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A survey on Beurling-Selberg majorants and some consequences of the Riemann hypothesis

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Abstract

This article briefly describes the recent advances on the Beurling-Selberg extremal problem in harmonic analysis and its connection with the theory of the Riemann zeta-function. In particular, under the Riemann hypothesis, this extremal tool provides improved bounds for the size of $\zeta(s)$ in the critical strip, for the argument function $S(t)$ and for its antiderivative, the function $S_1(t)$.

1 Introduction

This expository article is based on a lecture given at the *International Meeting on Differential Geometry and Partial Differential Equations - in honor to the 80th birthday of Professor Gervasio Colares*, held in Fortaleza - Brazil in August 2011.

1.1 Consequences of the Riemann hypothesis

Bernhard Riemann published his paper "*Über die Anzahl der Primzahlen unter einer gegebenen Grösse*" in the *Monatsberichte der Berliner Akademie* in November, 1859. There we find the statement that the function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

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initially defined for $\mathrm{Re}(s) > 1$, and then suitably extended meromorphically to the complex plane, "probably" has its complex zeros all aligned over the line $\Re(s) = 1/2$.

His hypothesis has not been proved until this day (despite the fact that modern computers can verify that the first 10^{12} zeros are on the critical line), but considerable effort has been put in order to understand the different objects in the theory of the Riemann zeta-function assuming its validity. This survey article briefly shows how we can approach some of these objects using tools from harmonic analysis and approximation theory.

J. E. Littlewood in 1924 [22] showed that under the Riemann hypothesis (RH) we have the following estimate:

$$\log\left|\zeta\left(\tfrac{1}{2}+it\right)\right| \leq \big(C+o(1)\big)\frac{\log t}{\log\log t},$$

for sufficiently large t. This estimate was never improved in its order of magnitude, and the advances have rather focused on diminishing the value of the admissible constant C. In [30] Ramachandra and Sankaranarayanan obtained $C = 0.466$, while in [32] Soundararajan improved this bound, obtaining $C = 0.373$. Recently, Chandee and Soundararajan in [10, Theorem 1] obtained another improvement, currently the best bound, as shown below.

Theorem 1.1 (Upper bound for $\zeta(s)$ in the critical line)**.** *Assume RH. For large real numbers t, we have*

$$\log\left|\zeta\left(\tfrac{1}{2}+it\right)\right| \leq \frac{\log 2}{2}\frac{\log t}{\log\log t} + O\left(\frac{\log t \log\log\log t}{(\log\log t)^2}\right).$$

A natural question here would be if such bounds can be obtained in an off-critical-line context using similar methods. The answer is yes and the following extension of Theorem 1.1 was obtained by Carneiro and Chandee in [3, Theorem 1].

Theorem 1.2 (Upper bound for $\zeta(s)$ in the critical strip)**.** *Assume RH and let*

$1/2 \leq \alpha \leq 1$. *For large real numbers t, we have*

$$\log|\zeta(\alpha+it)| \leq \begin{cases} \log\left(1+(\log t)^{1-2\alpha}\right)\frac{\log t}{2\log\log t} + O\left(\frac{(\log t)^{2-2\alpha}}{(\log\log t)^2}\right), \\ \qquad\qquad\qquad \text{if } \ (\alpha-1/2)\log\log t = O(1); \\ \log(\log\log t) + O(1), \qquad \text{if } \ (1-\alpha)\log\log t = O(1); \\ \left(\frac{1}{2}+\frac{2\alpha-1}{\alpha(1-\alpha)}\right)\frac{(\log t)^{2-2\alpha}}{\log\log t} + \log(2\log\log t) \\ +O\left(\frac{(\log t)^{2-2\alpha}}{(1-\alpha)^2(\log\log t)^2}\right), \qquad \text{otherwise.} \end{cases}$$

In the critical strip context we also have a lower bound given by [3, Theorem 2].

Theorem 1.3 (Lower bound for $\zeta(s)$ in the critical strip)**.** *Assume RH and let $1/2 < \alpha \leq 1$ For large real numbers t, we have*

$$\log|\zeta(\alpha+it)| \geq \begin{cases} \log\left(1-(\log t)^{1-2\alpha}\right)\frac{\log t}{2\log\log t} - O\left(\frac{(\log t)^{2-2\alpha}}{(\log\log t)^2(1-(\log t)^{1-2\alpha})}\right), \\ \qquad\qquad\qquad \text{if } \ (\alpha-1/2)\log\log t = O(1); \\ -\log(\log\log t) - O(1), \qquad \text{if } \ (1-\alpha)\log\log t = O(1); \\ -\left(\frac{1}{2}+\frac{2\alpha-1}{\alpha(1-\alpha)}\right)\frac{(\log t)^{2-2\alpha}}{\log\log t} - \log(2\log\log t) \\ -O\left(\frac{(\log t)^{2-2\alpha}}{(1-\alpha)^2(\log\log t)^2}\right), \qquad \text{otherwise.} \end{cases}$$

Another object of interest is the argument function defined by (here $t > 0$)

$$S(t) = \tfrac{1}{\pi}\arg\zeta\left(\tfrac{1}{2}+it\right),$$

where the argument is defined by a continuous variation along the line segments joining the points $2, 2+it$ and $\frac{1}{2}+it$, taking $\arg\zeta(2) = 0$, if t is not an ordinate of a zero of $\zeta(s)$. If t is an ordinate of a zero we set

$$S(t) = \tfrac{1}{2}\lim_{\epsilon\to 0}\left\{S(t+\epsilon)+S(t-\epsilon)\right\}.$$

This function appears for instance when counting the number of zeros $N(t)$ of $\zeta(s)$ with imaginary ordinate in the interval $[0,t]$

$$N(t) = \frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi} + \frac{7}{8} + S(t) + O\left(\frac{1}{t}\right).$$

In the work [22] Littlewood also showed that under RH we have

$$|S(t)| \leq \big(C + o(1)\big)\frac{\log t}{\log\log t},$$

and, as in the case of the size of $\zeta\big(\frac{1}{2}+it\big)$, this estimate has not been improved in its order of magnitude over the years. Efforts to bring down the value of the admissible constant C were carried out by Ramachandra and Sankaranarayanan [30] who proved that $C = 1.119$ is admissible, and later by Fujii [12] who obtained the result for $C = 0.67$.

The application of certain extremal functions of exponential type, that majorize and/or minorize the characteristic functions of intervals, to problems related to the theory of the Riemann zeta-function dates back to the works of Montgomery [27] and Gallagher [13], on the pair correlation of zeros of $\zeta(s)$. In [26] Goldston and Gonek were the first to realize a distinct connection between the Riemann hypothesis and these extremal functions, via the so called Guinand-Weil explicit formula (the method we shall be presenting here). Using this connection they obtained the following bound [26, Theorem 2] for the argument function

$$|S(t)| \leq \left(\frac{1}{2} + o(1)\right)\frac{\log t}{\log\log t}.$$

We shall present here a sharper version of this bound, recently obtained by Carneiro, Chandee and Milinovich in [4, Theorem 2].

Theorem 1.4 (Bound for $S(t)$)**.** *Assume RH. For t sufficiently large we have*

$$|S(t)| \leq \frac{1}{4}\frac{\log t}{\log\log t} + O\left(\frac{\log t\log\log\log t}{(\log\log t)^2}\right).$$

Finally, another important function in the theory of the Rieman zeta-function is the antiderivative of $S(t)$ defined by

$$S_1(t) = \int_0^t S(u)\,\mathrm{d}u.$$

There has been earlier work on establishing explicit bounds for $S_1(t)$. Littlewood [22] was the first to prove that $S_1(t) \ll \log t/(\log\log t)^2$ under the assumption of the Riemann hypothesis. More recently, Karatsuba and Korolëv [19] obtained that

$$\big|S_1(t)\big| \,\leq\, (40{+}o(1))\,\frac{\log t}{(\log\log t)^2},$$

and Fujii [12] obtained that

$$-\,(0.51{+}o(1))\,\frac{\log t}{(\log\log t)^2} \,\leq\, S_1(t) \,\leq\, (0.32{+}o(1))\,\frac{\log t}{(\log\log t)^2}.$$

We present here the following improvement obtained in [4, Theorem 1].

Theorem 1.5 (Bounds for $S_1(t)$)**.** *Assume RH. For t sufficiently large we have*

$$-\left(\frac{\pi}{24}{+}o(1)\right)\frac{\log t}{(\log\log t)^2} \,\leq\, S_1(t) \,\leq\, \left(\frac{\pi}{48}{+}o(1)\right)\frac{\log t}{(\log\log t)^2},$$

where the terms $o(1)$ in the above inequalities are $O(\log\log\log t/\log\log t)$.

The five theorems presented here have something in common: the strategy for their proofs is essentially the same. It consists of three steps: (i) expressing the considered object as a certain sum over the zeros of $\zeta(s)$; (ii) making use of suitable extremal majorants/minorants of exponential type; (iii) applying an appropriate explicit formula to evaluate the sums by taking advantage of the compactly supported Fourier transforms. We shall present here the proofs of Theorems 1.1, 1.4 and 1.5 to illustrate the method. For the other results and more details we refer the reader to the original sources.

1.2 The Beurling-Selberg extremal problem

We say that an entire function $K:\mathbb{C}\to\mathbb{C}$ has *exponential type* at most $2\pi\delta$ if, for every $\epsilon>0$, there exists a positive constant C_ϵ, such that the inequality

$$|K(z)|\leq C_\epsilon e^{(2\pi\delta+\epsilon)|z|}$$

holds for all $z\in\mathbb{C}$. These functions have distributional Fourier transforms compactly supported in the interval $[-\delta,\delta]$, as a consequence of the Paley-Wiener theorem, and sometimes are also referred as bandlimited functions.

The extremal problem we address here is the following: given a function $f:\mathbb{R}\to\mathbb{R}$ and $\delta>0$, we seek an entire function $K(z)$, of exponential type at most $2\pi\delta$, such that the integral

$$\int_{-\infty}^{\infty}|f(x)-K(x)|\,\mathrm{d}x \tag{1.1}$$

is minimized. This is a classical problem in harmonic analysis and approximation theory, considered by Bernstein, Akhiezer, Krein, Nagy and others, since at least 1938. In particular, Krein [20] in 1938 and Nagy [33] in 1939 published seminal papers solving this problem for a wide class of functions $f(x)$.

For applications to analytic number theory, it is convenient to consider an additional restriction: we ask that $K(z)$ is real on $\mathbb{R}$ and that $K(x) \geq f(x)$ for all $x \in \mathbb{R}$. In this case, a minimizer of the integral (1.1) is called an extremal majorant of $f(x)$ (or extremal upper one-sided approximation). Extremal minorants are defined analogously. Beurling started working on this one-sided extremal problem, independently, in the late 1930's, and obtained the solution for $f(x) = \operatorname{sgn}(x)$ and an inequality for almost periodic functions in an unpublished manuscript (see the survey [35] by J. D. Vaaler for a historical perspective). The one-sided extremals for the signum function were later used by Selberg [28, 31] to obtain the solution of the extremal problem for characteristic functions of intervals (of integer size, the general case was settled later, by B. Logan) and a sharp form of the large sieve inequality. In this article we are mostly interested! in the one-sided version of this problem and, therefore, we shall be referring to it as the *Beurling-Selberg extremal problem.*

The problem (1.1) is hard in the sense that there is no general known way to produce a solution given any $f : \mathbb{R} \to \mathbb{R}$. Besides the original examples $f(x) = \operatorname{sgn}(x)$ of Beurling and $f(x) = \chi_{[a,b]}(x)$ of Selberg, the solution for the exponential family $f(x) = e^{-\lambda|x|}$, $\lambda > 0$, was discovered by Graham and Vaaler in [16], with a first glimpse of the technique of integration on the free parameter λ to produce solutions for a family of even and odd functions. Later, the problem for $f(x) = x^n \operatorname{sgn}(x)$ and $f(x) = (x^+)^n$, where n is a positive integer, was considered by Littmann in [23, 24, 25]. Using the exponential subordination, Carneiro and Vaaler in [8, 9] extended the construction of extremal approximations for a class of even functions that includes $f(x) = \log|x|$, $f(x) = \log\big((x^2+1)/x^2\big)$ and $f(x) = |x|^\alpha$, with $-1 < \alpha < 1$. The analogous exponential subordination framework for truncated and odd functions was treated in [6].

Other classical applications of the solutions of these problems to analytic number theory include Hilbert-type inequalities [8, 16, 24, 29, 35], Erdös-Turán discrepancy inequalities [8, 21, 35], optimal approximations of periodic functions by trigonometric polynomials [2, 8, 9, 35] and Tauberian theorems [16]. The extremal problem in higher dimensions, with applications, is considered in [1, 17]. Approximations in L^p-norms with $p \neq 1$ are treated, for instance, in [14].

The recent advances in this theory include families of functions generated via a certain Gaussian subordination. Carneiro, Littmann and Vaaler in [7] found the solution of the extremal problem (1.1) for the Gaussian

$$G_\lambda(x) = e^{-\pi\lambda x^2}\,,$$

while Carneiro and Littmann in [5] obtained the solution for the odd Gausssian

$$G^o_\lambda(x) = \operatorname{sgn}(x)\, e^{-\pi\lambda x^2}\,,$$

where $\lambda > 0$ is a free parameter. These results, coupled with a refined technique for integration on the free parameter λ (both in the two-sided and one-sided settings), provide the solution of the extremal problem for a large class of even and odd functions, that includes most of the previously known examples in the literature plus a variety of new examples. Among the new examples we highlight here the ones that are connected to the theorems presented in the introduction. Firstly, the family of even functions

$$f_\alpha(x) = \log\left(\frac{x^2+1}{x^2+\left(\alpha-\frac{1}{2}\right)^2}\right), \tag{1.2}$$

is relevant to the proofs of Theorems 1.1, 1.2 and 1.3 (as a matter of fact, the case $\alpha = 1/2$, relevant to Theorem 1.1, also follows from [8]). Secondly, the odd function

$$g(x) = \arctan\left(\frac{1}{x}\right) - \frac{x}{1+x^2}, \tag{1.3}$$

shall be used in the proof of Theorem 1.4 and finally, the even function

$$h(x) = 1 - x\arctan\left(\frac{1}{x}\right), \tag{1.4}$$

will be the one relevant to the proof of Theorem 1.5. In the next section we will clarify the connection between the functions (1.2), (1.3), (1.4) and each of the theorems in the introduction.

2 Representation lemmas and the explicit formula

In this section we let

$$\xi(s) = \frac{1}{2}s(1-s)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

be Riemann's ξ-function. This function is an entire function of order 1 and satisfies the functional equation

$$\xi(s) = \xi(1-s).$$

Hadamard's factorization formula (cf. [11, Chapter 12]) gives us

$$\xi(s) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho},$$

where $\rho = \frac{1}{2} + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$. We have $B = -\sum_\rho \mathrm{Re}(1/\rho)$, with this sum being absolutely convergent. Under RH, γ is real.

Lemma 2.1 (Representation for $\log|\zeta(\alpha+it)|$)**.** *Assume RH and let $f_\alpha(x)$ be defined by* (1.2)*, where $\frac{1}{2} \leq \alpha \leq \frac{3}{2}$. For large t we have*

$$\log|\zeta(\alpha+it)| = \left(\frac{3}{4} - \frac{\alpha}{2}\right)\log t - \frac{1}{2}\sum_{\gamma} f_\alpha(t-\gamma) + O(1), \tag{2.1}$$

uniformly on α, where the sum runs over the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$.

Proof. We apply Hadamard's factorization formula at the points $s = \alpha + it$ and $s = \frac{3}{2} + it$ and divide. The absolute convergence of the product allows us to divide term by term to find

$$\left|\frac{\xi(\alpha+it)}{\xi\left(\frac{3}{2}+it\right)}\right| = \prod_{\rho=1/2+i\gamma} \left(\frac{(\alpha-\frac{1}{2})^2 + (t-\gamma)^2}{1+(t-\gamma)^2}\right)^{1/2},$$

and therefore

$$\log|\xi(\alpha+it)| = \log\left|\xi\left(\tfrac{3}{2}+it\right)\right| + \frac{1}{2}\sum_{\gamma}\log\left(\frac{(\alpha-\frac{1}{2})^2+(t-\gamma)^2}{1+(t-\gamma)^2}\right). \tag{2.2}$$

Recall Stirling's formula for the Gamma function [11, Chapter 10]

$$\log\Gamma(z) = \tfrac{1}{2}\log 2\pi - z + \left(z - \tfrac{1}{2}\right)\log z + O\left(|z|^{-1}\right),$$

for large $|z|$. Using Stirling's formula and the fact that $\left|\zeta\left(\frac{3}{2}+it\right)\right| \asymp 1$ in (2.2), we obtain (2.1). $\square$

Similar representations hold for the argument function $S(t)$ and for the function $S_1(t)$, as reported in [4].

Lemma 2.2 (Representation for $S(t)$)**.** *Assume RH and let $g(x)$ be defined by* (1.3)*. Then, for large t not coinciding with an ordinate of a zero of $\zeta(s)$, we have*

$$S(t) = \frac{1}{\pi} \sum_{\gamma} g(t-\gamma) + O(1), \tag{2.3}$$

where the sum runs over the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$.

Proof. For t not coinciding with an ordinate of a zero of $\zeta(s)$, we have

$$S(t) = -\frac{1}{\pi} \int_{\frac{1}{2}}^{\infty} \operatorname{Im} \frac{\zeta'}{\zeta}(\sigma + it)\, \mathrm{d}\sigma = \frac{1}{\pi} \int_{\frac{3}{2}}^{\frac{1}{2}} \operatorname{Im} \frac{\zeta'}{\zeta}(\sigma + it)\, \mathrm{d}\sigma + O(1).$$

We now replace the integrand on the right-hand side of the above expression by a sum over the non-trivial zeros of $\zeta(s)$. Let $s = \sigma + it$. If s is not a zero of $\zeta(s)$, then the partial fraction decomposition for $\zeta'(s)/\zeta(s)$ (cf. [11, Chapter 12]) and Stirling's formula

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z + O\big(|z|^{-1}\big), \tag{2.4}$$

valid for large $|z|$ with $\Re(z) > 0$, imply that

$$\begin{aligned} \frac{\zeta'}{\zeta}(s) &= \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) - \frac{1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{s}{2} + 1 \right) + O(1) \\ &= \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) - \frac{1}{2} \log t + O(1) \end{aligned} \tag{2.5}$$

uniformly for $\frac{1}{2} \le \sigma \le \frac{3}{2}$ and $t \ge 2$, where the sum runs over the non-trivial zeros ρ of $\zeta(s)$. From (2.5) and the Riemann hypothesis, it follows that

$$\begin{aligned} S(t) &= \frac{1}{\pi} \int_{\frac{3}{2}}^{\frac{1}{2}} \operatorname{Im} \frac{\zeta'}{\zeta}(\sigma + it)\, \mathrm{d}\sigma + O(1) \\ &= \frac{1}{\pi} \int_{\frac{3}{2}}^{\frac{1}{2}} \operatorname{Im} \left(\frac{\zeta'}{\zeta}(\sigma + it) - \frac{\zeta'}{\zeta}(\tfrac{3}{2} + it) \right) \mathrm{d}\sigma + O(1) \\ &= \frac{1}{\pi} \int_{\frac{1}{2}}^{\frac{3}{2}} \sum_{\gamma} \left\{ \frac{(t-\gamma)}{(\sigma - \frac{1}{2})^2 + (t-\gamma)^2} - \frac{(t-\gamma)}{1 + (t-\gamma)^2} \right\} \mathrm{d}\sigma + O(1) \\ &= \frac{1}{\pi} \sum_{\gamma} \int_{\frac{1}{2}}^{\frac{3}{2}} \left\{ \frac{(t-\gamma)}{(\sigma - \frac{1}{2})^2 + (t-\gamma)^2} - \frac{(t-\gamma)}{1 + (t-\gamma)^2} \right\} \mathrm{d}\sigma + O(1) \\ &= \frac{1}{\pi} \sum_{\gamma} \left\{ \arctan\left(\frac{1}{(t-\gamma)} \right) - \frac{(t-\gamma)}{1 + (t-\gamma)^2} \right\} + O(1), \\ &= \frac{1}{\pi} \sum_{\gamma} g(t-\gamma) + O(1), \end{aligned}$$

where the interchange of the integral and the sum is justified by dominated convergence since $g(x) = O(x^{-3})$. This proves the lemma. $\square$

Lemma 2.3 (Representation for $S_1(t)$). *Assume RH and let $h(x)$ be defined by* (1.4). *For large t we have*

$$S_1(t) = \frac{1}{4\pi}\log t - \frac{1}{\pi}\sum_{\gamma} h(t-\gamma) + O(1),$$

where the sum runs over the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$.

Proof. From [34, Theorem 9.9] we have

$$S_1(t) = \frac{1}{\pi}\int_{1/2}^{3/2} \log\left|\zeta(\alpha + it)\right| \mathrm{d}\alpha + O(1).$$

We replace the integrand by the absolutely convergent sum over the zeros of $\zeta(s)$ given by Lemma 2.1 and integrate term-by-term to obtain

$$S_1(t) = \frac{1}{4\pi}\log t - \frac{1}{\pi}\sum_{\rho} h(t-\gamma) + O(1),$$

where the interchange between integration and sum is justified since all terms are non-negative. Notice that we have used the fact that

$$h(x) = 1 - x\arctan\left(\frac{1}{x}\right) = \frac{1}{2}\int_{1/2}^{3/2} \log\left(\frac{x^2+1}{x^2 + \left(\alpha - \frac{1}{2}\right)^2}\right) \mathrm{d}\alpha. \tag{2.6}$$

This completes the proof of the lemma. $\square$

We note the similarity of the representations obtained on Lemmas 2.1, 2.2 and 2.3. We were able to write each of our objects (initially a function of t) as a simple function of t plus a sum over the zeros of $\zeta(s)$ plus a small error term. Naturally the hard part to be analyzed is the sum over the zeros of $\zeta(s)$, but fortunately for this matter we can invoke the following version of the Guinand-Weil explicit formula [18, Theorem 5.12] which connects sums over the zeros of $\zeta(s)$ to sums of the Fourier transforms evaluated at the prime powers.

Lemma 2.4 (Guinand-Weil explicit formula). *Let $\Phi(s)$ be analytic in the strip $|\mathrm{Im}\ s| \leq 1/2 + \varepsilon$ for some $\varepsilon > 0$, and assume that $|\Phi(s)| \ll (1 + |s|)^{-(1+\delta)}$ for some $\delta > 0$ when $|\mathrm{Re}\ s| \to \infty$. Let $\Phi(x)$ be a real-valued for real x, and set $\widehat{\Phi}(\xi) = \int_{-\infty}^{\infty} \Phi(x)\, e^{-2\pi i x\xi}\, \mathrm{d}x$. Then*

$$\sum_{\rho} \Phi(\gamma) \;=\; \Phi\left(\frac{1}{2i}\right) + \Phi\left(-\frac{1}{2i}\right) - \frac{1}{2\pi}\widehat{\Phi}(0)\log \pi$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(u)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)\mathrm{d}u$$
$$-\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\left(\widehat{\Phi}\left(\frac{\log n}{2\pi}\right)+\widehat{\Phi}\left(\frac{-\log n}{2\pi}\right)\right).$$

where Γ'/Γ *is the logarithmic derivative of the Gamma function, and* $\Lambda(n)$ *is the von Mangoldt function defined as*

$$\Lambda(n)=\begin{cases}\log p & \text{if } n=p^m,\ p \text{ prime},\ m\geq 1,\\ 0 & \text{otherwise.}\end{cases}$$

Observe however that we cannot apply the explicit formula to evaluate the sum of our particular functions f_α, g and h over the zeros of ζ, since f_α has singularities on the strip $|\mathrm{Im}\ s|\leq 1/2$ (if $\frac{1}{2}\leq\alpha\leq 1$), g is not continuous and h is not differentiable at the origin. To overcome this difficulty we adopt the following strategy:

(i) We want to replace each of our functions f_α, g and h by an appropriate majorant or minorant (to create an inequality), that satisfies the hypothesis of the explicit formula (a real entire function, integrable on $\mathbb{R}$).

Now that we believe we will be able to use the explicit formula, we might want to choose which of its expressions we would like to "keep" or "simplify". For this we will focus on two of its terms.

(ii) We will ask that the term $\widehat{\Phi}(0)$ for these majorants be as close as possible to the original $\widehat{f}(0)$, which is the same as saying that $\int_{\mathbb{R}}\{\Phi-f\}\,\mathrm{d}x$ should be minimal.

(iii) Finally, in order to simplify the sum of the Fourier transforms of over prime powers, we will consider the instances in which this sum is finite, i.e. $\widehat{\Phi}$ has compact support.

With this framework we are essentially asking for the solution of the Beurling-Selberg problem for each of the functions f_α, g and h.

3 Extremal one-sided approximations

Over the recent years considerable progress was accomplished in terms of understanding the Beurling-Selberg extremal problem (1.1), both via "hard analysis" techniques, that solve the problem for families of functions by using suitable integral representations, and via "soft analysis" techniques, in the sense that once one has the problem solved for a family of functions with a free parameter, one can integrate this parameter and produce the solution for new classes of functions. This general layout was traced by Graham and Vaaler in [16] and developed with more generality in [5, 6, 7, 8, 9].

In [8] this was achieved via a certain exponential subordination. To extract the result from that work that is relevant to our purposes here, we let ν be a non-negative Borel measure on $[0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{\lambda^2+1}\,\mathrm{d}\nu(\lambda) < \infty.$$

Then [8, Theorem 1.1] states that we can solve the extremal minorizing problem for the family of even functions given by

$$f_\nu(x) = \int_0^\infty \left\{e^{-\lambda|x|} - e^{-\lambda}\right\}\mathrm{d}\nu(\lambda).$$

In particular, it was observed in [10] that one can choose

$$\mathrm{d}\nu(\lambda) = \frac{2(1-\cos\lambda)}{\lambda}\,\mathrm{d}\lambda$$

to produce

$$f_\nu(x) = \int_0^\infty \left\{e^{-\lambda|x|} - e^{-\lambda}\right\}\mathrm{d}\nu(\lambda) = \log\left(\frac{x^2+1}{x^2}\right) - \log 2.$$

This leads to the following result concerning the extremal minorants for the function

$$f(x) := f_{1/2}(x) = \log\left(\frac{x^2+1}{x^2}\right), \tag{3.1}$$

as defined in (1.2). This shall be important in the proof of Theorem 1.1. Observe that there can be no discussion about real entire majorants for this function because of its singularity at the origin.

Lemma 3.1 (Extremal minorants for f)**.** *Let $1 \leq \Delta$ and f be defined by* (3.1)*. Then there is a unique real entire function $m_\Delta^- : \mathbb{C} \to \mathbb{C}$ satisfying the following properties:*

(i) For all real x we have

$$\frac{-C}{1+x^2} \leq m_\Delta^-(x) \leq f(x)$$

for some positive constant C. For any complex number $x+iy$ we have

$$\left|m_\Delta^-(x+iy)\right| \ll \frac{\Delta^2}{1+\Delta|x+iy|} e^{2\pi\Delta|y|}.$$

(ii) The Fourier transform of m_Δ^-, namely

$$\widehat{m}_\Delta^-(\xi) = \int_{-\infty}^{\infty} m_\Delta^-(x)\, e^{-2\pi i x\xi}\, \mathrm{d}x,$$

is a continuous real-valued function supported on the interval $[-\Delta,\Delta]$ and satisfies

$$\left|\widehat{m}_\Delta^-(\xi)\right| \ll 1$$

for each $\xi \in [-\Delta,\Delta]$.

(iii) The L^1-distance to f is given by

$$\int_{-\infty}^{\infty} \left\{f(x) - m_\Delta^-(x)\right\}\, \mathrm{d}x = \frac{2}{\Delta}\Big\{\log 2 - \log\big(1+e^{-2\pi\Delta}\big)\Big\}.$$

Proceeding with the developments of the extremal function theory, the recent works [5] and [7] contain a recipe to generate the solution of the Beurling-Selberg extremal problem (1.1) for a wide class of even and odd functions, now making use of a suitable Gaussian subordination. In the even case we start with the solution for the Gaussian

$$G_\lambda(x) = e^{-\pi\lambda x^2},$$

where $\lambda > 0$ is a free parameter, and for suitable non-negative Borel measures μ on $[0,\infty)$ we generate the solution of the extremal problem for the class of even functions

$$f_\mu(x) = \int_0^\infty e^{-\pi\lambda x^2}\, \mathrm{d}\mu(\lambda). \tag{3.2}$$

In particular any finite non-negative measure μ is admissible (these generate the class of positive definite f_μ, see [7, Section 11]) and we can take for instance, given $0 < a \le b$, the measure

$$\mathrm{d}\mu(\lambda) = \frac{\left\{e^{-\pi\lambda a^2} - e^{-\pi\lambda b^2}\right\}}{\lambda}\, \mathrm{d}\lambda$$

to produce

$$f_\mu(x) = \int_0^\infty e^{-\pi\lambda x^2} \frac{\left\{e^{-\pi\lambda a^2} - e^{-\pi\lambda b^2}\right\}}{\lambda}\, \mathrm{d}\lambda = \log\left(\frac{x^2+b^2}{x^2+a^2}\right). \tag{3.3}$$

Therefore the choice of $a = (\alpha - \frac{1}{2})$ and $b = 1$ for $\frac{1}{2} < \alpha \leq \frac{3}{2}$ will produce the function f_α defined in (1.2). As a matter of fact the method is more general and (3.2) only needs to hold in a certain distribution sense in the Fourier space (see [7, Theorems 14, 15 and 16]). This allows us to include the case $\alpha = 1/2$ treated in Lemma 3.1 as well. We state here the relevant facts for such optimal approximations from [7] in a convenient format. This is reported in [3, Lemmas 5 and 8].

Lemma 3.2 (Extremal functions for f_α)**.** *Let $1 \leq \Delta$, $\frac{1}{2} < \alpha \leq \frac{3}{2}$, and f_α defined by* (1.2)*. Then there are unique real entire functions $m^+_{\alpha,\Delta} : \mathbb{C} \to \mathbb{C}$ and $m^-_{\alpha,\Delta} : \mathbb{C} \to \mathbb{C}$ satisfying the following properties:*

(i) For real x we have

$$\frac{-C}{1+x^2} \leq m^-_{\alpha,\Delta}(x) \leq f_\alpha(x) \leq m^+_{\alpha,\Delta}(x) \leq \frac{C\left(1 + \left|\log\left(\alpha - \frac{1}{2}\right)\right|\right)}{1+x^2}, \tag{3.4}$$

for some positive constant C. Moreover, for any complex number $x+iy$ we have

$$\left|m^-_{\alpha,\Delta}(x+iy)\right| \ll \frac{\Delta^2}{1+\Delta|x+iy|} e^{2\pi\Delta|y|}$$

and

$$\left|m^+_{\alpha,\Delta}(x+iy)\right| \ll \left(1 + \left|\log\left(\alpha - \tfrac{1}{2}\right)\right|\right) \frac{\Delta^2}{1+\Delta|x+iy|} e^{2\pi\Delta|y|}. \tag{3.5}$$

(ii) The Fourier transforms of $m^\pm_{\alpha,\Delta}$, namely

$$\widehat{m}^\pm_{\alpha,\Delta}(\xi) = \int_{-\infty}^{\infty} m^\pm_{\alpha,\Delta}(x)\, e^{-2\pi i x\xi}\, \mathrm{d}x,$$

are continuous real-valued functions supported on the interval $[-\Delta, \Delta]$ and satisfy

$$\left|\widehat{m}^-_{\alpha,\Delta}(\xi)\right| \ll 1$$

and

$$\left|\widehat{m}^+_{\alpha,\Delta}(\xi)\right| \ll 1 + \left|\log\left(\alpha - \tfrac{1}{2}\right)\right|,$$

for each $\xi \in [-\Delta, \Delta]$. Moreover, for $0 \leq |\xi| \leq \Delta$, we have the explicit expressions

$$\widehat{m}^-_{\alpha,\Delta}(\xi) = \sum_{k=0}^{\infty} (-1)^k \left\{ \frac{k+1}{|\xi| + k\Delta} \left(e^{-2\pi(|\xi|+k\Delta)(\alpha-1/2)} - e^{-2\pi(|\xi|+k\Delta)} \right) \right.$$
$$\left. - \frac{k+1}{\Delta(k+2) - |\xi|} \left(e^{2\pi(|\xi|-(k+2)\Delta)(\alpha-1/2)} - e^{2\pi(|\xi|-(k+2)\Delta)} \right) \right\},$$

and

$$\widehat{m}^{+}_{\alpha,\Delta}(\xi) = \sum_{k=0}^{\infty}\left\{\frac{k+1}{|\xi|+k\Delta}\left(e^{-2\pi(|\xi|+k\Delta)(\alpha-1/2)} - e^{-2\pi(|\xi|+k\Delta)}\right)\right.$$
$$\left. - \frac{k+1}{\Delta(k+2)-|\xi|}\left(e^{2\pi(|\xi|-(k+2)\Delta)(\alpha-1/2)} - e^{2\pi(|\xi|-(k+2)\Delta)}\right)\right\}.$$

In particular, if $\xi = 0$, *we have*

$$\widehat{m}^{-}_{\alpha,\Delta}(0) = 2\pi\left(\frac{3}{2}-\alpha\right) - \frac{2}{\Delta}\log\left(\frac{1+e^{-(2\alpha-1)\pi\Delta}}{1+e^{-2\pi\Delta}}\right),$$

and

$$\widehat{m}^{+}_{\alpha,\Delta}(0) = 2\pi\left(\frac{3}{2}-\alpha\right) - \frac{2}{\Delta}\log\left(\frac{1-e^{-(2\alpha-1)\pi\Delta}}{1-e^{-2\pi\Delta}}\right).$$

(iii) The L^1*-distances between* $m^{\pm}_{\alpha,\Delta}$ *and* f_α *are equal to*

$$\int_{-\infty}^{\infty}\left\{f_\alpha(x) - m^{-}_{\alpha,\Delta}(x)\right\}\,\mathrm{d}x = \frac{2}{\Delta}\left\{\log\big(1+e^{-(2\alpha-1)\pi\Delta}\big) - \log\big(1+e^{-2\pi\Delta}\big)\right\},$$

and

$$\int_{-\infty}^{\infty}\left\{m^{+}_{\alpha,\Delta}(x) - f_\alpha(x)\right\}\,\mathrm{d}x = \frac{2}{\Delta}\left\{\log\big(1-e^{-2\pi\Delta}\big) - \log\big(1-e^{-(2\alpha-1)\pi\Delta}\big)\right\}.$$

In [5] we have the counterpart for odd and truncated functions of the Gaussian subordination method. We start solving the Beurling-Selberg extremal problem for the odd Gaussian

$$G^{o}_{\lambda}(x) = \operatorname{sgn}(x)\, e^{-\pi\lambda x^2},$$

where $\lambda > 0$ is a free parameter, and for finite non-negative Borel measures μ on $[0,\infty)$ we generate the solution of the extremal problem for the class of odd functions

$$f^{o}_{\mu}(x) = \operatorname{sgn}(x)\int_0^{\infty} e^{-\pi\lambda x^2}\,\mathrm{d}\mu(\lambda).$$

These are the odd counterparts of the positive definite functions. In particular, it was observed in [4] that the measure

$$\mathrm{d}\mu_g(\lambda) = \left\{\int_0^{\infty}\frac{t}{2\sqrt{\pi\lambda^3}}\,e^{-\frac{t^2}{4\lambda}}\left(\frac{1}{t}\sin\left(\sqrt{\pi}t\right) - \sqrt{\pi}\cos\left(\sqrt{\pi}t\right)\right)\mathrm{d}t\right\}\mathrm{d}\lambda.$$

is non-negative, finite and verifies

$$g(x) = \arctan\left(\frac{1}{x}\right) - \frac{x}{1+x^2} = \operatorname{sgn}(x)\int_0^{\infty} e^{-\pi\lambda x^2}\,\mathrm{d}\mu_g(\lambda).$$

We collect the facts from [5] on the extremal functions for $g(x)$ in the lemma below. This lemma shall be used in the proof of Theorem 1.4.

Lemma 3.3 (Extremal functions for g)**.** *Let $1 \leq \Delta$ and g be defined by (1.3). Then there are unique real entire functions $m_\Delta^+ : \mathbb{C} \to \mathbb{C}$ and $m_\Delta^- : \mathbb{C} \to \mathbb{C}$ satisfying the following properties:*

(i) For all real x we have

$$\frac{-C}{1+x^2} \leq m_\Delta^-(x) \leq g(x) \leq m_\Delta^+(x) \leq \frac{C}{1+x^2},$$

for some positive constant C. For any complex number $x+iy$ we have

$$\left|m_\Delta^\pm(x+iy)\right| \ll \frac{\Delta^2}{1+\Delta|x+iy|} e^{2\pi\Delta|y|}.$$

(ii) The Fourier transforms of $m_\Delta^\pm$ are continuous functions supported on the interval $[-\Delta, \Delta]$ and satisfy

$$\left|\widehat{m}_\Delta^\pm(\xi)\right| \ll 1$$

for each $\xi \in [-\Delta, \Delta]$.

(iii) The L^1-distances to g are given by

$$\int_{-\infty}^{\infty} \left\{m_\Delta^+(x) - g(x)\right\} \,\mathrm{d}x = \int_{-\infty}^{\infty} \left\{g(x) - m_\Delta^-(x)\right\} \,\mathrm{d}x = \frac{\pi}{2\Delta}.$$

For the final lemma in this section we return to the case of even functions and the Gaussian subordination (3.2). It was observed in [4] that one can consider the non-negative and finite measure given by

$$\mathrm{d}\mu_h(\lambda) = \int_{1/2}^{3/2} \left\{ \frac{e^{-\pi\lambda(\sigma-1/2)^2} - e^{-\pi\lambda}}{2\lambda} \right\} \mathrm{d}\sigma \,\mathrm{d}\lambda.$$

Using (2.6), (3.2) and (3.3) we arrive at

$$h(x) = 1 - x\arctan\left(\frac{1}{x}\right) = \int_0^\infty e^{-\pi\lambda x^2} \,\mathrm{d}\mu_h(\lambda).$$

We can therefore collect the relevant facts about the extremal functions for g from the general theorems in [7]. This was reported in [4] an shall be used in the proof of Theorem 1.5.

Lemma 3.4 (Extremal functions for h)**.** *Let $1 \leq \Delta$ and h be defined by (1.4). Then there are unique real entire functions $m_\Delta^+ : \mathbb{C} \to \mathbb{C}$ and $m_\Delta^- : \mathbb{C} \to \mathbb{C}$ satisfying the following properties:*

(i) For all real x we have

$$\frac{-C}{1+x^2} \leq m_\Delta^-(x) \leq h(x) \leq m_\Delta^+(x) \leq \frac{C}{1+x^2},$$

for some positive constant C. For any complex number $x+iy$ we have

$$\left|m_\Delta^\pm(x+iy)\right| \ll \frac{\Delta^2}{1+\Delta|x+iy|} e^{2\pi\Delta|y|}.$$

(ii) The Fourier transforms of $m_\Delta^\pm$ are continuous real-valued functions supported on the interval $[-\Delta,\Delta]$ and satisfy

$$\left|\widehat{m}_\Delta^\pm(\xi)\right| \ll 1$$

for each $\xi \in [-\Delta,\Delta]$.

(iii) The L^1-distances to h are given by

$$\int_{-\infty}^{\infty} \left\{h(x) - m_\Delta^-(x)\right\} \mathrm{d}x = \int_{1/2}^{3/2} \frac{1}{\Delta} \left\{\log\left(1+e^{-(2\sigma-1)\pi\Delta}\right) - \log\left(1+e^{-2\pi\Delta}\right)\right\} \mathrm{d}\sigma,$$

and

$$\int_{-\infty}^{\infty} \left\{m_\Delta^+(x) - h(x)\right\} \mathrm{d}x = \int_{1/2}^{3/2} \frac{1}{\Delta} \left\{\log\left(1-e^{-2\pi\Delta}\right) - \log\left(1-e^{-(2\sigma-1)\pi\Delta}\right)\right\} \mathrm{d}\sigma.$$

4 Proofs of the main theorems

We now make use of the extremal functions described on the last section, together with the representation formulas to provide the proofs of the main theorems.

4.1 Proof of Theorem 1.1

With f defined by (3.1) and m_Δ^- defined as in Lemma 3.1, we can use Lemma 2.1 to obtain

$$\begin{aligned} \log\left|\zeta\left(\tfrac{1}{2}+it\right)\right| &= \frac{1}{2}\log t - \frac{1}{2}\sum_\gamma f(t-\gamma) + O(1) \\ &\leq \frac{1}{2}\log t - \frac{1}{2}\sum_\gamma m_\Delta^-(t-\gamma) + O(1). \end{aligned} \tag{4.1}$$

We now apply the explicit formula (Lemma 2.4) with $\Phi(z) = m_\Delta^-(t-z)$. In this context we have $\widehat{\Phi}(\xi) = \widehat{m}_\Delta^-(-\xi)e^{-2\pi i\xi t}$ and therefore

$$\begin{aligned}\sum_\rho m_\Delta^-(t-\gamma) = \left\{m_\Delta^-\left(t-\tfrac{1}{2i}\right) + m_\Delta^-\left(t+\tfrac{1}{2i}\right)\right\} - \frac{1}{2\pi}\widehat{m}_\Delta^-(0)\log\pi \\ + \frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^-(t-u)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)\,\mathrm{d}u \\ - \frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\left\{n^{-it}\,\widehat{m}_\Delta^-\left(-\frac{\log n}{2\pi}\right) + n^{it}\,\widehat{m}_\Delta^-\left(\frac{\log n}{2\pi}\right)\right\}.\end{aligned} \tag{4.2}$$

Let us split this sum into four terms and quote each of these separately.

4.1.1 First term

From Lemma 3.1 (i) we see that

$$\left|m_\Delta^-\left(t-\tfrac{1}{2i}\right) + m_\Delta^-\left(t+\tfrac{1}{2i}\right)\right| \ll \frac{\Delta^2}{1+\Delta t}e^{\pi\Delta}. \tag{4.3}$$

4.1.2 Second term

From Lemma 3.1 (ii) we have

$$\left|\widehat{m}_\Delta^-(0)\right| \ll 1. \tag{4.4}$$

4.1.3 Third term

Using Stirling's formula (2.4), Lemma 3.1 (i) and (iii), and the fact that $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi$ we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^-(t-u)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)\,\mathrm{d}u = \\ = \frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^-(u)\left(\log t + O(\log(2+|u|)\right)\mathrm{d}u \\ = \log t - \frac{\log t}{\pi\Delta}\log\left(\frac{2}{1+e^{-2\pi\Delta}}\right) + O(1).\end{aligned} \tag{4.5}$$

4.1.4 Fourth term

Finally, we use the fact that the Fourier transform of m_Δ is compactly supported on the interval $[-\Delta, \Delta]$, as given in Lemma 3.1 (ii), to bound the sum over the prime powers

$$\begin{aligned}
&\left|\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\left\{n^{-it}\widehat{m}_{\Delta}^{-}\left(-\frac{\log n}{2\pi}\right)+n^{it}\widehat{m}_{\Delta}^{-}\left(\frac{\log n}{2\pi}\right)\right\}\right| \\
&\leq \frac{1}{2\pi}\sum_{n=2}^{e^{2\pi\Delta}}\frac{\Lambda(n)}{\sqrt{n}}\left\{\left|\widehat{m}_{\Delta}^{-}\left(-\frac{\log n}{2\pi}\right)\right|+\left|\widehat{m}_{\Delta}^{-}\left(\frac{\log n}{2\pi}\right)\right|\right\} \qquad (4.6)\\
&\ll \sum_{n=2}^{e^{2\pi\Delta}}\frac{\Lambda(n)}{\sqrt{n}} \ll e^{\pi\Delta},
\end{aligned}$$

where the last expression was evaluated via summation by parts.

4.1.5 Conclusion

Combining expressions (4.1)-(4.6) we arrive at

$$\log\left|\zeta\left(\tfrac{1}{2}+it\right)\right| \leq \frac{\log t}{2\pi\Delta}\log\left(\frac{2}{1+e^{-2\pi\Delta}}\right)+O\left(\frac{\Delta^2 e^{\pi\Delta}}{(1+\Delta t)}+e^{\pi\Delta}+1\right). \qquad (4.7)$$

Until now we did all of our estimates without prescribing any particular value for Δ. It turns out that the choice

$$\pi\Delta = \log\log t - 3\log\log\log t$$

in (4.7) concludes the proof of Theorem 1.1.

4.2 Proof of Theorem 1.4

This follows by a very similar argument. With g defined as in (1.3), and $m_{\Delta}^{\pm}$ defined as in Lemma 3.3, we can use Lemma 2.2 to obtain

$$\begin{aligned}
&\frac{1}{\pi}\sum_{\gamma} m_{\Delta}^{-}(t-\gamma)+O(1) \\
&\qquad \leq S(t) = \frac{1}{\pi}\sum_{\gamma} g(t-\gamma)+O(1) \leq \frac{1}{\pi}\sum_{\gamma} m_{\Delta}^{+}(t-\gamma)+O(1).
\end{aligned}$$

We then use the explicit formula with $m_{\Delta}^{\pm}$ and bound the first, second and fourth terms as done in the proof of Theorem 1.1, now using Lemma 3.3. For the third term we use Stirling's formula (2.4), Lemma 3.3 (i) and (iii), and the fact that

$\int_{-\infty}^{\infty} g(x)\,\mathrm{d}x = 0$ to get

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^{\pm}(t-u)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)\,\mathrm{d}u = \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^{\pm}(u)\,\big(\log t + O(\log(2+|u|)\big)\,\mathrm{d}u \\ &= \pm\frac{\log t}{4\Delta} + O(1).\end{aligned}$$

We thus arrive at

$$|S(t)| \le \frac{\log t}{4\pi\Delta} + O\left(\frac{\Delta^2 e^{\pi\Delta}}{(1+\Delta t)} + e^{\pi\Delta} + 1\right).$$

and again it is just a matter of choosing $\pi\Delta = \log\log t - 3\log\log\log t$ to conclude the proof of Theorem 1.4.

4.3 Proof of Theorem 1.5

Let h be defined as in (1.4), and $m_\Delta^{\pm}$ defined as in Lemma 3.4. From Lemma 2.3 we have

$$\begin{aligned}\frac{1}{4\pi}\log t - \frac{1}{\pi}\sum_{\gamma} m_\Delta^{+}(t-\gamma) + O(1) \\ \le S_1(t) = \frac{1}{4\pi}\log t - \frac{1}{\pi}\sum_{\gamma} h(t-\gamma) + O(1) \\ \le \frac{1}{4\pi}\log t - \frac{1}{\pi}\sum_{\gamma} m_\Delta^{-}(t-\gamma) + O(1).\end{aligned}$$

Once more we apply the explicit formula with $m_\Delta^{\pm}$ and bound the first, second and fourth terms as done in the proof of Theorem 1.1, now using Lemma 3.4. For the third term we use Stirling's formula (2.4), Lemma 3.4 (i) and (iii), and the fact that

$$\int_{-\infty}^{\infty} h(x)\,\mathrm{d}x = \frac{\pi}{2}$$

to get

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^-(t-u)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)\,\mathrm{d}u\\
&\qquad=\frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^-(u)\Big(\log t+O\big(\log(2+|u|)\big)\Big)\,\mathrm{d}u\\
&\qquad=\frac{1}{4}\log t-\frac{\log t}{2\pi\Delta}\int_{1/2}^{3/2}\Big(\log\big(1+e^{-(2\sigma-1)\pi\Delta}\big)-\log\big(1+e^{-2\pi\Delta}\big)\Big)\,\mathrm{d}\sigma+O(1)\\
&\qquad\geq\frac{1}{4}\log t-\frac{\log t}{2\pi\Delta}\int_{1/2}^{\infty}\log\big(1+e^{-(2\sigma-1)\pi\Delta}\big)\,\mathrm{d}\sigma+O(1)\\
&\qquad=\frac{1}{4}\log t-\frac{\log t}{2\pi^2\Delta^2}\int_{0}^{\infty}\log\big(1+e^{-2\alpha}\big)\,\mathrm{d}\alpha+O(1).
\end{aligned}$$

Now observe that (cf. [15, §4.291])

$$\int_0^\infty \log\big(1+e^{-2\alpha}\big)\,\mathrm{d}\alpha=\frac{1}{2}\int_0^1\frac{\log(1+u)}{u}\,\mathrm{d}u=\frac{\pi^2}{24}.$$

Therefore, by combining these estimates, we arrive at

$$S_1(t)\leq\frac{\log t}{48\pi\Delta^2}+O\left(\frac{\Delta^2e^{\pi\Delta}}{(1+\Delta t)}+e^{\pi\Delta}+1\right).$$

Choosing $\pi\Delta=\log\log t-3\log\log\log t$ in the inequality above gives us

$$S_1(t)\leq\frac{\pi}{48}\frac{\log t}{(\log\log t)^2}+O\left(\frac{\log t\log\log\log t}{(\log\log t)^3}\right),$$

which is the upper bound for $S_1(t)$ stated in Theorem 1.5. To prove the lower bound we proceed similarly by observing that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^+(t-u)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)\,\mathrm{d}u\\
&\qquad=\frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^+(u)\Big(\log t+O\big(\log(2+|u|)\big)\Big)\,\mathrm{d}u\\
&\qquad=\frac{1}{4}\log t-\frac{\log t}{2\pi\Delta}\int_{1/2}^{3/2}\Big(\log\big(1-e^{-(2\sigma-1)\pi\Delta}\big)-\log\big(1-e^{-2\pi\Delta}\big)\Big)\,\mathrm{d}\sigma\\
&\qquad+O(1)\\
&\qquad\leq\frac{1}{4}\log t-\frac{\log t}{2\pi\Delta}\int_{1/2}^{\infty}\log\big(1-e^{-(2\sigma-1)\pi\Delta}\big)\,\mathrm{d}\sigma+O(1)\\
&\qquad=\frac{1}{4}\log t-\frac{\log t}{2\pi^2\Delta^2}\int_{0}^{\infty}\log\big(1-e^{-2\alpha}\big)\,\mathrm{d}\alpha+O(1).
\end{aligned}$$

We now invoke the identity (cf. [15, §4.291])

$$\int_0^\infty \log\left(1-e^{-2\alpha}\right)\,\mathrm{d}\alpha = \frac{1}{2}\int_0^1 \frac{\log(1-u)}{u}\,\mathrm{d}u = -\frac{\pi^2}{12}$$

to arrive at

$$S_1(t) \geq -\frac{\log t}{24\pi\Delta^2} + O\left(\frac{\Delta^2 e^{\pi\Delta}}{(1+\Delta t)} + e^{\pi\Delta} + 1\right).$$

Finally, choosing $\pi\Delta = \log\log t - 3\log\log\log t$ in the inequality above gives us

$$S_1(t) \geq -\frac{\pi}{24}\frac{\log t}{(\log\log t)^2} + O\left(\frac{\log t \log\log\log t}{(\log\log t)^3}\right),$$

and this completes the proof of Theorem 1.5.

4.4 Final remarks

The proofs of Theorems 1.2 and 1.3 are more involved and make use of the expressions for the Fourier transforms $\widehat{m}^{\pm}_{\alpha,\Delta}$ given by Lemma 3.2 (ii) to evaluate the sum over the prime powers in the explicit formula, which will have a significant contribution in the case $\frac{1}{2} < \alpha \leq 1$. We refer the reader to [3, Theorems 1 and 2] for the details of these proofs. Let us register here an oversight in [3, Lemma 8 (i)], where the dependence of the constants on the parameter α is not explicit. The right statement is presented here in Lemma 3.2 (i) in equations (3.4) and (3.5). This, however, should not affect the proof of [3, Theorem 2].

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