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Some generalizations of Calabi compactness theorem

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Dedicated to Gervásio Colares for his 80th birthday

Abstract

In this paper we obtain generalized Calabi-type compactness criteria for complete Riemannian manifolds that allow the presence of negative amounts of Ricci curvature. These, in turn, can be rephrased as new conditions for the positivity, for the existence of a first zero and for the nonoscillatory-oscillatory behaviour of a solution $g(t)$ of $g'' + Kg = 0$, subjected to the initial condition $g(0) = 0$, $g'(0) = 1$. A unified approach for this ODE, based on the notion of critical curve, is presented. With the aid of suitable examples, we show that our new criteria are sharp and, even for $K \geq 0$, in borderline cases they improve on previous works of Calabi, Hille-Nehari and Moore.

1 Basic comparison and Myers type compactness result

Hereafter, we consider a connected, complete Riemannian manifold $(M, \langle\,,\,\rangle)$, and a chosen reference origin $o \in M$. Let $D_o = M\backslash(\{o\} \cup \mathrm{cut}(o))$ be the maximal domain

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of normal coordinates centered at o, and denote with $r(x)$ the distance function from o. The classical Bonnet-Myers theorem, showing the compactness of M under the condition

$$\mathrm{Ricc} \geq (m-1)B^2\langle\,,\,\rangle \tag{1.1}$$

for some $B > 0$, can be proved as a consequence of the Laplacian comparison theorem. Indeed, let us recall the following generalized form of this latter.

Theorem 1 (**Theorem 2.4 of [15]**). *Let M be as above. Assume that the radial Ricci curvature satisfies*

$$\mathrm{Ricc}(\nabla r, \nabla r)(x) \geq -(m-1)G(r(x)) \qquad \text{on } M, \tag{1.2}$$

for some function $G \in C^0(\mathbb{R}_0^+)$, and let $g \in C^2(\mathbb{R}_0^+)$ be a solution of

$$\begin{cases} g'' - Gg \geq 0 \\ g(0) = 0, \quad g'(0) = 1. \end{cases} \tag{1.3}$$

Let $(0, R_0)$ (possibly $R_0 = +\infty$) be the maximal interval where g is positive. Then,

$$D_o \subset B_{R_0} \tag{1.4}$$

and the inequality

$$\Delta r(x) \leq (m-1)\frac{g'(r(x))}{g(r(x))} \tag{1.5}$$

holds pointwise on D_o and weakly on M.

Suppose the validity of (1.1) so that $G(t) = -B^2$. A simple checking shows that $g(t) = B^{-1}\sin(Bt)$ solves (1.3). Its first positive zero is at $2\pi/B$. Then (1.4) gives that $\overline{D_o} \equiv M$ is bounded. Since M is closed, the Hopf-Rinow theorem implies that M is compact. In fact, we have also shown that $\mathrm{diam}(M) \leq 2\pi/B$, but since (1.1) is indipendent of the origin o we can improve the above to the sharp estimate $\mathrm{diam}(M) \leq \pi/B$.

Cleary the key point of our proof lies in the validity of the inclusion $D_o \subset B_{R_o}$. The way to prove this latter is as follows. Suppose to have shown (1.5) on $D_o \cap B_{R_o}$

A computation in normal coordinates gives

$$\Delta r = \frac{\partial}{\partial r}\log\sqrt{\tilde{g}(r,\theta)},$$

where $\tilde{g}(r,\theta)$ is the determinant of the metric in this coordinate system. Thus, (1.5) on $D_o \cap B_{R_0}$ reads

$$\frac{\partial}{\partial r}\log\sqrt{\tilde{g}(r,\theta)} \leq (m-1)\frac{g'(r)}{g(r)}. \tag{1.6}$$

Fix the unit vector θ and let γ_θ be the unit speed geodesic emanating from o with $\dot{\gamma}_\theta(o) = \theta$. γ_θ will stop to be minimizing after the first cut point attained at $t = c(\theta) > 0$.

With $\epsilon > 0$ sufficiently small, we integrate (1.6) on $[\epsilon, \min\{c(\theta), R_o\}]$, we let $\epsilon \to 0^+$ and we use the asymptotic behaviours in 0 to get

$$\sqrt{\tilde{g}(r,\theta)} \leq g(r)^{m-1},$$

Since $\tilde{g}(r,\theta) > 0$ on D_o, we have $R_0 \geq c(\theta)$, that is, $D_o \subset B_{R_0}$.

However, by a result of M. Morse, a complete manifold M is compact if and only if each unit speed geodesic γ_θ emanating from some fixed origin o ceases to be a segment i.e. length minimizing, for a value $c(t_o)$ of its parameter t which is finite. Thus, the above reasoning appears to be slightly redundant, in the sense that it provides a bound R_0 which is independent of the considered unit speed geodesic from o. This motivates the following result of Galloway [7].

Theorem 2. *Let $(M, \langle\,,\,\rangle)$ be a complete Riemannian manifold of dimension $m \geq 2$. Assume that, for some origin o and for every unit speed geodesic $\gamma : \mathbb{R}_0^+ \to M$ emanating from o, the solution g of*

$$\begin{cases} g'' + \dfrac{\mathrm{Ricc}(\gamma', \gamma')(t)}{m-1} g = 0, \\ \\ g(0) = 0, \quad g'(0) = 1 \end{cases} \tag{1.7}$$

has a first positive zero. Then, M is compact with finite fundamental group.

Proof. Let $r_0 > 0$ be the first positive zero of g solution of (1.7). Multiply the equation in (1.7) by g, integrate by parts and use the initial conditions to get

$$\int_0^{r_0} (g')^2 - \int_0^{r_0} \frac{\mathrm{Ricc}(\dot\gamma, \dot\gamma)}{m-1} g^2 = 0 \tag{1.8}$$

By Rayleigh characterization, this means that the operator

$$L = \frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{\mathrm{Ricc}(\dot\gamma, \dot\gamma)}{m-1}$$

satisfies

$$\lambda_1^L([0, r_0]) \leq 0,$$

and by monotonicity of eigenvalue

$$\lambda_1^L([0, r]) < 0 \qquad \forall\ r > r_0.$$

But L is the stability operator for the geodesic γ, and on $[0, T]$ γ is minimizing only if

$$\lambda_1^L([0, T]) \geq 0.$$

Thus if the value $c(\gamma)$ gives the cut-point di o along γ it must be $c(\gamma) \leq r_0$. By Morse result M is compact. The same procedure can also be applied to the Riemannian universal covering $\widetilde{M} \to M$, showing that $\widetilde{M}$ is compact and thus that $\Pi_1(M)$ is finite. □

If we ignore that L is the stability operator for the unit speed geodesic γ we can proceed with the following analytic alternative proof.

Let $p \in D_o$, and let $\gamma : [0, r(p)] \to M$ be the minimizing geodesic from o to p so that $r(\gamma(t)) = t$ and $\nabla r \circ \gamma = \dot{\gamma}$ for $t \in [0, r(p)]$. We fix a local orthonormal coframe $\{\theta^i\}$ to perform computations. Here $1 \leq i, j, \ldots \leq m$ and we use Einstein summation convention. Then for the distance function r on D_o we have

$$\mathrm{d}r = r_i \theta^i,$$

and Gauss lemma writes

$$r_i r_i \equiv 1. \tag{1.9}$$

Taking covariant derivative of (1.9) we obtain

$$r_{ij} r_i = 0 \tag{1.10}$$

that is,

$$\mathrm{Hess}\, r(\nabla r, \cdot) = 0. \tag{1.11}$$

Covariant differentiation of (1.10) yields

$$r_{ijk} r_i + r_{ij} r_{ik} = 0. \tag{1.12}$$

From the simmetry $r_{ij} = r_{ji}$ we deduce that $r_{ijk} = r_{jik}$, and by the Ricci commutation rules

$$r_{ijk} = r_{ikj} + r_t R_{tijk}$$

R_{tijk} the components of the Riemann tensor. Using this in (1.12) we get

$$0 = r_{ijk} r_i + r_{ij} r_{ik} = r_{jik} r_i + r_{ij} r_{ik} = r_{jki} r_i + r_t R^t_{jik} r_i + r_{ij} r_{ik}.$$

Thus, tracing with respect to j and k

$$r_i r_{kki} + r_t r_i R_{ti} + r_{ik} r_{ik} = 0,$$

with R_{ti} the components of the Ricci tensor. In other words

$$\langle \nabla \Delta r, \nabla r \rangle + \mathrm{Ricc}\,(\nabla r, \nabla r) + |\mathrm{Hess}(r)|^2 = 0$$

Computing along γ

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\Delta r \circ \gamma) + |\mathrm{Hess}(r)|^2 + \mathrm{Ricc}\,(\nabla r, \nabla r) = 0$$

on $[0, r(p)]$. Using (1.11) and Newton's inequality, we have

$$|\mathrm{Hess}(r)|^2 \geq \frac{(\Delta r)^2}{m-1},$$

and setting $\varphi(t) = \Delta r \circ \gamma(t)$ from the above we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) + \frac{\varphi(t)^2}{m-1} + \mathrm{Ricc}\,(\nabla r, \nabla r) \leq 0 \tag{1.13}$$

on $[0, r(p)]$. Furthermore, it is well known that

$$\Delta r = \frac{m-1}{r} + o(1) \qquad \text{as } r \to 0^+$$

Hence, since γ is minimizing

$$\frac{1}{m-1}\varphi(t) = \frac{1}{(r\circ\gamma)(t)} + o(1) = \frac{1}{t} + o(1) \qquad \text{as } t \to 0^+ \tag{1.14}$$

Defining

$$u(t) = t \exp\left\{\int_0^t \left(\frac{\varphi(s)}{m-1} - \frac{1}{s}\right) \mathrm{d}s\right\}$$

on $[0, r(p)]$, u is well defined because of (1.14) and a computation using (1.13) gives

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}u + \frac{\mathrm{Ricc}(\dot\gamma,\dot\gamma)}{m-1}u \le 0 \tag{1.15}$$

Let now h be any $C^1([0, r(p)])$ function such that $h(0) = 0 = h(r(p))$. Since $u > 0$ on $(0, r(p)]$ the function h^2u'/u is well defined on $(0, r(p)]$. Differentiating, using (1.15) and Young inequality we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(h^2\frac{u'}{u}\right) \le -\frac{\mathrm{Ricc}(\dot\gamma,\dot\gamma)}{m-1}h^2 - h^2\left(\frac{u'}{u}\right)^2 + 2hh'\frac{u'}{u} \le -\frac{\mathrm{Ricc}(\dot\gamma,\dot\gamma)}{m-1}h^2 + (h')^2$$

Fix $\epsilon > 0$ sufficiently small. Integration of the above on $[\epsilon, r(p)]$ gives

$$-h^2(\epsilon)\frac{u'(\epsilon)}{u(\epsilon)} \le \int_\epsilon^{r(p)} (h')^2 - \int_\epsilon^{r(p)} \frac{\mathrm{Ricc}(\dot\gamma,\dot\gamma)}{m-1}h^2$$

Since $h(\epsilon) = A\epsilon + o(1)$, for $\epsilon \to 0^+$ where $A \in \mathbb{R}$, letting $\epsilon \to 0^+$ we obtain

$$\int_0^{r(p)} (h')^2 - \int_0^{r(p)} \frac{\mathrm{Ricc}(\dot\gamma,\dot\gamma)}{m-1}h^2 \ge 0 \tag{1.16}$$

This contradicts (1.8) unless $r(p) \le r_0$.

Thus we have reduced the compactness problem for the complete manifold M to the problem of the existence of a first zero for solutions of the Cauchy problem

$$\begin{cases} g'' + K(t)g = 0 & \text{on } \mathbb{R}^+ \\ g(0) = 0, \quad g'(0) = 1. \end{cases} \tag{CP}$$

where in our geometric application

$$K(t) = K_\gamma(t) = \frac{\mathrm{Ricc}(\dot\gamma,\dot\gamma)}{m-1}(t) \tag{1.17}$$

We observe that the existence of a first zero is also "a posteriori" guaranteed via an oscillation result for the same equation, and that uniform upper estimate for the positioning of the first zero yields a diameter estimate. In this perspective the original result of Calabi can be stated as follows (see also Theorem 3.11 of [2]).

Theorem 3 (**Theorems 1 and 2 of [4]**). *Let M be as above, and assume that* $\mathrm{Ricc} \geq 0$ *on* M. *Suppose that for each unit speed geodesic* γ *emanating from* o *there exist* $0 < a < b$, *possibly depending on* γ, *such that*

$$\int_a^b \sqrt{\frac{\mathrm{Ricc}(\gamma', \gamma')(s)}{m-1}}\, \mathrm{d}s > \left\{ \left(1 + \frac{1}{2} \log \frac{b}{a} \right)^2 - 1 \right\}^{1/2}. \tag{1.18}$$

Then, M is compact and has finite fundamental group. In particular, this holds provided that

$$\limsup_{t \to +\infty} \left(\int_1^t \sqrt{\frac{\mathrm{Ricc}(\gamma', \gamma')(s)}{m-1}}\, \mathrm{d}s - \frac{1}{2} \log t \right) = +\infty. \tag{1.19}$$

Remark 1. As a matter of fact, under the assumption $\mathrm{Ricc} \geq 0$ on M, (1.19) gives an oscillation result for (CP).

In Calabi result the requirement $\mathrm{Ricc} \geq 0$ is essential. We stress that (1.18) is, to the best of our knowledge, the first instance of a condition in finite form for the existence of a first zero, that is, a condition involving the potential K only in a compact interval $[a, b]$. One of the main purpose of the present paper is to extend the result even when Ricci is negative somewhere.

2 The role of the critical curve

As we will see shortly, in order to extend Calabi result, we shall deal with a slightly different ODE. In particular, we are concerned with the following problems:

i) study the existence of a first zero of solutions $z(r)$ of

$$\begin{cases} (v(r)z'(r))' + A(r)v(r)z(r) = 0 & \text{on } \mathbb{R}^+ \\ z(0^+) = z_0 > 0, \end{cases} \tag{2.1}$$

with $A(t) \geq 0$, $v(t) > 0$ on $\mathbb{R}^+$;

ii) give an upper bound for the positioning of the first zero of z;

iii) study the oscillatory behavior of (2.1);

iv) extend the obtained result when $A(r)$ changes sign.

Towards these aims we introduce the "critical curve" $\chi(r)$ relative to (2.1) or to the next Cauchy problem

$$\begin{cases} (v(r)z'(r))' + A(r)v(r)z(r) = 0 & \text{on } [r_0, +\infty), \quad r_0 > 0 \\ z(r_0^+) = z_0 \in \mathbb{R}, \end{cases} \tag{2.2}$$

To do this we require the assumptions

$$0 \leq v(r) \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^{+}_{0}), \qquad \frac{1}{v(r)} \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^{+}), \qquad \lim_{r\to 0^{+}} v(r) = 0 \tag{V1}$$

(the last equation request is intended on a rapresentative of v) and the integrability condition

$$\frac{1}{v(r)} \in L^{1}(+\infty). \tag{V$_{\mathrm{L1}}$}$$

We set

$$\chi(r) = \left\{ 2v(r) \int_{r}^{+\infty} \frac{\mathrm{d}s}{v(s)} \right\}^{-2} = \left\{ \left(-\frac{1}{2} \log \int_{r}^{+\infty} \frac{\mathrm{d}s}{v(s)} \right)' \right\}^{2} \tag{2.3}$$

Fix $0 < R < r$, from the definition di χ we deduce

$$\int_{R}^{r} \sqrt{\chi(s)}\mathrm{d}s = \frac{1}{2} \log \left\{ \left(\int_{R}^{+\infty} \frac{\mathrm{d}s}{v(s)} \right) \Big/ \left(\int_{r}^{+\infty} \frac{\mathrm{d}s}{v(s)} \right) \right\} \qquad \forall\, 0 < R < r, \tag{2.4}$$

Thus letting $r \to +\infty$, we obtain

$$\sqrt{\chi(r)} \notin L^{1}(+\infty) \tag{2.5}$$

It is worth to stress that the function χ only depends on the weight v, not on A. Note that, although (CP) can be thought as a version of (2.1) with $v \equiv 1$, assumptions (V1), (V$_{\mathrm{L1}}$) are not satisfied. Thus, the next main Theorem 4 below cannot be directly applied to (CP).

The study of the Cauchy problem (2.1) turns out to be extremely useful in a number of different geometric problems, not only those described in this paper. For instance, a mainstream application of it is to derive spectral estimates for stationary Schrödinger tipe operators via radialization techniques. In this case, the role of v is played by the volume growth of geodesic spheres centered at o, for which (V1) is the highest regularity that we can in general guarantee. However, since there are natural upper and lower bounds coming from the Laplacian comparison theorems, it is worth to relate the critical curve with that of, say, an upper bound for v. More precisely, for f satisfying

$$f \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^{+}_{0}), \qquad \frac{1}{f} \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^{+}), \qquad 0 \leq v \leq f \quad \text{on } \mathbb{R}^{+}_{0} \tag{F1}$$

$$\frac{1}{f} \in L^{1}(+\infty) \tag{F$_{\mathrm{L1}}$}$$

we shall compare $\chi(r)$ with the critical curve $\chi_{f}(r)$ defined again via (2.3). We observe that, for any positive constant c, $\chi_{cf} = \chi_{f}$. This suggests that, in general, $v \leq f$ does not imply $\chi \leq \chi_{f}$. To recover this property we need a more stringent relation between v and f.

Proposition 1 (Proposition 4.13 of [2]). *Let v, f satisfy* (V1), (V$_{\mathrm{L1}}$) *on some interval $I = (r_{0}, +\infty) \subset \mathbb{R}^{+}$. Then,*

(i) If v/f is non-increasing on I, $\chi(r) \leq \chi_f(r)$ on I;

(ii) If v/f is non-decreasing on I, $\chi(r) \geq \chi_f(r)$ on I;

In the case $v(r) = \mathrm{vol}(\partial B_r)$, the above proposition fits well with the Bishop-Gromov comparison theorem for volumes ([15], Theorem 2.14). The interested reader may consult Chapter 4 of [2], where the authors give a detailed discussion on the critical curve, together with estimates on χ when $v(r) = \mathrm{vol}(\partial B_r)$, explicit examples, and many applications. For instance, the deep relationship between $\chi(r)$ and optimal weights for Hardy inequalities is discussed. Since, as we will see, in dealing with Calabi-type compactness results the role of v will be played by some suitable weight which has no direct relation with volumes, we shall not pursue this line of argument any further.

We now list the assumptions under which we will treat either of the Cauchy problems (2.1) or (2.2).

$$v(r)\int_r^a \frac{\mathrm{d}s}{v(s)}; \qquad \frac{1}{v(r)}\int_o^r v(s)\mathrm{d}s \ \in L^\infty([0,a]) \tag{V2}$$

for some $a \in \mathbb{R}^+$.

$$\frac{1}{v(r)}\int_0^r v(s)\mathrm{d}s = o(1) \qquad \text{as} \quad r \to 0^+ \tag{V3}$$

$$A(r) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+_0) \tag{A1}$$

Conditions

1. (A1), (V1), (V2) and (V3) guarantee the existence of a solution $z \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^+_0)$ of (2.1)
2. (A1), (V1) and (V2) its uniqueness
3. (A1), (V1) the fact that each solution $z \not\equiv 0$ has isolated zeros, if any.

Note that (V2) and (V3) are automatically satisfied if $v(r)$ is non-decreasing in a neighbourhood of 0.

The following theorem summarizes some of the results obtained in [3].

Theorem 4. *Let* (A1), (V1), (F1), ($\mathrm{V_{L1}}$) *be met, and let $z \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^+_0)$ be a solution of*

$$\begin{cases} (v(r)z'(r))' + A(r)v(r)z(r) = 0 & \text{on } \ \mathbb{R}^+, \\ z(0^+) = z_0 > 0. \end{cases} \tag{2.6}$$

Then,

*(1) [**Theorem 5.2 of [3]**] If $A(r) \leq \chi(r)$ on $\mathbb{R}^+$, then $z > 0$ on $\mathbb{R}^+$. Furthermore, there exists $r_1 > 0$ and a constant $C = C(r_1) > 0$ such that*

$$z(r) \geq -C\sqrt{\int_r^{+\infty} \frac{\mathrm{d}s}{f(s)}\,\log \int_r^{+\infty} \frac{\mathrm{d}s}{f(s)}} \qquad \text{on } [r_1, +\infty). \tag{2.7}$$

(2) *[**Corollary 5.4 of [3]**] If $A(r) \leq \chi(r)$ on $[r_0, +\infty)$, for some $r_0 > 0$, then z is nonoscillatory, that is, it has only finitely many zeroes (if any).*

(3) *[**Corollary 6.3 of [3]**] If $A \geq 0$ on $\mathbb{R}^+$, $A \not\equiv 0$ and there exist $r > R > 0$ such that $A \not\equiv 0$ on $[0, R]$ and*

$$\int_R^r \left(\sqrt{A(s)} - \sqrt{\chi_f(s)} \right) \mathrm{d}s > -\frac{1}{2} \left(\log \int_0^R A(s)v(s)\mathrm{d}s + \log \int_R^{+\infty} \frac{\mathrm{d}s}{f(s)} \right) \tag{2.8}$$

then z has a first zero. Moreover, this is attained on $(0, \overline{R}]$, where $\overline{R} > 0$ is the unique real number satisfying

$$\int_R^r \sqrt{A(s)}\mathrm{d}s = -\frac{1}{2} \log \int_0^R A(s)v(s)\mathrm{d}s - \frac{1}{2} \log \int_r^{\overline{R}} \frac{\mathrm{d}s}{f(s)} \tag{2.9}$$

(4) *[**Theorem 6.6 of [3]**] If $A \geq 0$ on $\mathbb{R}^+$ and, for some (hence any) $R > 0$ such that $A \not\equiv 0$ on $[0, R]$,*

$$\limsup_{r \to +\infty} \int_R^r \left(\sqrt{A(s)} - \sqrt{\chi_f(s)} \right) \mathrm{d}s = +\infty \tag{2.10}$$

then z is oscillatory, that is, it has infinitely many zeroes.

Remark 2. In fact, for (2) and (4) to hold, it is enough that z solves the Cauchy problem only on $[r_0, +\infty)$, for some $r_0 > 0$ and for some initial condition $z(r_0)$, $(vz')(r_0)$.

It is worth to make some observations on the conditions in the above theorem.

- In (1), $A \leq \chi$ cannot be replaced with $A \leq \chi_f$. The reason is that, as already observed, no relations between χ and χ_f can be deduced from the sole requirement $v \leq f$ in (F1). However, note that χ_f appears both in (2.8) and in (2.10). This is due to the technique developed for (3) and (4), which is different from that used for (1) and (2).
- The lower bound (2.7) is sharp. Indeed, it can be showed that if z is positive on $\mathbb{R}^+$ and $A \geq \chi$ on some $[r_0, +\infty)$, then necessarily z is bounded from above by the quantity on the RHS of (2.7), for some $C > 0$.
- The right hand side of (2.8) is independent both of r and of the behavior of A after R. Therefore, the left hand side of (2.8) represents how much must A exceed a critical curve modelled on f in the compact region $[R, r]$ in order to have a first zero for z, and it only depends on the behavior of A and f before R (the first addendum of the RHS), and on the growth of f after R. This is conceptually simpler than Calabi compactness condition, where the role of a, b is balanced between the two sides of (1.18).

Remark 3. The assumptions in (3) and (4) can be weakened. Indeed, it is enough that z solves the inequality $(vz')' + Avz \leq 0$ on $\mathbb{R}^+$, and that its initial condition satisfies

$$\frac{vz'}{z}(0^+) = 0.$$

Note that sufficiently mild singularities of z as $r \to 0^+$ are allowed, depending on the order of zero of $v(r)$ at 0.

Remark 4. Using (2.4) we see that (2.10) can be equivalently expressed as

$$\limsup_{r\to+\infty} \left\{ \int_R^r \sqrt{A(s)} + \frac{1}{2} \log \int_r^{+\infty} \frac{\mathrm{d}s}{f(s)} \right\} = +\infty. \tag{2.11}$$

The similarity between (2.11) and (1.19) is evident. Indeed, as a first application of Theorem 4 let us show that Calabi condition (1.19) implies that the solution of (CP), with

$$K(t) = \frac{\mathrm{Ricc}(\dot\gamma, \dot\gamma)}{m-1}(t) \geq 0, \tag{2.12}$$

is oscillatory.

Indeed, choose any v satisfying (V1), (V2), (V3) and $v^{-1} \in L^1(+\infty) \setminus L^1(0^+)$, for instance $v(r) = r^{m-1}$ for some $m \geq 3$. Let $r = r(t)$ be the inverse function of

$$t(r) = \left(\int_r^{+\infty} \frac{\mathrm{d}s}{v(s)} \right)^{-1} \tag{2.13}$$

and define

$$z(r) = \frac{g(t(r))}{t(r)} \tag{2.14}$$

Then z solves

$$\begin{cases} (vz')' + \dfrac{K(t(r))t^4(r)}{v^2(r)} v(r) z = 0 \qquad \text{on } \mathbb{R}^+ \\ z(0) = 1 \qquad (vz')(0) = 0 \end{cases} \tag{2.15}$$

where now differentiation is with respect to the variable r. If (2.11) holds with $f = v$ and

$$A(r) = \frac{K(t(r))t^4(r)}{v^2(r)} \geq 0,$$

then z oscillates and so does g. A change of variables shows that (2.11) is exactly (1.19).

The literature on the qualitative properties of solutions of (CP) is enormous, and considerable steps towards the comprehension of the matter have been made throughout all of the 20^{th} century. In particular, a number of sharp oscillatory and nonoscillatory conditions for g have been found. Here, we only quote two of the finest. The first is the so-called Hille-Nehari criterion, see [17], p.45 and [9], Theorem 5 and Corollary 1.

Theorem 5. *Let $K \in C^0(\mathbb{R}) \cap L^1(+\infty)$ be non-negative, and consider a solution g of $g'' + Kg = 0$. Denote with $k(t)$, k_* and k^* respectively the quantities*

$$k(t) = t\int_t^{+\infty} K(s)\mathrm{d}s, \qquad k_* = \liminf_{t\to+\infty} k(t), \qquad k^* = \limsup_{t\to+\infty} k(t).$$

We have:

- *if g is nonoscillatory, then necessarily $k_* \leq 1/4$ and $k^* \leq 1$;*
- *if $k(t) \leq 1/4$ for t large enough, in particular if $k^* < 1/4$, then g is nonoscillatory.*

As a consequence, $k_* > 1/4$ is a sufficient condition for g to be oscillatory.

Remark 5. If $K \notin L^1(+\infty)$, the result applies with $k_* = k^* = +\infty$, and g is thus oscillatory. This case is due to W.B. Fite [6].

Remark 6. Improving on an old criterion of Kneser, it can be showed (see [2], Proposition 2.23) that if $k(t) \leq 1/4$ on the whole $\mathbb{R}^+$, then the solution g of (CP) is positive and increasing on $\mathbb{R}^+$.

Remark 7. Hille-Nehari criterion detects the oscillation of g when $K(t) \geq B^2/(1+t^2)$ on $\mathbb{R}^+$, for some $B > 1/2$. In a geometrical context, this particular case has been investigated in [5], where the authors have also obtained upper bounds for the first zero of g solving (CP).

Remark 8. For every $B \in [0, 1/2]$, the Cauchy problem associated to the Euler equation

$$\begin{cases} g'' + \dfrac{B^2}{(1+t)^2} g = 0, \\ \\ g(0) = 0, \quad g'(0) = 1, \end{cases}$$

has the explicit, positive solution

$$g(s) = \begin{cases} \sqrt{1+t}\log(1+t) & \text{if } \ B = 1/2; \\ \\ \dfrac{1}{\sqrt{1-4B^2}}\left((1+t)^{B''} - (1+t)^{1-B''}\right) & \text{if } \ B \in [0, 1/2), \end{cases}$$

where

$$B'' = \frac{1+\sqrt{1-4B^2}}{2} \in (1/2, 1]$$

(see [17], p.45). For $B = 1/2$, this example shows that Hille-Nehari criterion is sharp.

When $k_* = k^* = 1/4$, Hille-Nehari criterion cannot grasp the behaviour of g. As we shall see, combining (2) and (4) of Theorem 4 in an iterative way, we can construct sharper and sharper oscillation and nonoscillation criteria that can detect the behaviour of g even in some cases when the Hille-Nehari theorem fails to give information.

The second result we quote allows sign-changing potentials K and is due to R. Moore (see [13], Theorem 2)

Theorem 6. *Let $K \in C^0(\mathbb{R})$. Each solution g of $g'' + Kg = 0$ is oscillatory provided that, for some $\lambda \in [0,1)$, there exists*

$$\lim_{t\to+\infty} \int_0^t s^\lambda K(s)\mathrm{d}s = +\infty, \tag{2.16}$$

Remark 9. Setting $\lambda = 0$ in Moore statement we recover a result of W. Ambrose [1] and A. Wintner [18] (one can also consult [8], Corollaries 3.5 and 3.6 for a different proof and a generalization). Remark 8 shows that in Moore result the interval of the parameter λ cannot be extended to $[0,1]$. Thus, Euler equation suggests that, when restricted to the case $K \geq 0$, Moore criterion is somehow weaker than that of Hille-Nehari.

Another observation on Moore result is that, although sharp from many points of view, it requires that the negative part of K be, loosely speaking, globally smaller than the positive part. This is the essence of the existence of the limit in (2.16). One of our goal in the next section will be to obtain an oscillation criterion that allows K to have a relevant negative part. Furthermore, with the aid of (2.8), we will also find a condition in finite form for the existence of a first zero that allows K to be negative somewhere. As far as we know, there is still no result in this direction besides some very recent work of P. Mastrolia, G. Veronelli and M. Rimoldi, which we recall here for the sake of completeness.

Theorem 7 (Theorem 5 of [12]). *Suppose that $K \in L^\infty(\mathbb{R}_0^+)$ satisfies $K \geq -B^2$, for some $B \geq 0$, and let g be a solution of* (CP). *Suppose that there exist $0 < a < b$ and $\lambda \neq 1$ for which either*

$$\int_a^b sK_\gamma(s)\mathrm{d}s > B\left\{b + a\frac{e^{2Ba}+1}{e^{2Ba}-1}\right\} + \frac{1}{4}\log\left(\frac{b}{a}\right) \tag{2.17}$$

or

$$\int_a^b s^\lambda K_\gamma(s)\mathrm{d}s > B\left\{b^\lambda + a^\lambda\frac{e^{2Ba}+1}{e^{2Ba}-1}\right\} + \frac{\lambda^2}{4(1-\lambda)}\left\{a^{\lambda-1} - b^{\lambda-1}\right\} \tag{2.18}$$

holds (if $B = 0$, this has to be intended in a limit sense). Then, g has a first zero.

Remark 10. The case $B = 0$ of the above result is due to Z. Nehari, see [14], p.432 (8), with an entirely different proof. We point out that, in [12], the authors also give an upper bound for the position of the first zero.

3 Extensions of Calabi compactness criterion

We shall now deal with (2.1) under the further assumption that A is possibly negative. Hereafter, we require the validity of (A1), (V1), (V2), (V3), (F1) . Let $z \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_0^+)$

be a solution of

$$\begin{cases} (vz')' + Avz = 0 & \text{on } \mathbb{R}^+, \\ z(0^+) = z_0 > 0, \end{cases} \tag{3.1}$$

or of the analogous problem on $[r_0, +\infty)$.

Choose a function $W \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+_0)$ such that

$$W \geq 0 \quad \text{a.e. on } \mathbb{R}^+, \qquad W + A \geq 0 \quad \text{a.e. on } \mathbb{R}^+. \tag{3.2}$$

For instance, W can be taken to be the negative part of A. To apply the results of the previous section, we need to produce, starting from (3.1) and W, a solution $\widetilde{z}$ of a linear ODE of the type $(\bar{v}\widetilde{z}')' + \bar{A}\bar{v}\widetilde{z} = 0$, for some new volume function $\bar{v}$ and some $\bar{A} \geq 0$. Towards this purpose, consider a solution $w(r) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^+_0)$ of

$$\begin{cases} (vw')' - Wvw \geq 0 & \text{on } \mathbb{R}^+ \\ w(0^+) = w_0 > 0. \end{cases} \tag{3.3}$$

Note that from

$$(vw')' \geq Wvw$$

we deduce $w' \geq 0$ a.e., hence w has a positive essential infimum on $\mathbb{R}^+_0$. Therefore, the function $\widetilde{z} = z/w$ is well defined on $\mathbb{R}^+_0$ and solves

$$\begin{cases} \big([vw^2]\widetilde{z}'\big)' + \big(A + W\big)[vw^2]\widetilde{z} \leq 0 & \text{on } \mathbb{R}^+ \\ \widetilde{z}(0) = z_0/w_0 > 0, \end{cases} \tag{3.4}$$

As observed in Remark 3, the inequality sign in (3.4) is irrelevant for the proofs of (3), (4) of Theorem 4. In this way, (3) and (4) can be extended to cover sign-changing potentials by simply replacing A with $A + W$, v with vw^2 and f with fw^2. The main problem therefore shifts to the search of explicit solutions w of (3.3), once v and W are given.

Up to taking some care when dealing with the initial condition, the same procedure can be carried on even when $v \equiv 1$. In this case, we are able to provide an explicit form for w when the potential W is a polynomial. This leads to the following theorem (see Theorem 6.41 of [2]). In the statement below, we denote with I_ν is the positive Bessel function of order ν.

Theorem 8 (**Compactness with sign-changing curvature**). *Let $(M, \langle\,,\,\rangle)$ be a complete m-dimensional Riemannian manifold. For each unit speed geodesic γ emanating from a fixed origin o, define*

$$K_\gamma(t) = \frac{\mathrm{Ricc}(\gamma', \gamma')(t)}{m-1}.$$

Assume that one of the following set of assumptions is met.

(*i*) *The function* $K_\gamma(t)$ *satisfies*

$$K_\gamma(t) \geq -B^2\big(1+t^2\big)^{\alpha/2} \qquad \text{on } \ \mathbb{R}^+,$$

for some $B > 0$ *and* $\alpha \geq -2$ *possibly depending on* γ. *Having set*

$$0 \leq A_\gamma(t) = K_\gamma(t) + B^2\big(1+t^2\big)^{\alpha/2},$$

suppose also that, for some $0 < S < t$ *such that* $A_\gamma \not\equiv 0$ *on* $[0,S]$,

$$\begin{aligned} &\int_S^t \left(\sqrt{A_\gamma(\sigma)} - \sqrt{\chi_{w^2}(\sigma)}\right) \mathrm{d}\sigma \\ &\qquad > -\frac{1}{2}\left(\log \int_0^S A_\gamma(\sigma) w^2(\sigma) \mathrm{d}\sigma + \log \int_S^{+\infty} \frac{\mathrm{d}\sigma}{w^2(\sigma)}\right), \end{aligned} \tag{3.5}$$

where

$$w(t) = \begin{cases} \sinh\left(\dfrac{2B}{2+\alpha}\left[(1+t)^{1+\frac{\alpha}{2}} - 1\right]\right) & \text{if } \ \alpha \geq 0; \\ t^{1/2} I_{\frac{1}{2+\alpha}}\left(\dfrac{2B}{2+\alpha} t^{1+\frac{\alpha}{2}}\right) & \text{if } \ \alpha \in (-2,0); \\ t^{B'} & \text{if } \ \alpha = -2, \end{cases} \tag{3.6}$$

and $B' = (1+\sqrt{1+4B^2})/2$.

(*ii*) *The function* $K_\gamma(t)$ *satisfies*

$$K_\gamma(t) \geq \frac{B^2}{(1+t)^2} \qquad \text{on } \ \mathbb{R}^+,$$

for some $B \in [0, 1/2]$ *possibly depending on* γ. *Having set*

$$0 \leq A_\gamma(t) = K_\gamma(t) - \frac{B^2}{(1+t)^2},$$

suppose also that, for some $0 < S < t$ *such that* $A_\gamma \not\equiv 0$ *on* $[0,S]$, *inequality* (3.5) *holds with*

$$w(t) = \begin{cases} (1+t)^{B''} - (1+t)^{1-B''} & \text{if } \ B \in [0, 1/2); \\ \sqrt{1+t}\log(1+t) & \text{if } \ B = 1/2, \end{cases} \tag{3.7}$$

and $B'' = (1+\sqrt{1-4B^2})/2$.

Then, M *is compact and has finite fundamental group.*

Remark 11. Note that, both for (3.6) and for (3.7), the critical curve related to w^2 exists since $1/w^2 \in L^1(+\infty)$.

Proof. By Theorem 2, it is enough to prove that, for every γ issuing from o, the solution g of

$$\begin{cases} g'' + K_\gamma(t)g = 0 \\ g(0) = 0, \quad g'(0) = 1 \end{cases} \tag{3.8}$$

has a first zero.

(i) A straightforward computation shows that the function w in (3.6) is a positive solution of

$$w'' - B^2(1+t^2)^{\alpha/2}w \geq 0 \qquad \text{on } \mathbb{R}^+$$

whose initial condition, in the cases $\alpha \in (-2, 0)$ and $\alpha \geq 0$, is

$$w(0) = 0, \qquad w'(0) = C > 0. \tag{3.9}$$

Consider $\widetilde{z} = g/w$. Then, $\widetilde{z}$ solves

$$(w^2\widetilde{z}')' + A_\gamma w^2 \widetilde{z} \leq 0 \qquad \text{on } \mathbb{R}^+. \tag{3.10}$$

In order to apply (3) of Theorem 4 to the differential inequality (3.10), we shall make use of Remark 3. From (3.9), in each of the cases of (3.6) we obtain

$$\frac{w^2\widetilde{z}'}{\widetilde{z}}(0^+) = \left(w^2\frac{g'}{g} - ww'\right)(0^+) = 0. \tag{3.11}$$

We can thus apply (3) of Theorem 4, and (3.5) implies that $\widetilde{z}$ (hence g) has a first zero. Case (ii) is analogous. Indeed, by Remark 8, w in (3.7) is a solution of the Cauchy problem

$$\begin{cases} w'' + \dfrac{B^2}{(1+t)^2}w = 0 \\ g(0) = 0, \quad g'(0) = C > 0. \end{cases}$$

□

Remark 12. We recall that, by (2.4), inequality (3.5) is equivalent to the somehow simpler

$$\int_S^t \sqrt{A_\gamma(\sigma)}\mathrm{d}\sigma > -\frac{1}{2}\left(\log\int_0^S A_\gamma(\sigma)w^2(\sigma)\mathrm{d}\sigma + \log\int_t^{+\infty}\frac{\mathrm{d}\sigma}{w^2(\sigma)}\right). \tag{3.12}$$

However, (3.5) put in evidence that the RHS does not depend on t, as opposed to conditions like (1.18) and (2.18) where both a and b appear in the LHS as well as in the RHS. Furthermore, although somehow complicated, (3.5) is entirely explicit once we are able to compute the critical curve related to w^2. In general, this can only be done numerically, but in some cases a closed expression can be given. For instance, this

is so for $m = 3$, $B = 1/2$ in (3.7), for $B = 0$ in (3.7) and for $\alpha = 0, -2$ in (3.6):

$$\int_t^{+\infty} \frac{\mathrm{d}\sigma}{w^2(\sigma)} = \begin{cases} \dfrac{t^{-\sqrt{1+4B^2}}}{\sqrt{1+4B^2}} & \text{for (3.6), } \alpha = -2 \text{ and for } B = 0; \\ B^{-1}\big[\coth(Bt) - 1\big] & \text{for (3.6), } \alpha = 0; \\ \dfrac{1}{\log(1+t)} & \text{for (3.7), } B = 1/2,\ m = 3. \end{cases}$$

Therefore, in the case $B = 0$, (3.12) reads

$$\int_S^t \sqrt{K_\gamma(\sigma)}\mathrm{d}\sigma > -\frac{1}{2}\left(\log \int_0^S \sigma^2 K_\gamma(\sigma)\mathrm{d}\sigma - \log t\right),$$

that should be compared to (1.18), while, for $\alpha = 0$, (3.12) becomes

$$\int_S^t \sqrt{K_\gamma(\sigma) + B^2}\mathrm{d}\sigma > -\frac{1}{2}\left(\log \int_0^S K_\gamma(\sigma)\sinh^2(B\sigma)\mathrm{d}\sigma + \log \frac{\coth(Bt) - 1}{B}\right),$$

that should be compared to (2.17) and (2.18).

Easier expressions can be obtained when considering oscillatory conditions. We state the result in analytic form.

Theorem 9 (**Generalized Calabi criterion**). *Let $K \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+_0)$, and let $g \not\equiv 0$ be a solution of $g'' + Kg = 0$. Then, g oscillates in each of the following cases:*

(1) *K satisfies*

$$K(t) \geq -B^2 t^\alpha \qquad \text{when } \ t > t_0, \tag{3.13}$$

for some $B > 0$, $\alpha \geq -2$ and $t_0 > 0$, and the following conditions hold:

$$\begin{aligned} &\text{for } \ \alpha = -2, \quad \limsup_{t\to+\infty} \left(\int_{t_0}^t \sqrt{K(\sigma) + \frac{B^2}{\sigma^2}}\mathrm{d}\sigma - \frac{\sqrt{1+4B^2}}{2}\log t\right) = +\infty; \\ &\text{for } \ \alpha > -2, \quad \limsup_{t\to+\infty} \left(\int_{t_0}^t \sqrt{K(\sigma) + B^2\sigma^\alpha}\mathrm{d}\sigma - \frac{2B}{\alpha+2}t^{\frac{\alpha}{2}+1}\right) = +\infty. \end{aligned} \tag{3.14}$$

(2) *K satisfies*

$$K(t) \geq \frac{B^2}{t^2} \qquad \text{when } \ t > t_0, \tag{3.15}$$

for some $B \in [0, 1/2]$, $t_0 > 0$, and the following conditions hold:

$$\begin{aligned} &\text{for } \ B < \tfrac{1}{2}, \quad \limsup_{t\to+\infty} \left(\int_{t_0}^t \sqrt{K(\sigma) - \frac{B^2}{\sigma^2}}\mathrm{d}\sigma - \frac{\sqrt{1-4B^2}}{2}\log t\right) = +\infty; \\ &\text{for } \ B = \tfrac{1}{2}, \quad \limsup_{t\to+\infty} \left(\int_{t_0}^t \sqrt{K(\sigma) - \frac{1}{4\sigma^2}}\mathrm{d}\sigma - \frac{1}{2}\log\log t\right) = +\infty; \end{aligned} \tag{3.16}$$

Proof. (1). The equation $w'' - B^2 t^\alpha w = 0$ on, say, $[1, +\infty)$ has the particular positive solution

$$\begin{aligned} w(t) &= \sqrt{t} I_{\frac{1}{2+\alpha}}\left(\frac{2B}{2+\alpha} t^{1+\frac{\alpha}{2}}\right) && \text{if } \alpha > -2; \\ w(t) &= t^{B'}, \quad B' = \frac{1+\sqrt{1+4B^2}}{2} && \text{if } \alpha = -2, \end{aligned} \tag{3.17}$$

where $I_\nu(t)$ is the Bessel function of order ν. From

$$I_\nu(t) = \frac{e^t}{\sqrt{2\pi t}}(1 + o(1)) \qquad \text{as } t \to +\infty$$

(see [11], p. 102), in both cases $\alpha = -2$ and $\alpha > -2$ we deduce that $1/w^2 \in L^1(+\infty)$. Moreover,

$$\int_t^{+\infty} \frac{\mathrm{d}\sigma}{w^2(\sigma)} \sim \begin{cases} C \exp\left(-\frac{4B}{2+\alpha} t^{1+\frac{\alpha}{2}}\right) & \text{if } \alpha > -2; \\ C t^{1-2B'} = C t^{-\sqrt{1+4B^2}} & \text{if } \alpha = -2. \end{cases} \tag{3.18}$$

Since the function $\widetilde{z} = g/w$ solves

$$(w^2 \widetilde{z}')' + (K + B^2 t^\alpha) w^2 \widetilde{z} \le 0 \qquad \text{on } [1, +\infty),$$

by (4) of Theorem 4, z (and hence g) oscillates provided

$$\limsup_{t \to +\infty} \int_{t_0}^t \left(\sqrt{K(\sigma) + B^2 \sigma^\alpha} - \sqrt{\chi_{w^2}(\sigma)}\right) \mathrm{d}\sigma = +\infty$$

which, by Remark 4, is equivalent to

$$\limsup_{t \to +\infty} \int_{t_0}^t \sqrt{K(\sigma) + B^2 \sigma^\alpha} \mathrm{d}\sigma + \frac{1}{2} \log \int_t^{+\infty} \frac{\mathrm{d}\sigma}{w^2(\sigma)} = +\infty \tag{3.19}$$

By (3.18), conditions (3.14) and (3.19) are equivalent, thus the conclusion.
(2). The proof is the same. Indeed, it is enough to consider the following positive solution w of $w'' + B^2 t^{-2} w = 0$:

$$\begin{aligned} w(t) &= t^{B''}, \quad B'' = \frac{1+\sqrt{1-4B^2}}{2} && \text{if } B \in [0, 1/2); \\ w(t) &= \sqrt{t} \log t && \text{if } B = 1/2. \end{aligned} \tag{3.20}$$

Again, in both cases $1/w^2 \in L^1(+\infty)$. □

Remark 13. Note that, for $B = 0$, we recover another proof of the original Calabi oscillation criterion, which is different from that described in the previous section.

Polynomial lower bounds for K are clearly chosen for their simplicity. Indeed, the statement in its full generality only requires a positive solution w of $w'' + Ww \ge 0$,

where the weight W has only to satisfy $K+W \geq 0$. In this way, arbitrary lower bounds for K are allowed, and up to finding a suitable positive w the oscillatory conditions are explicit. This improves on Moore oscillation criterion, where the existence of the limit in (2.16) is essential for the proof of Theorem 6 to work. The same discussion holds for Theorem 8, up to the further requirement that w is sufficiently well-behaved as $t \to 0^+$. From this perspective, Theorem 8 improves on Theorem 7, whose proof seems to us to be hardly generalizable when the lower bound for K is nonconstant.

The procedure described above, which loosely speaking allows to translate the potential up to inserting a weight, can be iterated. In this way, we can obtain finer and finer criteria in a very simple way. We now describe how to proceed in this direction. The first example is the following

Theorem 10 (**Positivity and nonoscillation criteria**). *Let $K \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+_0)$.*

(1) *Suppose that*

$$K(t) \leq \frac{1}{4(1+t)^2}\left[1+\frac{1}{\log^2(1+t)}\right] \qquad \text{on } \mathbb{R}^+. \tag{3.21}$$

Then, every solution g of

$$\begin{cases} g'' + K(t)g \geq 0 \\ g(0) = 0, \quad g'(0) = 1 \end{cases} \tag{3.22}$$

is positive on $\mathbb{R}^+$ and satisfies $g(t) \geq C\sqrt{t\log t}\log\log t$, for some $C > 0$ and for $t > 3$.

(2) *Suppose that*

$$K(t) \leq \frac{1}{4t^2}\left[1+\frac{1}{\log^2 t}\right] \qquad \text{on } [t_0, +\infty), \tag{3.23}$$

for some $t_0 > 0$. Then, every solution g of $g'' + Kg = 0$ is nonoscillatory.

Proof. (1). By Sturm argument, it is sufficient to prove the desired conclusion under the additional assumptions that g satisfies (3.22) with the equality sign, and that

$$K(t) \geq \frac{1}{4(1+t)^2}.$$

Let $w(t) = \sqrt{1+t}\log(1+t)$ be the solution of (3.22) with the equality sign and with $K(t) = [4(1+t)^2]^{-1}$. Then, $\widetilde{z} = g/w$ solves

$$\begin{cases} (w^2\widetilde{z}')' + \left[K(s) - \dfrac{1}{4(1+t)^2}\right] w^2\widetilde{z} = 0 \quad \text{on } \mathbb{R}^+ \\ \widetilde{z}(0) = 1, \qquad \widetilde{z}'(0) = 0. \end{cases} \tag{3.24}$$

Applying (1) of Theorem 4, $\widetilde{z}$ is positive provided

$$K(t) - \frac{1}{4(1+t)^2} \leq \chi_{w^2}(t) = \frac{1}{4(1+t)^2 \log^2(1+t)},$$

which is (3.21), and $\widetilde{z}$ satisfies

$$\widetilde{z}(t) \geq -C\sqrt{\int_t^{+\infty} \frac{\mathrm{d}\sigma}{w^2(\sigma)}} \log \int_t^{+\infty} \frac{\mathrm{d}\sigma}{w^2(\sigma)} = C\frac{\log\log t}{\sqrt{\log t}},$$

for some $C > 0$. The lower bound for g follows at once by the definition of $\widetilde{z}$.
To prove (2), again by Sturm argument we can assume that the inequality $K(t) \geq 1/[4t^2]$ holds. Proceeding along the same lines as for (1) with the choice $w = \sqrt{t}\log t$, and using (2) of Theorem 4, we reach the desired conclusion. □

The next prototype case illustrates the sharpness of our criteria. Let

$$K(t) = \frac{1}{4t^2} + \frac{c^2}{4t^2 \log^2 t}, \qquad \text{on } [2, +\infty),$$

where $c > 0$ is a constant. Then, applying (2) of Theorem 9, case $B = 1/2$ we deduce that g oscillates whenever $c > 1$. On the other hand, if $c \leq 1$, by Theorem 10 g is nonoscillatory. However, on $[2, +\infty)$

$$\frac{1}{4} < k(t) = t\int_t^{+\infty} K(\sigma)\mathrm{d}\sigma \leq \frac{1}{4} + t\frac{c^2}{4t}\int_t^{+\infty} \frac{\mathrm{d}\sigma}{\sigma \log^2 \sigma} = \frac{1}{4} + \frac{c^2}{4\log t},$$

hence the Hille-Nehari criterion cannot detect neither the oscillatory nor the nonoscillatory behaviour of g depending on c. Similarly, also Moore criterion is not sharp enough. The proof of Theorem 10 suggests an iterative improving procedure. In the general case, suppose that we are given an ordinary differential equation of the type $(vz')' + Avz = 0$, with v such that χ can be defined. By Sturm argument, there is no loss of generality if we assume that $A \geq \chi$. An explicit solution w of

$$(vw')' + \chi vw = 0$$

is given by

$$w(t) = -\sqrt{\int_t^{+\infty} \frac{\mathrm{d}s}{v(s)}} \log \int_t^{+\infty} \frac{\mathrm{d}s}{v(s)},$$

and it is positive on some intervall $[r_0, +\infty)$. Then, $\widetilde{z} = z/w$ solves

$$(\bar{v}\widetilde{z}')' + (A - \chi)\bar{v}\widetilde{z} = 0 \qquad \text{on } [r_0, +\infty),$$

where $\bar{v} = vw^2$, which implies that $\widetilde{z}$, and therefore z, are nonoscillatory if $(vw^2)^{-1} \in L^1(+\infty)$ and

$$A(r) - \chi(r) \leq \chi_{vw^2}(r),$$

and oscillatory if $(vw^2)^{-1} \in L^1(+\infty)$ and

$$\limsup_{t\to+\infty} \int_{t_0}^{t} \left(\sqrt{A(s)-\chi(s)} - \sqrt{\chi_{vw^2}(s)}\right) \mathrm{d}s = +\infty.$$

Now, the procedure can be pushed a step further by considering $\widetilde{z}$. This enables us to construct finer and finer critical curves. As an example, we refine Theorem 10. Suppose that

$$K(t) \geq \frac{1}{4t^2} + \frac{1}{4t^2 \log^2 t}$$

on, say, $[2, +\infty)$. Then, as in the proof of Theorem 10, define $w = \sqrt{t}\log t$ and $v = w^2 = t\log^2 t$. Since w is a positive solution of $w'' + (4t^2)^{-1}w = 0$ on some $[r_1, +\infty)$, $z = g/w$ is well defined and solves $(vz')' + Avz = 0$ on $[r_1, +\infty)$, where

$$A(t) = K(t) - \frac{1}{4t^2} \geq \frac{1}{4t^2\log^2 t} = \chi_{w^2}(t) = \chi(t).$$

Now, the function

$$w_2(t) = -\sqrt{\int_t^{+\infty} \frac{\mathrm{d}s}{v(s)}} \log \int_t^{+\infty} \frac{\mathrm{d}s}{v(s)} = \frac{\log\log t}{\sqrt{\log t}}$$

is a solution of $(vw_2')' + \chi v w_2 = 0$, positive after some $r_2 \geq r_1$. Setting

$$v_2(t) = v(t)w_2(t)^2 = t\log t \log^2 \log t,$$

then

$$\frac{1}{v_2(t)} \in L^1(+\infty),$$

and the function $z_2 = z/w_2$ is a solution of $(v_2 z_2')' + A_2 v_2 z_2 = 0$ on $[r_2, +\infty)$, where

$$A_2(t) = A(t) - \chi(t) = K(t) - \frac{1}{4t^2} - \frac{1}{4t^2\log^2 t} \geq 0.$$

Thus z_2, and hence z and g, is nonoscillatory provided

$$A_2(t) \leq \chi_{v_2}(t), \qquad \text{that is,} \qquad K(t) \leq \frac{1}{4t^2} + \frac{1}{4t^2\log^2 t} + \frac{1}{4t^2\log^2 t \log^2\log t},$$

and, by (2.11), it is oscillatory if

$$\limsup_{t\to+\infty} \left(\int_{t_2}^{t} \sqrt{K(\sigma) - \frac{1}{4\sigma^2} - \frac{1}{4\sigma^2\log^2\sigma}}\,\mathrm{d}\sigma - \frac{1}{2}\log\log\log t \right) = +\infty.$$

Remark 14. We mention that, with the aid of the change of variables (2.13) and (2.14), Theorems 8, 9 and 10 can be applied to get sharp extensions of index estimates for stationary Schrödinger operators on $\mathbb{R}^m$, $m \geq 3$, that highly improve on classical results of M. Reed and B. Simon [16], and W. Kirsch and B. Simon [10]. The interested reader can consult [2], Theorem 6.50.

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