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ON ISOMETRIC IMMERSIONS INTO COMPLEX SPACE FORMS

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In this paper we consider the question of whether an isometric immersion of a connected Kaehler manifold into a non-flat complex space form is holom orphic. The main purpose of this paper is to study isometric immersions of connected Kaehler manifolds into non-flat complex space forms. The question of when such an isometric immersion is holomorphic is an interesting problem. This question has been successfully studied in the more general setting of harmonic maps by several authors (see Siu and Yau [S-Y], Burns et al [B], and Carlson and Toledo [C-T]), where they require the manifolds to be compact. In our case, a positive result is obtained under local conditions only.

Recall that given an isometric immersion $M \to \tilde{M}$, the *index of relative nullity* $\nu(x)$ of f at $x \in M$ is defined as $\nu(x) = \dim \triangle(x)$ where

$$\triangle(x) = \{X \in T_xM \alpha(X,Y) = 0 \quad \text{for all} \quad Y \in T_xM\};$$

here α denotes the second fundamental form of the immersion f with values in the normal bundle $TM^\perp$. Let us denote by Q_c^N (respectively, Q_c^N) a simply connected complex (respectively, real) space form of constant holomorphic (respectively, sectional) curvature c.

The following is our main result.

Theorem 1. *Let $M^{2n} \to Q_c^N$, $n \geq 2$, be an isometric immersion of a Kaehler manifold into a complex space form of constant holomorphic curvature $c \neq 0$. If $\nu(x) > 0$ everywhere, then f is holomorphic.*

The case where the ambi ent space is flat has been treated in [D–R]. The above local theorem has several applications. We have the following one for low codimension.

Theorem 2. *Let $M^{2n} \to Q_c^N$, $n \geq 2$, be an isometric immersion of a Kaehler manifold into a complex space form of constant holomorphic curvature $c \neq 0$. If at one point the sectional curvature of M satisfies $K_M \leq c/4$ and $N < \frac{3}{2}n$, then f is holomorphic.*

Combining Theorem 1 with a result of Abe [Ab], we have the following global consequence.

Corollary 3. *Let $M^{2n} \to P_c^N$, $n \geq 2$, be an isometric immersion of a complete Kaehler manifold into the complex projective space. If $\nu(x) > 0$ everywhere, then $M^{2n} = P_c^n$ and f is totally geodesic.*

An immediate application of the proof of the above theorem to the case of immersions of Kaehler manifolds into real space forms is the following local rather surprising result.

Theorem 4. *Let $M^{2n} \to Q_c^N$, $n \geq 2$, be an isometric immersion of a Kaehler manifold into a real space form with $c \neq 0$. Then $\nu(x) = 0$ everywhere.*

The above result has the following consequence.

Corollary 5. Assmue that M^{2n} is a Kaehler manifold whose sectional curvature at some point satisfies $K_M(x_0) \leq c$. Then there exists no isometric immersion $f : M^{2n} \to Q_c^{2n+p}$ for $p < n$ and $c \neq 0$.

We also study isometric immersions of Riemannian manifolds whose first normal space $N_1(x)$ form a proper subbundle of low rank of the normal bundle. Recall that the *first normal space* $N_1(x)$ is the linear subspace of the normal space spanned by the image of the second fundamental form. First we prove a general result on reduction of condimension extending the one of Cecil [Ce] (we do not require the immersion to be holomorphic). This result together with Theorem 1 has several application. One of them ios the following.

Theorem 6. *Let* $f : M^{2n} \to CQ_c^N, n \geq 2$ *and* $c \neq 0$, *be an isometric immersion of a Kaehler manifold with first normal bundle of rank 1. Then f is a standard imbedding of an open set of one of the following:*

i) $S_{c_1}^2 \times S_{c_2}^2$ contained in a totally geodesic P_c^5,

ii) $S^2 2_{c_1} \times H_{c_2}^2$ contained in a totally geodesic H_c^2,

iii) $\mathbf{R}^{2n}$ as an umbilical hypersurface of H_c^{2n+1}.

Here $1/c = 1/c_1 + 1/c_2$.

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