torus-invariant divisors corresponding to one-cones in the fan of X, and E_0 is a collection of sections of $\mathcal{O}_X(D_\rho)$, which are toric analogues of $\mathcal{O}_{\mathbb{P}^n}(1)$. Taking $X = \mathbb{P}^1 \times \mathbb{P}^1$, for example, the sequence becomes

$$0 \longrightarrow \mathcal{O}_X^2 \xrightarrow{E_0} \mathcal{O}_X(1,0)^2 \oplus \mathcal{O}_X(0,1)^2 \longrightarrow T_X \longrightarrow 0,$$

where the map is

$$E_0 = \begin{pmatrix} x_0 & 0 \\ x_1 & 0 \\ 0 & x_2 \\ 0 & x_3 \end{pmatrix}.$$

A deformation of T_X may be obtained by choosing a different collection of sections for the map. For example, selecting the map to be

$$E = \begin{pmatrix} x_0 & \epsilon_1 x_0 + \epsilon_2 x_1 \\ x_1 & \epsilon_3 x_0 \\ \gamma_1 x_2 + \gamma_2 x_3 & x_2 \\ \gamma_3 x_2 & x_3 \end{pmatrix} \tag{2.3}$$

as in [GK] gives a convenient basis for the space of deformations of the tangent bundle ($\epsilon_i, \gamma_i \in \mathbb{C}$). Therein, several of the invariants $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle_\beta$ were computed for the bundle $\mathcal{E} \to \mathbb{P}^1 \times \mathbb{P}^1$ defined as the cokernel of the morphism in equation (2.3). These were then used to deduce the quantum sheaf cohomology of $\mathcal{E}$;

$$QH^\bullet(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{E}) \simeq \frac{\mathbb{C}[\psi, \widetilde{\psi}][\![q_1, q_2]\!]}{\left\langle \begin{array}{c} \psi^2 + \epsilon_1 \psi\widetilde{\psi} - \epsilon_2\epsilon_3\widetilde{\psi}^2 - q_1, \\ \widetilde{\psi}^2 + \gamma_1\psi\widetilde{\psi} - \gamma_2\gamma_3\psi^2 - q_2 \end{array} \right\rangle}. \tag{2.4}$$

These computations were confirmed in [MM1] using physics techinques. Note that as $\epsilon_i, \gamma_i \to 0$, the quantum sheaf cohomology in equation (2.4) limits to the ordinary quantum cohomology of $\mathbb{P}^1 \times \mathbb{P}^1$ in equation (1.4).

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