Shadow boundaries in space forms

Oscar Palmas(iD)   Gabriel Ruiz-Hernández *

Dedicated to Professor Gervásio Colares on the occasion of his eightieth birthday.

Abstract

We study shadow boundaries of submanifolds of Riemannian manifolds admitting a closed conformal vector field. As applications we give a method to find a principal direction in a compact hypersurface and a characterization of totally umbilical hypersurfaces in space forms.

1 Introduction

Given a Riemannian manifold N, an immersed submanifold M and a vector field Y in N, the notion of the shadow boundary of M is a natural one: It is the set of points of M such that Y is tangent to M. This concept appears already in 1990, in an article [2] by J. Choe, under the name of *horizon* (see Definition 2.1 there) where he applied it to the study of minimal surfaces.

In [4], M. Ghomi investigated a very close concept, the *shadow*. Given a fied unit vector in $\mathbb{R}^3$, or equivalently, a parallel vector field v, the shadow of an orientable surface in $\mathbb{R}^3$ with respect to v is the set of points of the surface such that the sign of the angle between v and a given nowhere zero global normal vector field to the surface does not change. The relation between these concepts is that a shadow is an open set in the complement of a shadow boundary.

In a more general setting, we have to impose some conditions on the ambient space N and on the vector field Y in order to obtain some relevant geometrical information. For example, the paper [6] of the second named author contains some properties of shadow

boundaries of submanifolds with respect to a parallel vector field in a Riemannian manifold.

In this work we investigate shadow boundaries of hypersurfaces with respect to a closed conformal vector field in the ambient, a concept which includes as a particular case that of a parallel vector field. In fact, we show that space forms are the first examples of manifolds admitting plenty of closed conformal vector fields; see Proposition 3.4. Our work is mainly inspired in the paper [3] about compact shadow boundaries of a hypersurface in Euclidean space. Here we extend some of the results in [3] to more general spaces, in particular to any space form.

In general, a shadow boundary is a closed subset of a hypersurface M. In the case here considered using closed conformal vector fields to construct shadow boundaries, we give first in Proposition 3.1 a condition on the shape operator of the immersion $M \subset N$ for a shadow boundary to be a submanifold of M. As a particular case of this proposition, we obtain for a hypersurface with nowhere zero Gauss-Kronecker curvature that every shadow boundary is a submanifold of M.

In Proposition 4.1 we give a characterization of a principal direction of a hypersurface M in a space form via shadow boundaries. Namely, given a closed conformal vector field Y, the vector $Y_p \in T_p M$ dfines a principal direction of M if and only if the (regular) shadow boundary $S\partial(M, Y)$ is orthogonal to the vector field Y. A useful corollary of this result (see Corollary 3.8) says that a surface in a three dimensional manifold with the property that every shadow boundary is a line of curvature must be totally umbilical.

Our main result here is Theorem 4.5 and relates shadow boundaries to the geometry of a submanifold: Given a compact hypersurface with nowhere zero Gauss-Kronecker in a space form, if for each point of M and every direction Y_p there exists a corresponding shadow boundary making a constant angle with respect to Y, then M must be totally umbilical.

2   Preliminaries

In this section we fix our notation. Our ambient space $(N^{n+1}, \langle \, , \, \rangle)$ will be a Riemannian manifold with connection D.

Definition 2.1. Let M be an immersed submanifold of N, and let $Y : N \to TN$ be a vector field in N. The *shadow boundary* of M with respect to Y is the following subset of M:

$$S\partial(M, Y) = \{p \in M \mid Y_p \in T_p M\}. \tag{1}$$

In [2], J. Choe gave the above definition of shadow boundary of Riemannian submanifolds, calling it horizon. Using the generalized Morse index theorem, he related this concept with the index of stability of a complete minimal surface in $\mathbb{R}^3$.

In this paper we will work with Riemannian manifolds which admit a closed conformal vector field Y.

Definition 2.2. We say that Y is a closed conformal vector field if there exist $\varphi : N \to \mathbb{R}$ smooth such that for every vector field X in N we have

$$D_X Y = \varphi X,$$

where D is the Levi-Civita connection of N.

For example, if Y is parallel we can take $\varphi = 0$. In the Euclidean space $\mathbb{R}^{n+1}$, if Y is a radial vector field, the corresponding φ is constant equal to one. This means that in particular, our results hold for constant (i.e., parallel) and radial vector fields in $\mathbb{R}^{n+1}$.

Hereafter Y will denote a closed conformal vector field in N. We will suppose that Y does not vanish on the submanifold M.

3 Closed conformal vector fields and shadow boundaries

In general a shadow boundary is just a closed subset of M. The next result says that the shadow boundary is a smooth submanifold of M in a region when the Gauss-Kronecker curvature is different from zero.

Proposition 3.1. *Let M be an oriented immersed hypersurface in N and Y a closed conformal vector field in N. Let p be a point in $S\partial(M,Y)$ where either of the following conditions hold:*

- *The shape operator satisfies $A(Y_p) \neq 0$; or*
- *The Gauss-Kronecker curvature of M is not zero at p.*

Then there exists a neighbourhood U of p in M such that $S\partial(M,Y) \cap U$ is a hypersurface of M.

Proof. Let ξ be a unit vector field which is everywhere normal to M. Recall that the Gauss-Kronecker curvature is given by $\det A$, where $A(X) = -D_X \xi$ is the shape operator of M relative to ξ. Note that if $\det A_p \neq 0$, then A_p is a linear isomorphism and then $A(Y_p) \neq 0$, so we just analyze this case.

Let U be a neighbourhood of p where the following conditions hold for each point q in U:

- Y_q is not orthogonal to M, i.e. $Y_q^T \neq 0$.
- $A(Y_q^T) \neq 0$; here Y^T denotes the projection of Y into TM.

It is clear that the above neighbourhood U exists. Let us define the smooth function $F : U \to \mathbb{R}$ by $F = \langle Y, \xi \rangle$. Therefore, $S\partial(M, Y) \cap U = F^{-1}(0)$. We will prove that zero is a regular value of F. If X denotes a vector field tangent to M and ∇F denotes the gradient of F, then

$$
\begin{aligned}
\langle \nabla F, X \rangle &= XF = \langle D_X Y, \xi \rangle + \langle Y, D_X \xi \rangle \\
&= \langle \varphi X, \xi \rangle - \langle Y^T, A(X) \rangle = -\langle A(Y^T), X \rangle.
\end{aligned}
$$

Note that the above implies that $\nabla F = -A(Y^T)$. By our assumptions, $A(Y^T) \neq 0$ and therefore ∇F does not vanish in U. In particular, 0 is a regular value of F and therefore $S\partial(M, Y) \cap U$ is a hypersurface.   □

We now express the properties of the shadow boundary in terms of the second fundamental form of M.

Lemma 3.2. *Let M be an immersed hypersurface in N with second fundamental form α and Y be a closed conformal vector field in N. If $p \in L = S\partial(M, Y)$ and $A(Y_p) \neq 0$, then $\alpha(Y_p, X_p) = 0$ for every $X_p \in T_p L$.*

Proof. By Proposition 3.1, we now that under the given hypotheses L is a hypersurface of N, at least in a neighbourhood of p. Let ξ be a local unit normal vector field defined in such a neighbourhood of p and γ a smooth curve in L such that $\gamma(0) = p$ and $\gamma'(0) = X_p$. Since every point of γ belongs to L we have $\langle Y, \xi \rangle = 0$; taking the derivative with respect to X we obtain

$$
0 = \langle D_X Y, \xi \rangle + \langle Y, D_X \xi \rangle = \langle \varphi X, \xi \rangle - \langle Y, A(X) \rangle = -\langle Y, A(X) \rangle
$$

as in the proof of Proposition 3.1. Since $Y_p = Y_p^T$, we have

$$
0 = -\langle Y_p, A(X_p) \rangle = -\langle A(Y_p), X_p \rangle = -\langle \alpha(Y_p, X_p), \xi \rangle;
$$

the above implies that $\alpha(Y_p, X_p) = 0$.   □

Definition 3.3. Let M be an immersed hypersurface in N. Given a tangent vector $v_p \in T_p M \setminus \{0\}$, we say that a shadow boundary $S\partial(M, Y)$ is *generated* by v_p if Y is a closed conformal vector field satisfying the initial condition $Y_p = v_p$.

As an important example, we analyze shadow boundaries of hypersurfaces in space forms in the next Proposition.

Proposition 3.4. *Let N be a space form $\mathbb{Q}_c^{n+1}$. Given an immersed hypersurface M of $\mathbb{Q}_c^{n+1}$ and any vector $v_p \in T_p M \setminus \{0\}$, there exists a shadow boundary $S\partial(M, Y)$ generated by v_p.*

Proof. Let M and $v_p \in T_pM$ be as in the statement. Define a point $p_0 \in M$ by $p_0 = \exp_p(-v_p)$ when $c \leq 0$ and by $p_0 = \exp_p(-\lambda v_p)$ for a suitable $\lambda > 0$ such that $\lambda|v_p| < \pi/\sqrt{c}$, the diameter of $\mathbb{Q}_c^{n+1}$, if $c > 0$.

Consider the gradient vector field ∇d of the distance function $d(\cdot, p_0)$. As usual, ∇d is defined only in $\mathbb{Q}_c^{n+1} \setminus \{p_0\}$ (or $\mathbb{Q}_c^{n+1} \setminus \{p_0, -p_0\}$ for $c > 0$). In this domain, following [1], p. 205, we define the position vector Y in $\mathbb{Q}_c^{n+1}$ relative to p_0 by

$$Y_q = S_c(d(q, p_0))\nabla d_q,$$

where

$$S_c(s) = \begin{cases} s, & c = 0; \\ \sin(s\sqrt{c})/\sqrt{c}, & c > 0; \\ \sinh(s\sqrt{-c})/\sqrt{-c}, & c < 0. \end{cases}$$

Y is a closed conformal vector field. In fact, it is well known that the gradient of a distance function satisfies $D_{\nabla d}\nabla d = 0$, from which we obtain

$$D_{\nabla d}Y = S_c'\nabla d, \quad S_c' = \frac{dS_c}{ds};$$

on the other hand, in [1], p. 207, it is proved (see equation (2.2) there) that

$$D_X Y = S_c' X$$

for every vector field X transversal to ∇d. The last two equations imply that Y is closed conformal.

From the very definition of p_0 we have that Y_p is a scalar multiple of v_p, so that by multiplying by a suitable constant we obtain a closed conformal vector field assuming the vaue v_p at p. $\qquad \square$

Proposition 3.5. *Let M be an immersed hypersurface in N with second fundamental form α. Let $p \in M$ be a point where the Gauss-Kronecker curvature of M is different from zero. Then for every n-dimensional subspace V of T_pM there exists a vector $v_p \in T_pM$ such that the tangent space at p of every shadow boundary $S\partial(M, Y)$ generated by v_p is equal to V.*

Proof. Let ξ be a local normal unit vector field of M around p and A its associated shape operator. Since $\det A_p \neq 0$, $A : T_pM \to T_pM$ is invertible. Thus we may take $v_p \in T_pM$ such that $A(v_p)$ spans $V^\perp$, the orthogonal complement of V in T_pM.

Let Y be any closed conformal vector field which takes the value v_p at p. By Proposition 3.1 we know that the shadow boundary $L := S\partial(M, Y)$ is an embedded hypersurface of M around p.

By Lemma 3.2, $\alpha(Y_p, X_p) = 0$ for every $X_p \in T_pL$. Therefore,

$$0 = \langle \alpha(Y_p, X_p), \xi \rangle = \langle A(Y_p), X_p \rangle = \langle A(v_p), X_p \rangle.$$

Since $A(v_p)$ spans $V^\perp$, the above proves that $T_pL = V$. $\qquad \square$

Let us recall that a point p in a hypersurface $M \subset N$ is an umbilic point of M if and only if every tangent vector to M at p is a principal direction of the shape operator A of M at p. As noted in the abstract, we may use shadow boundaries in order to detect umbilic points in a hypersurface. The following results show how this can be done.

Definition 3.6. *A submanifold L of a hypersurface $M \subset N$ is* invariant *under the shape operator A of M if for every point $p \in L$, we have that $A(T_pL) \subset T_pL$.*

Proposition 3.7. *Let M be an immersed hypersurface in N. Let $p \in M$ be a point where the Gauss-Kronecker curvature is different from zero, and such that for every $v_p \in T_pM \setminus \{0\}$ there exists a shadow boundary $S\partial(M, Y)$ generated by v_p which is invariant under the shape operator of M. Then p is an umbilic point of M.*

Proof. Since the Gauss-Kronecker curvature of M does not vanish at p, the shape operator A of M relative to a unit normal vector field ξ is invertible.

Fix a vector v_p and let Y be as in the hypotheses. We will show that $L = S\partial(M, Y)$ is orthogonal to Y at p. By Lemma 3.2, $\alpha(Y_p, X_p) = 0$ for every $X_p \in T_pL$. Since A is invertible and L is invariant under A, then $A(T_pL) = T_pL$. So, for every $Z_p \in T_pL$ there exists $X_p \in T_pL$ such that $Z_p = A(X_p)$. Therefore,

$$\langle Z_p, Y_p \rangle = \langle A(X_p), Y_p \rangle = \langle \alpha(Y_p, X_p), \xi \rangle = 0.$$

Since v_p was arbitrarily chosen, by Corollary 4.2 we conclude that p is an umbilic point of M.   □

We have the following straightfoward application of Proposition 3.7

Corollary 3.8. *Given a surface M with nowhere zero Gauss-Kronecker curvature in a three dimensional Riemannian manifold N, if for every $p \in M$ and every $v_p \in T_pM \setminus \{0\}$ there exists a shadow boundary $S\partial(M, Y)$ generated by v_p which is a line of curvature of M, then M is totally umbilical in N.*

4   Totally umbilical hypersurfaces in space forms

In this section we will apply our previous results in order to characterize the totally umbilical hypersurfaces of the space forms $\mathbb{Q}_c^{n+1}$ in terms of shadow boundaries.

Proposition 4.1. *Let M be an immersed hypersurface in $\mathbb{Q}_c^{n+1}$. Let $p \in M$ be any point where the Gauss-Kronecker curvature of M is different from zero. A vector $v_p \in T_pM \setminus \{0\}$ determines a principal direction of M if and only if there exists a shadow boundary $S\partial(M, Y)$, generated by v_p, which is orthogonal to Y at p.*

Proof. Let us assume first that the shadow boundary $S\partial(M, Y)$ generated by v_p is orthogonal to the closed conformal vector field Y at p. By Lemma 3.2, $\langle A(Y_p), X_p \rangle = \langle \alpha(Y_p, X_p), \xi \rangle = 0$ for every $X_p \in T_p L$. This says that $A(Y_p)$ is orthogonal to $T_p L$. But Y_p is also orthogonal to $T_p L$; therefore, $A(Y_p)$ is a multiple of Y_p; since $Y_p = v_p$, this means that v_p determines a principal direction.

Conversely, let us assume that a vector v_p determines a principal direction, so that $A(v_p) = \lambda v_p$. Note that since $\det A_p \neq 0$, we have $\lambda \neq 0$.

By Proposition 3.4, we may construct a closed conformal vector field Y such that $Y_p = v_p$. By Lemma 3.2, we have that

$$\lambda \langle Y_p, X_p \rangle = \langle \lambda Y_p, X_p \rangle = \langle A(Y_p), X_p \rangle = \langle \alpha(Y_p, X_p), \xi \rangle = 0,$$

where $X_p \in T_p L$. This proves that $Y_p = v_p$ is orthogonal to the shadow boundary L generated by v_p. $\qquad \square$

The following result is an immediate consequence of Proposition 4.1.

Corollary 4.2. *Let M be an immersed hypersurface in $\mathbb{Q}_c^{n+1}$. Let $p \in M$ be any point where the Gauss-Kronecker curvature of M is non zero. The point p is an umbilic point of M if and only if for every $v_p \in T_p M \setminus \{0\}$, there exists a shadow boundary $S\partial(M, Y)$, generated by v_p, which is orthogonal to Y at p.*

In order to prove our main results, we apply Proposition 2 and Remark 2 in [5], which describe the structure of a complete manifold possessing a globally defined closed conformal vector field. For completeness we rephrase here the facts relevant in our setting.

Proposition 4.3 (Montiel). *Let Y be a non-trivial closed conformal vector field defined globally in the space form $\mathbb{Q}_c^{n+1}$. Then Y has at most two zeroes and*

1. *If Y has exactly one zero q, then $\mathbb{Q}_c^{n+1} \setminus \{q\}$ is isometric to a warped product $\mathbb{R}^+ \times_f \mathbb{S}^n$. If $(r, p) \in \mathbb{R}^+ \times \mathbb{S}^n$ represent the polar coordinates of a point, then $Y_{(r,p)} = f(r)p$. The spheres $\{r\} \times \mathbb{S}^n$ give a foliation of $\mathbb{Q}_c^{n+1} \setminus \{q\}$.*

2. *If Y has two zeroes $q, -q$, then $\mathbb{Q}_c^{n+1} \setminus \{q, -q\}$ is isometric to a warped product $(0, \pi) \times_f \mathbb{S}^n$. The spheres $\{r\} \times \mathbb{S}^n$ give a foliation of $\mathbb{Q}_c^{n+1} \setminus \{q - q\}$.*

3. *If Y has no zeroes, then $\mathbb{Q}_c^{n+1}$ is isometric to a warped product $\mathbb{R} \times_f \mathbb{Q}_d^n$, where $\mathbb{Q}_d^n$ is a space form of curvature d. In this case, $Y = f(r)\partial_r$ and the slices $\{r\} \times \mathbb{Q}_d^n$ foliate $\mathbb{Q}_c^{n+1}$.*

We say that a regular curve L in a surface $M \subset \mathbb{Q}_c^3$ contains a principal direction of M at $p \in L$ if the tangent line of L at p is generated by a principal direction of the shape operator of M in $\mathbb{Q}_c^3$.

Proposition 4.4. *Let M be an immersed surface in $\mathbb{Q}_c^3$. Then every compact regular shadow boundary $L := S\partial(M, Y)$ of M relative to a globally defined closed conformal vector field Y contains at least two principal directions of M at two different points of L. In particular, if M is a compact surface with nowhere zero Gaussian curvature then every shadow boundary of M contains at least two principal directions of M at two different points.*

Proof. By Proposition 4.1, we have to prove the existence of two points in L where L is orthogonal to Y.

Suppose the first case in Proposition 4.3 holds; that is, suppose that Y has exactly one zero q and that $\mathbb{Q}_c^{n+1} \setminus \{q\} = \mathbb{R}^+ \times_f \mathbb{S}^n$. Since M is compact and L is a closed subset of M, there are real numbers r_1, r_2 such that L is contained in a slab $[r_1, r_2] \times \mathbb{S}^n$. If $[r_1, r_2]$ is the smallest interval with this property, then each sphere $\{r_i\} \times \mathbb{S}^n$ is tangent to the shadow boundary at their contact points; since the spheres are orthogonal to Y, the same happens to the shadow boundary; that is, the shadow boundary is orthogonal to Y at its contact points with the mentioned spheres.

A completely similar argument holds for the cases where Y has two or no zeroes. $\square$

Finally, we will prove our main result.

Theorem 4.5. *Let M be a compact hypersurface with nowhere zero Gauss-Kronecker curvature in $\mathbb{Q}_c^{n+1}$. If for every $p \in M$ and every direction $Y_p \in T_p M \setminus \{0\}$ there exists a shadow boundary $S\partial(M, Y)$ generated by Y_p which makes a constant angle with respect to the globally defined closed conformal vector field Y, then M is totally umbilical.*

Proof. By Proposition 3.1, every shadow boundary is a hypersurface of M. Moreover, since M is compact and each shadow boundary $L = S\partial(M, Y)$ is closed in M, we have that each L is compact. The idea of the proof is to use compactness to prove that the constant angle between $S\partial(M, Y)$ and Y should be $\pi/2$, i.e., that every shadow boundary $S\partial(M, Y)$ is orthogonal to Y.

The argument here is analogous to that of Proposition 4.4. Let p be any point of M, $Y_p \in T_p M$ be any non zero tangent vector and Y the corresponding closed conformal vector field.

By Proposition 4.3, the region where Y has no zeroes has a decomposition by slices $\{r\} \times \mathbb{Q}_d^n$, each one orthogonal to Y. By compactness, the shadow boundary L is tangent to one of these slices at, say, a point p. Therefore, the angle between Y_p and $T_p S\partial(M, Y)$ is $\pi/2$. Now we apply Corollary 4.2 to conclude that the point p is an umbilic point. $\square$

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Oscar Palmas
Facultad de Ciencias, UNAM, D.F., México
e-mail: oscar.palmas@ciencias.unam.mx

Gabriel Ruiz-Hernández
Instituto de Matemáticas, UNAM, D.F., México
e-mail: gruiz@matem.unam.mx