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THE INDEX OF CONSTANT MEAN CURVATURE SURFACES IN HYPERBOLIC 3-SPACE

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This is a report on a joint work with Pierre Berard and Walcy Santos. Before stating the results, let me fix the notation and present some perspective on the problem.

Let $x : M^2 \to H^3(-1)$ be a complete noncompact surface with constant mean curvature H in the hyperbolic 3-space $H^3(-1)$ of constant sectional curvature -1. Let T_pM be the tangent plane of $M^2 = M$ at a point $p \in M$, and let $A : T_pM \to T_pM$ be the linear operator associated to the second fundamental form of x. When dealing with constant mean curvature, it is convenient to introduce the operator $\phi = HI - A$.

The question 1 want to talk about has to do with the Morse index of x. For completeness, let me present the pertinent definitions.

On M introduce the differential operator

$$L = \Delta + | A |^2 - 2$$

acting on the compactly supported smooth functions $C_0^\infty(M)$ of M. Here Δ is the (positive) Laplacian in the induced metric of M and $| A |$ is the norm of A. Let $I(f) = -\int_M fLf$ be the quadratic form on the vector space $C_0^\infty(M)$ associated to L. Given a compact domain $K \subset M$, the *Morse index* Ind K of K is the index of the quadratic form I restricted to the functions with support in K, and the index Ind M of M is, by definition,

$$\mathrm{Ind} M = \sup_{K \subset M} \mathrm{Ind}\, K \, .$$

The geometric meaning of Ind K is that it measures the number of (linearly independent) functions supported in K that can be used as variations that "decrease area" (cf. [1]). Although it can be proved that Ind K is finite, Ind M may well become infinite. It is therefore an interesting question to find conditions that ensure that Ind $M < \infty$. For instance, in the case of complete minimal surfaces in R^3, it is a theorem that Ind M (which can be defined in a way similar to the above; cf. [1]) is finite if and only if $\int_M |A|^2 < \infty$.

In the present situation, the following results were known: 1) If $H^2 > 1$, then Ind $M < \infty$ if and only if M is compact (cf. [4]); 2) If $H^2 = 1$, then Ind $M < \infty$ if and only if $\int_M |\phi|^2 < \infty$ (cf. [2]). In 1990, I proposed the following conjecture (cf. [1]) to complete the above picture.

Conjecture. *If $H^2 < 1$, then*

$$\int |\phi|^2 < \infty \Rightarrow \text{Ind } M < \infty ,$$

and the converse is false.

That the converse is false follows easily from an example in [4] p. 635. Futhermore, in the particular case of minimal surfaces in $H^3(-1)$, the conjecture was shown to hold true by G. de Oliveira [3]. Recently, in a joint work of Berard, do Carmo and Santos, the conjecture was shown to be true in general. Thus we can replace in the above statement the word conjecture by theorem.

We will now describe the main steps in the proof of the above theorem. Complete details will appear elsewhere.

Assume that $\int_M |\phi|^2 < \infty$, $H^2 < 1$ and set $a = 2(1 - H^2) > 0$. We must prove the following assertions.

Assertion 1. *There exists a compact domain $K \subset M$ such that for all $f \in C_0^\infty(M - K)$ we have*

$$I(f) \geq \frac{a}{2} \int_M f^2.$$

The proof of Assertion 1 is inspired in a proof of a weaker assertion given in [3]. It depends on establishing a version of Simon's inequality for the tensor ϕ which in our case reads

$$\Delta | \phi |^2 \leq | \phi |^4 + a | \phi |^2 .$$

Assertion 2. *There exists a real number β such that for all $f \in C_0^\infty(M)$ we have*

$$I(f) \geq \beta \int_M f^2.$$

The proof of assertin 2 depends only on the fact that $I(f) \geq 0$ for $f \in C_0^\infty(M - K)$, where K is some compact domain in M, or, in other words, that M is stable outside some compact domain. This follows from (but it is weaker than) Assertion 1. The stronger form os Assertion 1 will, however, be used later.

Assertion 3. *The operator L (which is defined in $C_0^\infty(M)$) has a unique self-adjoint extension $\overline{L}$ to the Hilbert space $L^2(M)$.*

We now need to recall a few definitions. From Assertion 3, we can talk about the spectrum $Sp\overline{L}$ of $\overline{L}$, namely,

$$Sp\overline{L} = \{\lambda \in R;\ \overline{L} - \lambda I \text{ has no continuous inverse}\}.$$

We say that $\lambda \in Sp\overline{L}$ is an *eigenvalue* of $\overline{L}$ if Kern $(\overline{L} - \lambda I) \neq 0$ and the dimension of kernel is the *multiplicity* of the eigenvalue λ. Futhermore, a point $\lambda \in Sp\overline{L}$ is called *nonessential* if both λ is isolated and dim Ker$(\overline{L} - \lambda I)$ is finite. The *essential spectrum* $Sp_{ess}\overline{L}$ of $\overline{L}$ is defined as

$$Sp_{ess}\overline{L} = Sp\overline{L} - \{\text{nonessential points}\}.$$

Assertion 4. $\inf Sp_{ess}\overline{L} \geq \frac{a}{2} > 0$.

To prove Assertion 4, we first show that

$$\inf Sp_{ess}\overline{L} = \sup_{K \subset M} \{\inf I(f),\ f \in C_0^\infty(M - K)\ ,\ \textstyle\int_M f^2 = 1\}.$$

and use this together with Assertion 1.

To conclude the proof, we observe that, by Assertion 2, $Sp_{ess}\overline{L} \subset [\beta,\infty)$. Therefore, by Assertion 4, the number of negative eigenvalues of $\overline{L}$ (counted with multiplicities) is finite, and this is easily seen to be equivalent to the finiteness of the index of M.

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