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COMPLETE MINIMAL SURFACES IN $\mathbf{R}^3$ OF FINITE TOPOLOGY AND INFINITE TOTAL CURVATURE.

Celso J. Costa

The construction of examples with prescribed geometry or topology is a basic problem in the theory of minimal surfaces in $\mathbf{R}^3$. The aim of this work is to construct complete minimal surfaces in $\mathbf{R}^3$ with finite topology and infinite total curvature. In Theorem A we take a perturbation of a Catenoid to obtain a proper complete minimal surface in $\mathbf{R}^3$ with one end of Catenoid type and one end of infinite total curvature. In Theorem B we construct a one-parameter family of complete minimal surfaces in $\mathbf{R}^3$ given by a perturbation of Costa's surface and in Theorem C we prove that every compact Riemann Surface of genus one punctured at a point can be immersed in $\mathbf{R}^3$ as a complete minimal surface with infinite total curvature.

More precisely, we prove the following results.

Theorem A: (Perturbation of the Catenoid) *For every $\alpha > 0$ we have a Weierstrass representation of a complete minimal immersion $X : \mathbf{C} \setminus \{0\} \to \mathbf{R}^3$ of infinite total curvature and with one end of Catenoid type given by*

$$g(z) = ze^{\alpha z^2}, \eta = \frac{e^{-\alpha z^2}}{z^2} dz. \tag{1}$$

Furthermore, X is proper and for each fixed $r > 0$ the curves $\beta_r := X(re^{i\theta}), 0 \leq \theta \leq 2\pi$ lie in a plane P_r parallel to the (x_1, x_2)-plane of $\mathbf{R}^3$ and $P_r \cap P_{r'} = \emptyset$, if $r \neq r'$.

Theorem B: (Perturbation of Costa's surface) *There exists a one-parameter family of complete minimal surfaces in* $\mathbf{R}^3$ *of genus one, three ends and infinite total curvature. Furthermore, each member of this family has two ends of finite total curvature, respectively a planar end and a Catenoid type end.*

Theorem C: *Let* $\overline{M_1}$ *be a compact Riemann surface of genus one and* $p \in \overline{M_1}$ *a point. Then, there exists a complete minimal immersion* $X : \overline{M_1} \setminus \{p\} \to \mathbf{R}^3$ *of infinite total curvature.*

We will use Weierstrass representations to construct examples of complete minimal surfaces in $\mathbf{R}^3$ with the properties required by Theorems A, B, and C. We summarize this procedure in Theorem 1 below.

Theorem 1: *Let* M *be a non compact, connected Riemann surface* M *and let* (g, η) *be a pair, where* g *is a meromorphic function and* η *a holomorphic differential on* M. *Suppose that* (g, η) *satisfies:*

(c_1) *a point* $p \in M$ *is a pole of order* m *of* g *if and only if* p *is a zero of order* $2m$ *of* η,

(c_2) *for every closed curve* γ *in* M

$$Re \int_\gamma g\eta = 0, \int_\gamma g^2\eta = \overline{\int_\gamma \eta},$$

(c_3) *For every divergent curve* l *in* M

$$\int_l (1 + | g |^2) | \eta | = +\infty.$$

Then, $X : M \to \mathbf{R}^3$,

$$X(z) = Re \int_{z_0}^{z} ((1 - g^2)\eta, i(1 + g^2\eta), 2g\eta), z_0 \in M, \tag{2}$$

is a complete minimal immersion of M *in* $\mathbf{R}^3$. *The pair* (g, η) *is called* the Weierstrass representation *of* X.

In order to prove Theorems B and C, we need some basic facts and notations about elliptic functions. In $\mathbf{C}$ we consider the set FM

$$FM = \{\tau = x + iy \in \mathbf{C}; x^2 + y^2 \geq 1, | x |\leq 1/2, y > 0\}$$

and for $\tau \in FM$ the lattices $L(\tau) = \{m + n\tau, m, n \in \mathbf{Z}\}$. Then $\mathbf{C}/L(\tau)$ is a genus one compact Riemann surface. Also, it is well known that if $\overline{M}$ is a compact Riemann surface of genus one, then there exists $\tau \in FM$ such that $\mathbf{C}/L(\tau)$ is conformally equivalent to $\overline{M}$. Associated to the lattices $L(\tau)$ we have the P-function of Weierstrass. Also, we define the complex numbers

$$e_j = P(w_j), j = 1, 2, 3; g_2 = -4 \sum_{j<k} e_j e_k, g_3 = 4e_1 e_2 e_3, \tag{3}$$

where

$$w_1 = 1/2, w_2 = -\frac{1+\tau}{2}, w_3 = \tau/2$$

and

$$2\eta_j = -\int_{l_j} P(z)dz, j = 1, 3, \tag{4}$$

where, $l_j : [0, 1] \to \mathbf{C}$ are the paths

$$l_1(t) = \tau/3 + t, l_3(t) = 1/3 + t\tau. \tag{5}$$

Observe that l_j is a basis of the first group of homology of $\mathbf{C}/L(\tau)$.

1. Proof of Theorem A.

We observe that if $\alpha = 0$ the pair (g, η) given in (1) is the Weierstrass representation of the Catenoid. Nevertheless, by using the hypothesis $\alpha > 0$ we obtain

$$g\eta = \frac{dz}{z} \text{ and } Res_z g^2\eta = Res_z \eta = 0, \tag{6}$$

for every $z \in \mathbf{C} \cup \{\infty\}$, where $Res\varphi$ represents the residue of the differential φ. So, (g, η) satisfies (c_2) of Theorem 1. On the other hand, g is holomorphic in $\mathbf{C} \setminus \{0\}$ and $\eta(z) \neq 0$ for every $z \in \mathbf{C} \setminus \{0\}$. So, (g, η) satisfies (c_1) of Theorem 1. This shows that $X : \mathbf{C} \setminus \{0\} \to \mathbf{R}^3$ defined by (g, η) as in (2) is a minimal immersion. Also, from (6) we find that

$$X_3(z) = Re \int_{z_0}^{z} g\eta = \log \left| \frac{z}{z_0} \right|,$$

where X_3 is the third coordinate of X. Then X is proper and complete. Furthermore, if $r > 0$,

$$X_3(re^{i\theta}) = \log \frac{r}{| z_0 |}.$$

This completes the proof of Theorem A.

2. Proof of Theorem B.

Let $L(iy), y \geq 1$ are the lattices $L(iy) = \{m + niy \in \mathbf{C}; m, n \in \mathbf{Z}\}$. We will prove that there exist $\varepsilon > 0$ and real positive functions $\alpha = \alpha(y), \lambda = \lambda(y), 1 < y < 1 + \varepsilon$ such that

$$g_y = \lambda e^{\alpha P} \tilde{g}_y, \eta_y = e^{-\alpha P} \tilde{\eta}_y, \tag{7}$$

where

$$\tilde{g}_y = \frac{P'}{(P - e_1)^2}, \tilde{\eta}_y = \frac{(P - e_1)^2}{P - e_2} dz \tag{8}$$

are Weierstrass representations of complete minimal immersions of

$$M_y = \mathbf{C}/L(iy) - \{\pi(0), \pi(w_1), \pi(w_2)\} \tag{9}$$

in $\mathbf{R}^3$ with the desired properties. Here $\pi : \mathbf{C} \to \mathbf{C}/L(iy)$ is the canonical projection, P is the function of Weierstrass of $L(iy)$ and $e_j, j = 1, 2, 3$ is as given in (3).

Also, we will prove that $\lim_{y \to 1} \alpha(y) = 0$ and $\lim_{y \to 1} \lambda(y) = e_1 \sqrt{1/2\pi}$. With these limit values for α and λ, (7) is exactly the Weierstrass representation of Costa's surface. In this sense, the one-parameter family that we obtain is a perturbation of this surface (see Remark 1 after the proof of Theorem B).

We observe that g_y and η_y are analytic in M_y and η_y does not have zeros. So, (g_y, η_y) satisfies (c_1) of Theorem 1. Also, by using that

$$(P')^2 = 4 \prod_{j=1}^{3} (P - e_j) = 4P^3 - g_2 P - g_3, \tag{10}$$

we find that

$$| \eta_y | + | g_y^2 \eta_y | = \left[e^{-\alpha ReP} \frac{| P - e_1 |^2}{| P - e_2 |} + 4\lambda^2 e^{\alpha ReP} \frac{| P - e_3 |}{| P - e_1 |} \right] | dz | .$$

As P is an even function with a double pole at $z = 0$, we conclude that (7) satisfies (c_3) of Theorem 1.

On the other hand, if $\lambda = \lambda_y$ is a real number and $l \subset M_y$ is a closed curve, we obtain

$$Re \int_l g_y \eta_y = \lambda \log | P(z) - e_2 |_l = 0. \tag{11}$$

So, (g_y, η_y) satisfies the first equality in (c_2) of Theorem 1.

Also, from (8) and (10) we conclude that

$$\tilde{\eta}_y = \left[P - e_2 + 2(e_2 - e_1) + \frac{(e_2 - e_1)^2}{P - e_2} \right] dz \tag{12}$$

and

$$(\tilde{g}_y)^2 \tilde{\eta}_y = 4 \left(1 + \frac{e_1 - e_3}{P - e_1} \right) dz. \tag{13}$$

So, from (7) and (12) we find that

$$\begin{aligned}
\eta_y &= \left[P + (e_2 - 2e_1) + \frac{(e_2 - e_1)^2}{P - e_2} \right] \cdot \left[1 - \alpha P + \alpha^2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \alpha^{n-2} P^n \right] dz = \\
&= \left[P - \alpha P^2 + (e_2 - 2e_1) - \alpha(e_2 - 2e_1)P + \frac{(e_2 - e_1)^2}{P - e_2} - \alpha \frac{(e_2 - e_1)^2}{P - e_2} P \right] dz \\
&+ \alpha^2 \left[\sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \alpha^{n-2} P^n \right] \tilde{\eta}_y = \\
&= \{ P - \alpha(\frac{1}{6} P'' + \frac{g_2}{12}) + (e_2 - 2e_1) - \alpha(e_2 - 2e_1)P + \\
&+ \frac{e_2 - e_1}{e_2 - e_3} [P(z - w_2) - e_2] - \alpha(e_2 - e_1)^2 - \alpha \frac{e_2(e_2 - e_1)}{e_2 - e_3} [P(z - w_2) - e_2] \} dz \\
&+ \alpha^2 \left[\sum_{n=2}^{\infty} \frac{(-1)^n \alpha^{n-2}}{n!} P^n \right] \tilde{\eta}_y,
\end{aligned}$$

where g_2 is given in (3). Then we can write

$$\eta_y = [-\frac{\alpha}{6} P'' + \gamma_0 + \gamma_1 P + \gamma_2 P(z - w_2)] dz + \alpha^2 \delta_2, \tag{14}$$

where

$$\begin{aligned}
\gamma_0 &= \frac{-g_2}{12} \alpha + \frac{e_2^2 + 2e_1 e_3}{e_2 - e_3} - \frac{e_2 - e_1}{e_2 - e_3} (e_2^2 + e_1 e_3) \alpha, \\
\gamma_1 &= 1 - \alpha(e_2 - 2e_1), \\
\gamma_2 &= \frac{e_2 - e_1}{e_2 - e_3} (1 - \alpha e_2) \text{ and } \delta_2 = \left[\sum_{n=2}^{\infty} (-1)^n \frac{\alpha^{n-2}}{n!} P^n \right] \tilde{\eta}_y.
\end{aligned} \tag{15}$$

Also, from (7) and (13) we can obtain

$$\begin{aligned}
\frac{1}{4\lambda^2} g_y^2 \eta_y &= \left[1 + \alpha P + \frac{e_1 - e_3}{P - e_1} + \alpha \frac{e_1 - e_3}{P - e_1} P + \alpha^2 \left(1 + \frac{e_1 - e_3}{P - e_1}\right) \sum_{n=2}^{\infty} \frac{\alpha^{n-2}}{n!} P^n \right] d \\
&= \{1 + \alpha P + \frac{P(z - w_1) - e_1}{e_1 - e_2} + \alpha(e_1 - e_3) + \frac{\alpha e_1}{e_1 - e_2}[P(z - w_1) - e_1] \\
&+ \alpha^2 \left(1 + \frac{e_1 - e_3}{P - e_1}\right) \sum_{n=2}^{\infty} \frac{\alpha^{n-2}}{n!} P^n\} dz.
\end{aligned}$$

So,

$$g_y^2 \eta_y = 4\lambda^2 [\beta_0 + \beta_1 P + \beta_2 P(z - w_1) + \alpha^2 \delta_1] dz, \tag{16}$$

where,

$$\begin{aligned}
\beta_0 &= \frac{1}{e_1 - e_2} [\alpha(e_1^2 + e_2 e_3) - e_2], \\
\beta_1 &= \alpha, \quad \beta_2 = \frac{1}{e_1 - e_2}(1 + \alpha e_1) \text{ and} \\
\delta_1 &= \left(1 + \frac{e_1 - e_3}{P - e_1}\right) \sum_{n=2}^{\infty} \frac{\alpha^{n-2}}{n!} P^n.
\end{aligned} \tag{17}$$

From (14) and (16) and by using that P is an even function we can conclude that

$$Res_z g_y^2 \eta_y = Res_z \eta_y = 0 \tag{18}$$

for every $z \in M_y$, where M_y is defined in (9). Now we need an assertion:

Assertion 1. There exist real functions $A_j = A_j(y), B_j = B_j(y), y \geq 1$, such that

$$\int_{l_k} \delta_j = 2\eta_k A_j + 2w_k B_j, j = 1, 2, k = 1, 3,$$

where l_k is defined in (5) and δ_j are given in (15) and (17).

Proof. From [7] vol 3 page 59 and vol. 4 page 109, we find that

$$\int_{l_k} P^n dz = -2\eta_k B_{n-1}^n + 2w_k B_n^n, \quad k = 1, 3, n = 1, 2 \cdots, \tag{19}$$

where $B_{n-1}^n = B_{n-1}^n(y), B_n^n = B_n^n(y), y \geq 1$ are the c^∞ functions given by

$$B_0^1 = 1, B_1^1 = 0, B_1^2 = 0, B_2^2 = \frac{g_2}{12}, B_2^3 = \frac{3g_2}{20}, B_3^3 = \frac{g_3}{10}, \tag{20}$$

and

$$B_r^{n+1} = \frac{2n-1}{4(2n+1)} B_{r-2}^{n-1} g_2 + \frac{n-1}{2(2n+1)} B_{r-3}^{n-2} g_3, r = n, n+1 \text{ and } n \geq 3. \quad (21)$$

Here, we observe that in the rectangular lattices $L(iy), y \geq 1, g_2 = g_2(y)$ and $g_3 = g_3(y)$ as defined in (3) are real functions of variable y.

On the other hand, from (12), (15), and (17) we can write

$$\delta_j = \left(\sum_{n=0}^{\infty} a_n^j P^n + \frac{b^j}{P - e_j}\right) dz = \left[\sum_{n=0}^{\infty} a_n^j P^n + b^j \frac{P(z - w_j) - e_j}{(e_k - e_j)(e_t - e_j)}\right] dz, j = 1, 2,$$

where (k, t, j) is a permutation of $(1, 2, 3)$ and $a_n^j = a_n^j(\alpha), n = 0, 1 \cdots, b^j = b^j(\alpha)$ are c^∞ functions of α given by convergent series. By integrating term by term δ_j on curves l_k and by using (5), (19), (20), and (21) we conclude the proof of Assertion 1.

Now we continue the proof of Theorem B. By using Assertion 1 together with (4), (14), (16), and Legendre's relation we obtain

$$\int_{l_1} g_y^2 \eta_y = 4\lambda^2[\beta_0 - 2\eta_1(\beta_1 + \beta_2) + \alpha^2(2\eta_1 A_1 + B_1)], \quad (22)$$

$$\int_{l_1} \eta_y = \gamma_0 - 2\eta_1(\gamma_1 + \gamma_2) + \alpha^2(2\eta_1 A_2 + B_2), \quad (23)$$

$$\begin{aligned} \int_{l_3} g_y^2 \eta_y &= 4\lambda^2\{[\beta_0 - 2\eta_1(\beta_1 + \beta_2) + \alpha^2(2\eta_1 A_1 + B_1)]iy + \\ &+ 2\pi i(\beta_1 + \beta_2 - \alpha^2 A_1)\}, \end{aligned} \quad (24)$$

and

$$\begin{aligned} \int_{l_3} \eta_y &= [\gamma_0 - 2\eta_1(\gamma_1 + \gamma_2) + \alpha^2(2\eta_1 A_2 + B_2)]iy + \\ &+ 2\pi i(\gamma_1 + \gamma_2 - \alpha^2 A_2). \end{aligned} \quad (25)$$

Now for a small $\varepsilon^* > 0$ we consider the c^∞ function $H : (0, \infty) \times \mathbf{R} \times (1 - \varepsilon^*, \infty) \to \mathbf{C}$ given by

$$H(\lambda, \alpha, y) = H_1(\lambda, \alpha, y) + iH_2(\lambda, \alpha, y),$$

where $H_j = H_j(\lambda, \alpha, y), j = 1, 2$ are the functions

$$H_1 = \int_{l_1} g_y^2 \eta_y - \overline{\int_{l_1} \eta_y}, H_2 = -i\left[\int_{l_3} g_y^2 \eta_y - \overline{\int_{l_3} \eta_y}\right].$$

Observe that at the point

$$\tilde{P} = \left(e_1\sqrt{1/2\pi}, 0, 1\right)$$

we have $H_1(\tilde{P}) = H_2(\tilde{P}) = 0$ and (g_1, η_1) is the Weierstrass representation of Costa's surface. Also, we recall that in the lattice $L(i)$ we have $2\eta_1 = \pi$, $e_1 = -e_3 > 0, e_2 = 0$. So, from (15), (17), (22), (23), (24), and (25) we find that

$$\begin{aligned}
\frac{\partial H_1}{\partial \lambda}(\tilde{P}) &= -4\sqrt{2\pi}, \frac{\partial H_2}{\partial \lambda}(\tilde{P}) = 4\sqrt{2\pi}, \\
\frac{\partial H_1}{\partial \alpha}(\tilde{P}) &= \frac{2e_1}{\pi}(e_1^2 - \frac{4\pi}{3}e_1 + \pi^2) \text{ and} \\
\frac{\partial H_2}{\partial \alpha}(\tilde{P}) &= \frac{2e_1}{\pi}(e_1^2 + \frac{4\pi}{3}e_1 + \pi^2).
\end{aligned}$$

By using that $e_1 > 2\pi$ we find that

$$\det \frac{\partial(H_1, H_2)}{\partial(\lambda, \alpha)} = -\frac{16}{\pi}\sqrt{2\pi}(e_1^2 + \pi^2) < 0.$$

Hence, from Implicit Theorem's Function there exist $\varepsilon > 0$, open sets $I = (1-\varepsilon, 1+\varepsilon), J = (e_1\sqrt{1/2\pi} - \varepsilon, e_1\sqrt{\frac{1}{2\pi}} + \varepsilon) \times (-\varepsilon, \varepsilon)$ and a c^∞ function $\theta : I \to J, \theta(y) = (\lambda(y), \alpha(y), y)$ such that

$$H(\theta(y), y) = 0. \tag{26}$$

So, by using (11) and (18) we conclude that if $1 < y < 1 + \varepsilon$ and $\lambda = \lambda(y), \alpha = \alpha(y)$ are given by (26), then the pair (g_y, η_y) satisfies the condition (c_2) of Theorem 1. This completes the proof of the existence of c^∞ functions $\alpha(y)$ and $\lambda(y), 1 < y < 1+\varepsilon$, such that $(g_y \eta_y)$ given in (7) are Weierstrass representations of a one-parameter family of complete minimal immersions $X_y : M_y \to \mathbf{R}^3$.

We recall that M_y are given by (9). Observe that at $z = w_j, j = 1, 2$, each g_y has, respectively, a pole of order three and a zero of order one. Also, at these points, each η_y has respectively a zero of order four and a double pole. So

$z = w_3$ are ends of Catenoid type and $z = w_1$ are planar ends of the immersions X_y. Finally, we affirm that g_y has essential singularities at $z = 0$. That is, $\alpha(y) \neq 0$ for every $1 < y < 1 + \varepsilon$. In fact, suppose that $\alpha(y) = 0$ for some $1 < y < 1 + \varepsilon$. Then, at $z = 0$, g_y has a single zero and η_y has a double pole, respectively. In this situation $(g_y \eta_y)$ is a Weierstrass representation of a complete minimal surface in $\mathbf{R}^3$ of genus one with finite total curvature and three ends. Furthermore, two ends are of Catenoid type, one end is planar of order two, and the normal vectors at the ends are parallel. On the other hand, as $y > 1$, Theorem 1 in [1] shows that there does not exist an immersion in $\mathbf{R}^3$ with these properties. This contradiction proves that $z = 0$ are ends of infinite total of the complete minimal immersions X_y for every $1 < y < 1 + \varepsilon$. This completes the proof of Theorem B.

Remark 1. Suppose for every $1 \leq y < 1 + \varepsilon$ the complete minimal immersions $Z_y : M_1 \rightarrow \mathbf{R}^3$, $Z_y = X_y \circ \varphi_y$ where $\varphi_y : M_1 \rightarrow M_y$ are the diffeomorphism $\varphi_y(\pi_1(u + iv)) = \pi_y(u + iyv)$. Observe that X_1 is Costa's surface. By using properties of the P-function it is not hard to prove that on compact sets $K \subset M_1$, Z_y converges smoothly to Z_1.

3. Proof of Theorem C. In order to prove Theorem C, we need a lemma.

Lemma 1: *Let $M_1 = \mathbf{C}/L(\tau)$ be a genus one compact Riemann surface, where $\tau \in FM$ and $L(\tau) = \{m + n\tau \in \mathbf{C}; m, n \in \mathbf{Z}\}$. Then there exist $\varepsilon > 0$ and differentiable functions $\lambda, \beta : D(\varepsilon) \rightarrow \mathbf{C}$, where $D(\varepsilon) = \{\alpha \in \mathbf{C}; 0 <\mid \alpha \mid< \varepsilon\}$ and such that*

$$\int_{l_k} g_\alpha^2 \eta_\alpha = \overline{\int_{l_k} \eta_\alpha}, \tag{27}$$

where g_α and η_α are, respectively, the meromorphic functions and the holomorphic differentials on $M_1 \setminus \{\pi(0)\}$ given by

$$g_\alpha = [\lambda e^{(\alpha/2)P(z)} + \beta] \frac{1}{P'(z)}, \eta_\alpha = P'^2(z) dz \tag{28}$$

and l_k is as defined in (5). Furthermore,

$$\lambda(\alpha) \neq 0 \text{ and } \lambda(\alpha) + \beta(\alpha) e^{(\alpha/2)e_j} \neq 0 \tag{29}$$

for every $\alpha \in D(\varepsilon)$ and $j = 1, 2, 3$.

Proof: From (3), (10), and (28) we find that

$$\eta_\alpha = 4P^3 - g_2 P - g_3$$

and

$$g_\alpha^2 \eta_\alpha = \left\{ \lambda^2 \left[1 + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} P^n(z) \right] + 2\lambda\beta \left[1 + \sum_{n=1}^{\infty} \frac{\alpha^n}{2^n n!} P^n(z) \right] + \beta^2 \right\} dz.$$

So, by using (4), (19) and (20) we conclude that (27) is true if and only if

$$\begin{aligned} \lambda^2 \quad & [2w_k - 2\eta_k Q_1 + 2w_k S_1] + \\ + \quad & 2\lambda\beta[2w_k - 2\eta_k Q_0 + 2w_k S_0] + 2w_k\beta^2 = \\ = \quad & \overline{4(-2\eta_k B_2^3 + 2w_k B_3^3) + 2\eta_k g_2 - 2w_k g_3}, \quad k = 1, 3, \end{aligned} \tag{30}$$

where

$$\begin{aligned} Q_1 &= \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} B_{n-1}^n, S_1 = \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} B_n^n, \\ Q_0 &= \sum_{n=1}^{\infty} \frac{\alpha^n}{2^n n!} B_{n-1}^n, S_0 = \sum_{n=1}^{\infty} \frac{\alpha^n}{2^n n!} B_n^n \end{aligned} \tag{31}$$

and B_{n-1}^n, B_n^n are defined in (20) and (21). By using Legendre's relation, $\eta_3 = \eta_1 \tau - \pi i$, we conclude that equation (30) is equivalent to

$$\lambda^2[1 - 2\eta_1 Q_1 + S_1] + 2\lambda\beta[1 - 2\eta_1 Q_0 + S_0] + \beta^2 = \bar{A} \tag{32}$$

and

$$\lambda^2 Q_1 + 2\lambda\beta Q_0 = -E/\pi, \tag{33}$$

where

$$A = 4(-2\eta_1 B_2^3 + B_3^3) + 2\eta_1 g_2 - g_3 \tag{34}$$

and

$$E = y\bar{A} + \pi\overline{(4B_2^3 - g_2)}. \tag{35}$$

We observe that $(A, E) \neq (0, 0)$. In fact, if $A = E = 0$ then from (20), (34), and (35) we find

$$g_2 = g_3 = 0.$$

But for every lattice $L(\tau), \tau \in FM$, we know that $(g_2, g_3) \neq (0,0)$. This contradiction proves that $(A, E) \neq (0,0)$. Also, equations (32) and (33) are equivalent to

$$\lambda^2 Q_1 + 2\lambda\beta Q_0 + E/\pi = 0 \tag{36}$$

and

$$\lambda^2(1+S_1) + 2\lambda\beta(1+S_0) + \beta^2 - F = 0, \tag{37}$$

where

$$F = \bar{A} - 2\eta_1(E/\pi).$$

Notice that $(A, E) \neq (0,0)$ implies that

$$(E, F) \neq (0,0). \tag{38}$$

Also, by using (20), (21), and (31) we conclude that $S_j = S_j(\alpha), Q_j = Q_j(\alpha)$ are holomorphic functions,

$$S_j(0) = Q_j(0) = 0, j = 0,1 \tag{39}$$

and there exists $\tilde{\varepsilon} > 0$ such that

$$S_j(\alpha) \neq 0, Q_j(\alpha) \neq 0, j = 0,1, 0 <\mid \alpha \mid< \tilde{\varepsilon}. \tag{40}$$

Now we will prove that there exist $\varepsilon > 0$ such that (36) and (37) have solutions $(\lambda(\alpha), \beta(\alpha))$ with $\lambda(\alpha) \neq 0$ for every $0 <\mid \alpha \mid< \varepsilon$. We prove this by considering separately the case $E = 0$ and the case $E \neq 0$.

In the first case, it suffices to find $\varepsilon > 0$ and $(\lambda(\alpha), \beta(\alpha))$ with $\lambda(\alpha) \neq 0$ for every $0 <\mid \alpha \mid< \varepsilon$ and such that

$$\lambda Q_1 + 2\beta Q_0 = 0 \tag{41}$$

and

$$\lambda^2(1+S_1) + 2\lambda\beta(1+S_0) + \beta^2 - F = 0. \tag{42}$$

Now let $\varphi = \varphi(\alpha)$ be the holomorphic function

$$\varphi(\alpha) = (1+S_1)Q_0^2 - (1+S_0)Q_1Q_0 + 1/4Q_1^2, \tag{43}$$

defined in a neighbourhood of $\alpha = 0 \in \mathbf{C}$. After a long calculation and by using (20), (21), and (31) we conclude that

$$\frac{d^k\varphi}{d\alpha^k}(0) = 0, k = 0,1,2,3, \quad \frac{d^4\varphi}{d\alpha^4}(0) = g_2/8 \tag{44}$$

and

$$\frac{d^5}{d\alpha^5}\varphi(0) = \frac{3g_3}{8}. \tag{45}$$

As $(g_2, g_3) \neq (0,0)$ for every lattice $L(\tau), \tau \in FM$, we conclude that there exist $\varepsilon > 0$ such that $\varphi(\alpha) \neq 0$ for all $0 <| \alpha |< \varepsilon$. This result together with (38) shows that (41) and (42) have solutions $(\lambda(\alpha), \beta(\alpha))$ with $\lambda(\alpha) \neq 0$ for every $0 <| \alpha |< \varepsilon$ and completes the proof of the first case.

In order to prove the existence of solutions $(\lambda(\alpha), \beta(\alpha))$ of (36) and (37) with $\lambda(\alpha) \neq 0$ and under the hypothesis of the second case, that is $E \neq 0$, we introduce homogeneous coordinates (λ, β, γ) in $\mathbf{P}^2\mathbf{C}$. From (36) and (37) we define in $\mathbf{P}^2\mathbf{C}$ a family (depending on the complex parameter $\alpha \neq 0$) of algebraic curves

$$\lambda^2 Q_1 + 2\lambda\beta Q_0 + E/\pi\gamma^2 = 0 \tag{46}$$

and

$$\lambda^2(1+S_1) + 2\lambda\beta(1+S_0) + \beta^2 - F\gamma^2 = 0. \tag{47}$$

As $(E, F) \neq (0,0)$, we conclude that (46) and (47) have no common components. So, from Bezout's Theorem, these curves are four points of intersection (counting with multiplicities). Let $g_1 = g_1(\alpha), g_1 = (\lambda_1, \beta_1, \gamma_1) \in \mathbf{P}^2\mathbf{C}$, one of these points of intersection. We will prove that $\gamma_1 \neq 0$.

We reason by contradiction to prove this last statement. If $\gamma_1 = 0$, then from (40), (46), and (47) we conclude that,

$$\lambda_1 \neq 0, \beta_1 \neq 0, \lambda_1 Q_1 + 2\beta_1 Q_0 = 0 \tag{48}$$

and

$$\lambda_1^2(1+S_1) + 2\lambda_1\beta_1(1+S_0) + \beta_1^2 = 0. \tag{49}$$

These equations, together with (40) imply that

$$\lambda_1^2[(1+S_1)Q_0^2 - (1+S_0)Q_1Q_0 + 1/4Q_1^2] = \lambda_1^2\varphi(\alpha) = 0, \tag{50}$$

where $\varphi(\alpha)$ is the holomorphic fuction defined in (43). But, from the first case, we know that $\alpha = 0$ is an isolated zero of φ. This contradicts (48) and (50). So, the points of intersection $g_1 = g_1(\alpha)$ are such that $\gamma_1 \neq 0$.

As $E \neq 0$, and g_1 is a solution of (46), we conclude that $\lambda_1 \neq 0$. Then, we can write in homogeneous coordinates $g_1 = (\lambda, \beta, 1)$ and then (λ, β) with $\lambda \neq 0$ are the desired solutions of (36) and (37) for the second case.

In order to finish the proof of Lemma 1, we need only to complete the proof of (29). That is, we need to show that for small $\alpha \neq 0$,

$$\lambda e^{(\alpha/2)e_j} + \beta \neq 0, j = 1, 2, 3. \tag{51}$$

We observe that $\lambda = \lambda(\alpha), \beta = \beta(\alpha)$, are solutions of (36) and (37), where $\lambda \neq 0$. This implies that

$$\left[1 + S_1 + 2(1 + S_0)\frac{\beta}{\lambda} + \left(\frac{\beta}{\lambda}\right)^2\right]\frac{E}{\pi} + \left[Q_1 + 2Q_0\frac{\beta}{\lambda}\right]F = 0. \tag{52}$$

Now suppose that (51) is not true. That is, for some $j \in \{1, 2, 3\}$ and for a sequence $\alpha_n, \alpha_n \to 0$, the solutions $\lambda_n = \lambda(\alpha_n), \beta_n = \beta(\alpha_n)$ are such that

$$-\beta = \lambda e^{(\alpha/2)e_j}.$$

Then from (20), (21), (31), and (52) we can define in a neighbourhood of $0 \in \mathbf{C}$ a holomorphic function $\theta_j(\alpha)$, by the expression

$$\theta_j(\alpha) = \left[1 + S_1 - 2(1 + S_0)e^{(\alpha/2)e_j} + e^{\alpha e_j}\right]\frac{E}{\pi} + \left[Q_1 - 2Q_0 e^{(\alpha/2)e_j}\right]F. \tag{53}$$

Notice that $\theta_j(\alpha_n) = 0$ implies that $\theta_j(\alpha) = 0$ for every α in a neighbourhood of $\alpha = 0$. Observe that for a fixed lattice $L(\tau)$, E and F are constants. Suppose that $E \neq 0$. Then $f(\alpha) = \frac{\pi}{E}\theta_j(\alpha)$ is such that $f^{(k)}(0) = 0$ for every $k = 0, 1, 2, \ldots$, where $f^{(k)}$ is the k-derivative of the holomorphic fuction $f(\alpha)$. Now by using that $e_1 + e_2 + e_3 = 0$ together with (3), (20), (21), (31), (48) and $f^{(2)}(0) = 0$, we conclude that

$$g_2 = 12e_j(2\delta - e_j) \text{ and } g_3 = -8e_j^2(3\delta - e_j), \tag{54}$$

where $\delta = \pi F/E$. These results together with $f^{(3)}(0) = 0$ show that $e_j = 0$ or $e_j = \delta$. In this situation, we find from (54) that $g_2 = g_3 = 0$ or $g_2 - 12e_j^2 = (e_j - e_k)(e_j - e_t) = 0$, where (j, k, t) is a permutation of $(1, 2, 3)$. But, for every lattice $L(\tau), \tau \in FM$ we have $(g_2, g_3) \neq (0, 0)$ and $e_j \neq e_k$, if $j \neq k$. This contradiction shows that (51) is true with the hypothesis $E \neq 0$. In the same way we can prove (51) for the case $F \neq 0$. This finishes the proof of Lemma 1.

Proof of Theorem C. Let $\tau \in FM$ fixed. We will prove that for each $0 < |\alpha| < \varepsilon$, the pair (g_α, η_α) given by Lemma 1 is a Weierstrass representation of a complete minimal immersion of $M = \mathbf{C}/L(\tau) \setminus \{\pi(0)\}$ in $\mathbf{R}^3$ with the properties of Theorem C. From (29) we can conclude that $\Lambda = \{\pi(w_j), j = 1, 2, 3\}$ is, respectively, the set of poles of g_α (with multiplicity one) and the set of zeros of η_α (with multiplicity two) on M. So, (g_α, η_α) satisfies (c_1) of Theorem 1.

On the other hand, as $g_\alpha^2 \eta_\alpha$ and η_α are holomorphic differentials on M with only a singularity at $\pi(0) \in \mathbf{C}/L(\tau)$, we conclude that

$$Res_{\pi(0)} g^2 \eta = Res_{\pi(0)} \eta = 0. \tag{55}$$

Also, $g_\alpha \eta_\alpha$ are exact differentials on M. So,

$$\int_l g_\alpha \eta_\alpha = 0, \tag{56}$$

for every closed curve $l \subset M$. From (53), (54), and by using Lemma 1 we can conclude that (g_α, η_α) satisfy (c_2) of Theorem 1. Also, as η_α have poles of order six at $\pi(0)$, we conclude that (g_α, η_α) satisfy (c_3) of Theorem 1. This completes the proof of Theorem C.

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Departamento de Geometria
Universidade Federal Fluminense
Praça do Valonguinho
24020-005 Niteroi-RJ
Brazil