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SUBMANIFOLDS OF CONSTANT NON NEGATIVE CURVATURE

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1. Introduction

The n-dimensional submanifolds $M^n(K)$ of the euclidean space $\mathbb{R}^{2n-1}$ with constant sectional positive curvature $K > 0$ and no weak umbilic points, are in correspondence with solutions of a system of differential equations called the *intrinsic generalized elliptic sinh-Gordon equation*. Similarly, flat submanifolds with no weak umbilic points M^n of the hyperbolic space H^{2n-1} correspond to solutions of the *intrinsic generalized Laplace equation*. The correspondence between the submanifolds $M^n(K)$ contained in a space form $\tilde{M}^{2n-1}(\tilde{K})$ such that $K > \tilde{K}$, with no weak umbilic points, and the solutions of the equations mentioned above was obtained by Tojeiro [15] inspired by the work of Moore [8] and the dual results for the case $K < \tilde{K}$ considered by Aminov [1], Tenenblat-Terng [13] for $\tilde{K} = 0$ and by Tenenblat [12] and Beals-Tenenblat [2] for any $\tilde{K}$. The submanifolds in the latter case are in correspondence with the solutions of the intrinsic generalized equation ($K = 0$) and the intrinsic generalized sine-Gordon equation ($K \neq 0$).

In [10] (see also [9]) Rabelo-Tenenblat considered the special solutions of these equations which depend only on one variable and proved that the associated submanifolds are toroidal submanifolds. Such submanifolds are generated by a curve in such a way that each point of the curve describes an $(n-1)$-dimensional torus. Moreover, they also provided a classification of the toroidal submanifolds M^n of $\mathbb{R}^{2n-1}$ and the toroidal flat submanifolds M^n of S^{2n-1}.

In this note, we consider the special solutions of the intrinsic generalized elliptic sinh-Gordon equation and the intrinsic generalized Laplace equation that depend on one variable (Theorems 3.1, 3.2, 4.1, and 4.2) and we obtain the associated submanifolds, which in contrast with the dual case considered in [10], they are not always toroidal submanifolds.

In the case of constant positive curvature the special solutions exist only for $n \leq 3$, and the associated submanifolds are toroidal submanifolds generated either by curves which are given explicitly in terms of a family of elliptic functions or by a family of helices in $\mathbb{R}^3$. These provide a classification of the constant positive curvature toroidal submanifolds with no weak umbilic points (see Theorems 3.2-3.5). As an immediate consequence, one concludes that there are no complete toroidal submanifolds with no weak umbilic points M^n in $\mathbb{R}^{2n-1}$, with constant sectional curvature 1.

The flat n-dimensional submanifolds of the hyperbolic space H^{2n-1}, with no weak umbilic points, which correspond to the special solutions of the intrinsic generalized Laplace equation are given explicitly in Theorems 4.3 and 4.4. We observe that not all of them are toroidal submanifolds. However, those submanifolds which are toroidal are the only flat toroidal submanifolds of the Lorentzian space L^{2n} contained H^{2n-1} with no weak umbilic points. In particular, we conclude that the only complete flat toroidal submanifold of H^{2n-1}, with no umbilic points, is generated by a plane curve (see Theorem 4.5).

We want to observe that besides the solutions given in this paper for the intrinsic generalized elliptic sinh-Gordon and Laplace equations, one can also obtain explicit solutions by the method of Backlund Transformations (see [4]).

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2. Preliminary Results

We consider $M^n(K)$ an n-dimensional manifold of constant sectional curvature K. The following theorem for submanifolds $M^n(K) \subset \tilde{M}^{2n-1}(\tilde{K})$, $K > \tilde{K}$, are known (see [4], [8] and [15]).

Theorem 2.1. *Let $M^n(K)$ be a riemannian manifold isometrically immersed in $\tilde{M}^{2n-1}(\tilde{K})$, such that $K > \tilde{K}$. Then in a neighboorhood with no weak umbilic points there exist local coordinates $(x_1, \ldots, x_n)$ such that the first and second fundamental forms are given by*

$$I = \sum_{i=1}^{n} \frac{a_{1i}^2}{K - \tilde{K}} dx_i^2 \qquad II = \sum_{i=1,j=2}^{n} \frac{a_{ji}a_{1i}}{\sqrt{K - \tilde{K}}} dx_i^2 e_{n+j-1}, \tag{2.1}$$

where $a_{11}^2 - \sum_{j=2}^{n} a_{1j}^2 = 1$ and e_{n+j-1}, $2 \le j \le n$, is an orthonormal frame normal to M.

Without loss of generality, we normalize the curvatures by considering $K - \tilde{K} = 1$. Under the conditions of Theorem 2.1, one can show [15] that the $n \times n$ matrix function $a = (a_{ij})$ satisfies the following system of equations

$$aJa^t = J \tag{2.2}$$

$$\frac{\partial a_{1i}}{\partial x_j} = a_{1j}h_{ji}\,, \quad i \neq j \tag{2.3}$$

$$\frac{\partial h_{ij}}{\partial x_i} + \frac{\partial h_{ji}}{\partial x_j} + \sum_{s \neq i,j} h_{si}h_{sj} = -K a_{1i}a_{1j}\,, \quad i \neq j \tag{2.4}$$

$$\frac{\partial h_{ij}}{\partial x_s} = h_{is}h_{sj}\,, \quad i, j, s \text{ distinct} \tag{2.5}$$

$$\frac{\partial a_{j\ell}}{\partial x_i} = a_{ji}h_{i\ell}\,, \quad i \neq \ell, \ j \ge 2 \tag{2.6}$$

where the off diagonal matrix function $h = (h_{ij})$ is defined by (2.3) and $J = (J_{ij})$ is the $n \times n$ diagonal matrix

$$J = diag(1, -1, \ldots, -1). \tag{2.7}$$

When $K = 1$ this is the *Generalized Elliptic Sinh-Gordon Equation* (GESGE) and when $K = 0$ this is the *Generalized Laplace Equation* (GLE).

The above equations are equivalent to the Gauss and Codazzi equations. As a consequence of the fundamental theorem for submanifolds, given a matrix function a, which satisfies the equations (2.2)-(2.6), defined on a simply connected open subset $\Omega \subset \mathbb{R}^n$, there exists an isometric immersion $X : \Omega \to \tilde{M}^{2n-1}(\tilde{K})$ whose first and second fundamental forms are given by (2.1).

In the two-dimensional case, the Codazzi equation (2.6) is a consequence of the Gauss equation (2.4) and (2.5). Motivated by this result, intrinsic generalizations for the elliptic sinh-Gordon and Laplace equations are introduced. The following result is essentially due to Bianchi [3] (see also [2], [15]

Theorem 2.2. *Let Ω be a simply connected open subset of $\mathbb{R}^n$ with coordinates $x_1, \ldots, x_n$ endowed with a riemannian metric $g = (g_{ij})$ of constant sectional curvature K. Suppose g is diagonal and $g_{11} - \sum_{j=2}^{n} g_{jj} = 1$. Then the smooth function $v : \Omega \to \mathbb{R}^n$, $\mathrm{v} = (\mathrm{v}_1, \ldots, \mathrm{v}_n)$, defined by $v_i^2 = g_{ii}$ satisfies*

$$vJv^t = 1 \tag{2.8}$$

$$\frac{\partial v_i}{\partial x_j} = v_j h_{ji}, \quad i \neq j \tag{2.9}$$

$$\frac{\partial h_{ij}}{\partial x_i} + \frac{\partial h_{ji}}{\partial x_j} + \sum_{s \neq i,j} h_{si} h_{sj} = -K v_i v_j\,, \quad i \neq j \tag{2.10}$$

$$\frac{\partial h_{ij}}{\partial x_s} = h_{is} h_{sj}\,, \quad i, j, s \text{ distinct} \tag{2.11}$$

where $1 \leq i, j, s \leq n$, the off diagonal matrix function $h = (h_{ij})$ is defined by (2.9) and J is the diagonal matrix given by (2.7). Conversely, given a solution of (2.8)-(2.11) such that $v_i(x) \neq 0$, for all $x \in \Omega$, then $g_{ij} = \delta_{ij} v_i^2$ defines a metric on Ω which satisfies the above conditions.

One can show that, whenever $v_i(x) \neq 0$, for all $x \in \Omega$ and $1 \leq i \leq n$, then

v and h satisfy

$$J_{ii}\frac{\partial v_i}{\partial x_i} + \sum_{j\geq 1} J_{jj}v_j h_{ij} = 0 \tag{2.12}$$

$$J_{ii}\frac{\partial h_{ji}}{\partial x_i} + J_{jj}\frac{\partial h_{ij}}{\partial x_j} + \sum_{s\neq i,j} J_{ss}h_{is}h_{js} = 0, \quad i \neq j\,. \tag{2.13}$$

The system (2.8)-(2.13) is called the *Intrinsic Generalized Elliptic Sinh-Gordon Equation* (IGESGE) when $K = 1$ and the *Intrinsic Generalized Laplace Equation* (IGLE) when $K = 0$.

In what follows, we consider solutions v_i of these equations, which define metrics as in Theorem 2.2. Therefore, the intrinsic equations reduce to (2.8)-(2.11). Under these conditions, the relation between the generalized equations is stated in the following result, whose proof follows the same arguments used in [2] (see also [15]).

Theorem 2.3.

(i) *If a is a solution of the GLE, then each row of a, whose elements do not vanish, is a solution of the IGLE.*

(ii) *Suppose a is a solution of the GESGE. Then the first row of a is a solution of the IGESGE, whenever its elements do not vanish.*

(iii) *Conversely, if v is a solution of the IGESGE (resp. IGLE) whose coordinate functions do not vanish on a simply connected domain $\Omega \subset \mathbb{R}^n$, then there exists a solution a on Ω for the GESGE (resp. GLE) whose first row is v. Moreover, if a and $\tilde{a}$ are two these solutions, then $\tilde{a} = PJa$, where $P \in O(1, n-1)$ is a constant matrix and J is the diagonal matrix given by (2.7).*

It follows from the above results that the solutions v of the IGESGE (resp. IGLE), which do not vanish on a simply connected open subset $\Omega \subset \mathbb{R}^n$ are in correspondence with the isometric immersion $X : \Omega \to \mathbb{R}^{2n-1}$ (resp. $X : \Omega \to H^{2n-1}$) of constant curvature $K = 1$ (resp. $K = 0$). Such an immersion is determined up to rigid motions and is called *the immersion associate to* v.

We denote by L^{2n} the space of vectors $(x_1, \dots, x_{2n})$, $x_j \in \mathbb{R}$, endowed with the Lorentzian metric $\langle\langle x, y\rangle\rangle = -x_1 y_1 + \sum_{j=2}^{2n} x_j y_j$, and now we introduce the definition of toroidal submanifolds.

Definition.

(i) *Let* $\alpha(t) = (f_1(t), \dots, f_n(t))$, $t \in I \subset \mathbb{R}$, *be a parametrization of a regular curve em* $\mathbb{R}^n$, $n \geq 3$, *such that for all* $j \geq 2$, f_j *do not vanish in the open interval* I. *The submanifold which up to a rigid motion of* $\mathbb{R}^{2n-1}$ *is given by*

$$(f_1(t),\ f_2(t) cos\, x_1,\ f_2(t) sin\, x_1, \dots, f_n(t) cos\, x_{n-1},\ f_n(t) sin\, x_{n-1})$$

is called a toroidal submanifold M^n *of* $\mathbb{R}^{2n-1}$ *generated by the curve* α.

(ii) *Let* $\beta(t) = (f_0(t),\ f_1(t), \dots, f_n(t))$, $t \in I \subset \mathbb{R}$, *be a parametrization of a regular curve in* $\mathbb{R}^{n+1}$, $n \geq 3$, *such that, for all* $j \geq 2$, f_j *do not vanish in the open interval* I. *The submanifold which up to a rigid motion of* L^{2n} *is given by*

$$(f_0(t),\ f_1(t),\ f_2(t) cos\, x_1,\ f_2(t) sin\, x_1, \dots, f_n(t) cos\, x_{n-1},\ f_n(x_1) sin\, x_{n-1})$$

is called a toroidal submanifold M^n *of* L^{2n} *generated by the curve* β.

A toroidal submanifold is generated by a curve in such a way that each point of the curve describes a flat $(n-1)$-dimensional torus T^{n-1} contained in $\mathbb{R}^{2n-2}$.

3. Submanifolds of Curvature 1 of Euclidean Space

The solutions of the intrinsic generalized elliptic sinh-Gordon equation which depend only on one independent variable are given in the following two result. Without loss of generality we may assume that the independent variable on which the solutions depend is x_1 or x_2. The proofs in this section follow by modifying conveniently the proofs in [10].

Proposition 3.1. *Let $v = (v_1, \ldots, v_n)$, $n \geq 2$, be a solution of the IGESGE which depends only on x_1, such that $v_i(x_1) \neq 0$, $1 \leq i \leq n$, for x_1 in an open interval $I \subset \mathbb{R}$.*

Then $n \leq 3$ and

(i) If $n = 2$,

$$\begin{aligned}(v_1')^2 &= (v_1^2 - 1)(r^2 - v_1^2)\\ v_2^2 &= v_1^2 - 1\end{aligned} \tag{3.1}$$

where $1 < r \in \mathbb{R}$;

(ii) *if $n = 3$ and $p^2 = 1$,*

$$\begin{aligned}v_1 &= m\\ v_2 &= pb\, sen(mx_1 - \delta)\\ v_3 &= b\cos(mx_1 - \delta)\end{aligned} \tag{3.2}$$

where $b, m, p, \delta \in \mathbb{R}$, $\mathrm{m}^2 > 1$, $\mathrm{m}^2 - \mathrm{b}^2 = 1$, *and* $x_1 \in I$ *such that* $\ell\pi/2 < mx_1 - \delta < (\ell + 1)\pi/2$, $\ell \in \mathbb{Z}$;

(iii) *if $n = 3$ and $p^2 \neq 1$,*

$$\begin{aligned}(v_1')^2 &= \left(v_1^2 - \frac{p^2 + q^2 - 1}{p^2}\right)(q^2 - v_1^2)\\ v_2^2 &= \frac{p^2}{p^2 - 1}\left(v_1^2 - \frac{p^2 + q^2 - 1}{p^2}\right)\\ v_3^2 &= \frac{1}{p^2 - 1}\left(q^2 - v_1^2\right)\end{aligned} \tag{3.3}$$

where $p, q, \in \mathbb{R}^*$, $\mathrm{p}^2 \neq 1$, $\mathrm{q}^2 > 1$.

Proof: It follows from the hypothesis that equations (2.8)-(2.11) reduce to

$$\begin{aligned}
vJv^t &= 1 && (3.4)\\
v_1' &= \sum_{s\geq 2} v_s h_{1s} && (3.5)\\
v_j' &= v_1 h_{1j}, \quad j \geq 2 && (3.6)\\
h_{ij} &= 0, \quad i \geq 2 && (3.7)\\
h_{1i}h_{1j} &= -v_i v_j, \quad i \neq j, \quad i,j \geq 2 && (3.8)\\
h_{1j}' &= -v_1 v_j, \quad j \geq 2, && (3.9)
\end{aligned}$$

where $1 \leq i,j,s \leq n$.

If $n = 2$, it is easy to see that this system is equivalent to (3.1).

If $n \geq 3$, using (3.6), the equations (3.9) and (3.8) reduce to

$$\left(\frac{v_j'}{v_1}\right)' = -v_1 v_j, \quad j \geq 2, \tag{3.10}$$

$$\frac{v_i'}{v_1} \cdot \frac{v_j'}{v_1} = -v_i v_j\,, \quad i \neq j, \quad i,j \geq 2. \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$v_j' = c_{ji} v_1 v_i\,, \quad i \neq j, \quad i,j \geq 2, \tag{3.12}$$

where c_{ji} is a nonzero real constant.

If $n = 3$, then the equations (3.12) reduce to

$$v_2' = p v_1 v_3 \tag{3.13}$$

$$v_3' = -\frac{1}{p} v_1 v_2, \tag{3.14}$$

$p \in \mathbb{R}^*$. Therefore, we have

$$v_1' = \left(p - \frac{1}{p}\right) v_2 v_3. \tag{3.15}$$

If $p^2 = 1$, then v_1 is a constant m with $m^2 > 1$ and it is easy to obtain the solutions (3.2).

If $p^2 \neq 1$, then it follows from the equations (3.14) and (3.15) that

$$v_1 v_1' + (p^2 - 1) v_3 v_3' = 0,$$

i.e. $v_3^2 = \frac{1}{p^2-1}(q^2 - v_1^2), \quad q \in \mathbb{R}, \quad q^2 > 1.$

Now, using (3.4) we have

$$v_2^2 = \frac{p^2}{p^2-1}\left(v_1^2 - \frac{p^2+q^2-1}{p^2}\right).$$

Hence, from the equation (3.15) we obtain

$$(v_1')^2 = \left(v_1^2 - \frac{p^2+q^2-1}{p^2}\right)(q^2 - v_1^2).$$

Finally, if $n \geq 4$ from the equation (3.12) we have

$$v_j(x_1) = c_j f(x_1), \quad j \geq 2, \tag{3.16}$$

where c_j is a nonzero constant and f is a nowhere zero real function on I. We observe that it follows from (3.10) that $v_\ell' \neq 0, \quad \ell \geq 2$. Substituting (3.16) into (3.11), we obtain

$$(f')^2 + v_1^2 f^2 = 0,$$

i.e. $f = 0$.

■

Similar arguments provide the solutions which depend only on x_2.

Proposition 3.2. *Let $v = (v_1, \ldots, v_u)$, $n \geq 2$, a solution of the IGESGE which depends only on x_2, such that $v_i(x_2) \neq 0$, $1 \leq i \leq x$, for x_2 in an open interval $I \subset \mathbb{R}$.*

Then $n \leq 3$ and

i) *If $n = 2$,*

$$\begin{aligned} (v_2')^2 &= (1+v_2^2)(r^2 - v_2^2) \\ v_1^2 &= 1 + v_2^2 \end{aligned} \tag{3.17}$$

where $0 < r \in \mathbb{R}$;

(ii) *if* $n = 3$,

$$\begin{aligned} (v_2^1)^2 &= (v_2^2 + q^2)(p^2q^2 - p^2 - 1 - v_2^2) \\ v_1^2 &= \frac{p^2}{p^2+1}(v_2^2 + q^2) \\ v_3^2 &= \frac{1}{p^2+1}(p^2q^2 - p^2 - 1 - v_2^2) \end{aligned} \tag{3.18}$$

where $p, q \in \mathbb{R}^*$, $q^2 > 1$.

One can show that the isometric immersions $M^n \hookrightarrow \mathbb{R}^{2n-1}$ of constant curvature $K \equiv 1$ associated to these solutions are, up to rigid motions, toroidal submanifolds.

Theorem 3.3. *The submanifolds $M^n \subset \mathbb{R}^{2n-1}$ with constant sectional curvature $K \equiv 1$, associated to the solutions of the IGESGE given in Proposition 3.1 are, up to a rigid motion,*

(i) *the surface of rotation generated by the curve*

$$\left(\frac{1}{r}\int v_1^2 dx_1,\ \frac{v_2}{r}\right), \quad \text{if} \quad n = 2,$$

where v_1 e v_2 are definid by (3.1);

(ii) *the toroidal submanifolds generated by the helices*

$$(x_1 - x_1^0, \frac{v_2}{m}, \frac{v_3}{m}), \quad \text{if} \quad n = 3,$$

where $m^2 - b^2 = 1$, $p^2 = 1$, $\ell\pi/2 < mx_1 - \delta < (\ell+1)\pi/2$, $\ell \in \mathbb{Z}$, and v_1, v_2, v_3 are defined by (3.2);

(iii) *the toroidal submanifolds generated by the curves*

$$\left(\frac{p}{q\sqrt{p^2+q^2-1}}\int v_1^2 dx_1,\ \frac{v_2}{q},\ \frac{pv_3}{\sqrt{p^2+q^2-1}}\right)$$

where v_1, v_2, v_3 are defined by (3.3).

Proof: Let $v = (v_1, \dots, v_n)$ be a solution of the IGESGE as in Proposition 3.1. By Theorem 2.3 there exists a solution (a_{ij}) of the GESGE such that $a_{1j} = v_j$, for $1 \leq j \leq n$. Moreover, a_{ij}, $2 \leq i \leq n$, also depend only on x_1.

The submanifold associated to v is determined by the first and second fundamental forms given by

$$I = \sum_{i=1}^{n} a_{1i}^2 dx_i^2 \qquad \text{and} \qquad II = \sum_{j=2}^{n} \left(\sum_{i=1}^{n} a_{ji} a_{1i} dx_i^2 \right) e_{n+j-1}$$

where e_{n+j-1}, $2 \leq j \leq n$, is an orthonormal frame for the normal bundle. The proof of the theorem follows from the fundamental theorem for submanifolds of the euclidean space, which says that the immersion $X : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{2n-1}$, where Ω is a simply connected domain, is determined, up to a rigid motion, by solving the following system of differential equations for the vector fields X_{x_i} and e_{n+s-1}, where $1 \leq i \leq n$, $2 \leq s \leq n$,

$$\begin{aligned} X_{x_i x_j} &= \sum_{k=1}^{n} \Gamma_{ij}^k X_{x_k} + \sum_{r=2}^{n} a_{1i} a_{ri} \delta_{ij} e_{n+r-1} \\ e_{n+s-1, x_i} &= -\frac{a_{si}}{a_{1i}} X_{x_i} . \end{aligned}$$

Since the riemannian metric g is diagonal and the functions $g_{ii} = a_{ii}^2$ depend only on x_1, the Chritoffel symbols are given by

$$\begin{aligned} \Gamma_{ij}^k &= 0, \quad i, j, k \quad \text{distinct} \\ \Gamma_{ij}^i &= \Gamma_{ji}^i = \Gamma_{ii}^j = 0, \quad j \geq 2 \\ \Gamma_{i1}^i &= \Gamma_{1i}^i = \frac{a_{1i}'}{a_{1i}} \\ \Gamma_{ii}^1 &= -\frac{a_{1i} a_{1i}'}{a_{11}^2}, \quad i \geq 2. \end{aligned}$$

Therefore the parametrization X is obtained by solving the following system of partial differential equations,

$$
\begin{aligned}
X_{x_1x_1} &= \frac{a'_{11}}{a_{11}}X_{x_1} + a_{11}\sum_{s\geq 2} a_{s1}e_{n+s-1} \\
X_{x_1x_j} &= \frac{a'_{1j}}{a_{1j}}X_{x_j}\,, \quad j \geq 2 \\
X_{x_ix_i} &= -\frac{a_{1i}a'_{1i}}{a_{11}^2}X_{x_1} + a_{1i}\sum_{s\geq 2} a_{si}e_{n+s-1}\,, \quad i \geq 2 \\
X_{x_ix_j} &= 0\,, \quad i \neq j\,, \quad i,j \geq 2 \\
e_{n+s-1,x_i} &= -\frac{a_{si}}{a_{1i}}X_{x_i}\,, \quad s \geq 2, \quad i \geq 1,
\end{aligned}
$$

where $1 \leq i \leq j \leq n$ and $2 \leq s \leq n$, subject to appropriate initial conditions. ■

Similarly, in the following result we obtain the submanifolds of the Euclidean space associated to the solutions given in Proposition 3.2.

Theorem 3.4. *The submanifolds $M^n \subset \mathbb{R}^{2n-1}$ with constant sectional curvature $K \equiv 1$ associated to the solutions of the IGESGE given in Proposition 3.2. are up to a rigid motion,*

i) *the surface of rotation generated by the curve*

$$\left(\frac{1}{r}\int v_2^2 dx_2\,,\ \frac{v_1}{r}\right),$$

where v_1 and v_2 are defined by (3.17);

ii) *the toroidal submanifolds generated by the curves*

$$\left(\frac{1}{q\sqrt{p^2q^2-p^2-1}}\int v_2^2 dx_2\,,\ \frac{v_1}{\sqrt{p^2q^2-p^2-1}}\,,\ \frac{v_3}{q}\right)$$

We can show that these submanifolds characterize the toroidal submanifolds of $\mathbb{R}^{2n-1}$ with constant curvature 1. Consequently, we conclude that there is no complete toroidal submanifolds $M^n \subset \mathbb{R}^{2n-1}$ with $K \equiv 1$.

Theorem 3.5.

(i) *The submanifolds* $M^n \subset \mathbb{R}^{2n-1}$, $n \geq 3$, *given by Theorem 3.3 and 3.4 are, up to a rigid motion, the only toroidal submanifolds of* $\mathbb{R}^{2n-1}$ *with constant sectional curvature* $K \equiv 1$ *and no weak umbilic points.*

(ii) *There is no complete toroidal submanifold* $M^n \subset \mathbb{R}^{2n-1}$, *with* $K \equiv 1$ *and no weak umbilic points.*

Proof: A toroidal submanifold is locally given by

$$X = (f_1(t), f_2(t)\cos x_1, f_2(t)\sin x_1, \ldots, f_n(t)\cos x_{n-1}, f_n(t)\sin x_{n-1}),$$

where $t \in I \subset \mathbb{R}$, $-\pi < x_j < \pi$ and $f_j(t) \neq 0$, $\forall\, t \in I$, $1 \leq j \leq n-1$. We will show that there exists a change of coordinates such that the metric (g_{ij}) is a diagonal matrix whose diagonal satisfies the condition $g_{11} - \sum_{j=2}^{n} g_{jj} = 1$.

We consider the following change of coordinates

$$\tilde{x}_1 = t(s) \qquad \tilde{x}_j = x_{j-1}, \quad 2 \leq j \leq n,$$

where

$$s = \int_0^t \sqrt{\frac{\sum_{j=1}^{n} (f_j')^2}{1 + \sum_{j=2}^{n} f_j^2}}\, dt.$$

Therefore,

$$X_{\tilde{x}_1} \cdot X_{\tilde{x}_1} = 1 + \sum_{j=2}^{n} f_j^2, \quad X_{\tilde{x}_i} \cdot X_{\tilde{x}_j} = \delta_{ij} f_i^2$$

and, consequently, $g_{11} - \sum_{j=2}^{n} g_{jj} = 1$.

The first fundamental form of this parametrization is given by

$$I = \sum_{i=1}^{n} v_i^2(\tilde{x}_1) d\tilde{x}_i^2,$$

where $v_1^2 = 1 + \sum_{j=2}^{n} f_j^2$ and $v_j^2 = f_j^2$, $j \geq 2$. Moreover, $v_1^2 - \sum_{j=2}^{n} v_j^2 = 1$. Since the manifold has curvature $K \equiv 1$, it follows that $v = (v_1, \ldots, v_n)$ is a solution of IGESGE which depends only on one independent variable. Therefore, we are in the cases (ii), (iii) of Proposition 3.1 or ii) Proposition 3.2. The proof of (ii) is an immediate consequence of the part (i) and from the fact that for each solution given in Proposition 3.1 and Proposition 3.2 there exists a value $x_0 \in \mathbb{R}$ such that $v_j \to 0$ when $x \to x_0$.

■

4. Flat Submanifolds of the Hyperbolic Space

We obtain the results analogous to those in section 3, by considering the correspondence between the flat submanifolds of the hyperbolic space and the solutions of the IGLE. The solutions of the IGLE which depend only on one independent variable are given by the following results. Without loss of generality we may assume the independent variable to be x_1 or x_2.

Proposition 4.1. *Let $v = (v_1, \ldots, v_n)$, $n \geq 2$, be a solution of the IGLE which depends only on x_1, such that $v_i(x_1) \neq 0$, $1 \leq i \leq n$, for x_1 in an open interval $I \subset \mathbb{R}$. Then either v is constant or there exists $j_0 \geq 2$ such that*

$$\begin{aligned} v_1 &= \epsilon\lambda cosh(mx_1 - \delta) \\ v_{j_0} &= \lambda sinh(mx_1 - \delta) \\ v_j &= b_j, \quad j \neq j_0, \quad j \geq 2, \end{aligned} \tag{4.1}$$

where $m, b_j, \lambda, \delta \in \mathbb{R}$, $\epsilon^2 = 1$, $\sum_{\substack{j=2 \\ j\neq j_0}}^{n} b_j^2 = \lambda^2 - 1$, $b_j \neq 0$, $m \neq 0$ and I does not contain δ/m.

Proposition 4.2. *Let $v = (v_1, \ldots, v_n)$, $n \geq 2$, be a non constant solution of the IGLE which depends only on x_2, such that $v_i(x_2) \neq 0$, $1 \leq i \leq n$, for x_2 in an open interval $I \subset \mathbb{R}$. Then*

i)
$$\begin{aligned} v_1 &= e\lambda\cosh\,(mx_2-\delta) \\ v_2 &= \lambda\,\mathrm{senh}(mx_2-\delta) \\ v_j &= b_j \quad , \quad j \geq 3, \end{aligned} \tag{4.2}$$

where $m, b_j, \lambda, \delta \in \mathbb{R}$, $\epsilon^2 = 1$, $\sum_{j=3}^{n} b_j^2 = \lambda^2 - 1$, $b_j \neq 0$, $m \neq 0$ *and* I *does not contain* δ/m.

ii) $\exists\, j_0 \geq 3$ *such that*

$$\begin{aligned} v_2 &= \epsilon\lambda\cos(m_2x_2-\delta) \\ v_{j_0} &= \lambda\,\sin(mx_2-\delta) \\ v_j &= b_j\ ,\ j \neq j_0\ ,\ j \neq 2 \end{aligned} \tag{4.3}$$

where $m, b_j, \lambda, \delta \in \mathbb{R}$, $\epsilon^2 = 1$, $\sum_{\substack{j\neq 2\\ j\neq j_0}} J_{jj} b_j^2 = \lambda^2 + 1$, $b_j \neq 0$, $m \neq 0$ *and* $x_2 \in I$ *is such that* $\ell\pi/2 < mx_2 - \delta < (\ell+1)\frac{\pi}{2}$, $\ell \in \mathbb{Z}$.

As in Section 3, one can show that the flat isometric immersions in the hyperbolic space, $M^n \subset H^{2n-1} \subset L^{2n}$, associated to the solutions given in Proposition 4.1 are toroidal submanifolds. We observe that H^{2n-1} can be caracterized by the subset of vectors is L^{2n} such that $\langle\langle X, X\rangle\rangle = -1$.

Theorem 4.3. *The flat submanifolds* $M^n \subset H^{2n-1}$, *associated to the solutions of the IGLE given in Proposition 4.1 are, up to a rigid motion, the toroidal submanifolds generated by the curves*

(i) $(b_1 \cosh x_1,\ b_1 \sinh x_1,\ b_2, \ldots, b_n)$,
where $v_j = b_j \neq 0$, $1 \leq j \leq n$, *whenever* $m = 0$.

(ii) $(\lambda f_0, \lambda f_1, \lambda f_2,\ b_3, \ldots, b_n)$,
where f_0, f_1, f_2 *are real functions defined by expressions*

$$\begin{aligned} f_0 &= \frac{r}{m} f_2' \cosh r x_1 - m f_2 \,\mathrm{senh}\, r x_1 \\ f_1 &= \frac{r}{m} f_2' \sinh r\, x_1 - m f_2 \cosh r x_1 \\ f_2 &= \frac{1}{r}\sinh(mx_1 - \delta), \end{aligned} \tag{4.4}$$

with $r^2 = 1 + m^2$ *and* $\sum_{j=3}^{n} b_j^2 = \lambda^2 - 1$, *whenever* $m \neq 0$.

Proof: Let $v = (v_1, \ldots, v_n)$ be a solution of the IGLE given by Proposition 4.1, where without loss of generality we will assume $j_0 = 2$. As in the proof of Theorem 3.2 such submanifolds are given, up to a rigid motion, locally by immersions $X : \Omega \subset \mathbb{R}^n \to \mathrm{H}^{2n-1} \subset \mathrm{L}^{2n}$, where Ω is simply connected. In order to obtain X, we consider the adapted orthonormal frame in L^{2n}

$$e_i = \frac{1}{a_{1i}} X_{x_i}, \quad 1 \leq i \leq n, \qquad e_{n+j-1}, \quad 2 \leq j \leq n, \qquad e_{2n} = X.$$

It is not difficult to see that X has to satisfy the system

$$\begin{cases} X_{x_i x_j} &= \sum_{k=1}^{n} \Gamma_{ij}^k X_{x_k} + \sum_{s=2}^{n} (a_{1i} a_{si} \delta_{ij}) e_{n+s-1} + a_{1i} a_{1j} \delta_{ij} X, \\ e_{n+s-1, x_i} &= -\frac{a_{si}}{a_{1i}} X_{x_i}, \end{cases}$$

where $1 \leq i, j \leq n$ and $2 \leq s \leq n$.

By choosing the initial conditions appropriately we conclude that, up to a rigid motion, the immersion is given by

$$X = (b_1 \cosh x_1, b_1 \sinh x_1, b_2 \cos x_2, b_2 \sin x_2, \ldots, b_n \cos x_n, b_n \sin x_n),$$

where $b_j \neq 0, \quad 1 \leq j \leq n$; or

$$X = (\lambda f_0, \lambda f_1, \lambda f_2 \cos r_2 x_2, \lambda f_2 \sin r_2 x_2, \ldots, b_j \cos x_j, b_j \sin x_j, \ldots) \quad j \geq 3.$$

where $\sum_{j=3}^{n} b_j^2 = \lambda^2 - 1$ and f_0, f_1, f_2 are defined by (4.4).

∎

In the following result we obtain the flat isometric immersion $M^n \subset H^{2n-1} \subset L^{2n}$ associated to the solutions given in Proposition 4.2.

Theorem 4.4. *The flat submanifolds* $M^n \subset H^{2n-1}$, *associated to the solutions of the IGLE given in Proposition 4.2 are, up to a rigid motion,*

i) a) *the toroidal submanifold given by*

$$X = (f_0(x_2), f_1(x_2), \frac{v_1}{r}\cos rx_1, \frac{v_1}{r}\sin rx_1, ..., b_j\cos x_j, b_j\sin x_j, ...) \quad j \geq 3$$

where f_0 *and* f_1 *are defined by the expressions*

$$\begin{aligned} f_0 &= -v_2 f + \epsilon\frac{v_1 m f'}{r^2} \\ f_1 &= -v_2\frac{f'}{r} + \epsilon\frac{v_1 m f}{r} \\ f &= \sinh\, rx_2 \end{aligned}$$

if $r^2 = m^2 - 1 > 0$, *and* v_1, v_2 *given by* (4.2);

b) *the submanifolds parametrized by*

$$X = \Big(\frac{v_1}{r}\cosh rx_1,\ \frac{v_1}{r}\sinh rx_1,\ f_0(x_2),\ f_1(x_2),\ \cdots, b_j\cos x_j,\ b_j\ \sin x_j,\ \cdots,$$

where

$$\begin{aligned} f_0 &= v_2\sin rx_2 + \epsilon\frac{mv_1}{r}\cos rx_2, \\ f_1 &= -v_2\cos rx_2 + \frac{\lambda m v_1}{r}\sin\ rx_2 \end{aligned}$$

if $r^2 = 1 - m^2 > 0$ *and,* v_1, v_2 *are given by* (4.2.)

c) *the submanifolds parametrized by*

$$X = (v_1 + L, -\epsilon L, x_1 v_1, \epsilon x_2 v_1 - v_2, ..., b_j\cos x_j, b_j\sin x_j, ...),\ j \geq 3$$

where

$$L(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)v_1 - \epsilon x_2 v_2,$$

if $m^2 - 1 = 0$ *and,* v_1, v_2 *are given by* (4.2.).

ii) *the submanifolds generated by*

$$X = \Big(b_1\cosh x_1, b_1\sinh x_1, f_0, f_1, \frac{v_{j_0}}{r}\sin rx_{j_0}, \frac{v_{j_0}}{r}\cos rx_{j_0}, ..., b_j\cos x_j, b_j\sin x_j, ...$$

where

$$\begin{aligned} f_0(x_2) &= -v_2\cos rx_2 - \frac{\epsilon m v_{j_0}}{r}\sin\ rx_2 \\ f_1(x_2) &= v_2\sin rx_2 - \frac{\epsilon m v_{j_0}}{r}\cos rx_2, \end{aligned}$$

and v_2 v_{j_0} *are given by* (4.3)

The following theorem gives a classification result analogous to Theorem 3.5.

Theorem 4.5.

(i) *The submanifolds $M^n \subset H^{2n-1}$ given by Theorem 4.3 and Theorem 4.4(i)a are, up to a rigid motion, the only toroidal flat submanifolds of L^{2n} contained in H^{2n-1}, with no weak umbilic points.*

(ii) *The only complete flat toroidal submanifolds $M^n \subset H^{2n-1}$, with no weak umbilic points, is generated by the plane curve*

$$(b_1 cosh\, x_1,\ b_1 sinh\, x_1,\ b_2, \ldots, b_n), \quad b_j \in \mathbb{R}^*, \quad 1 \leq j \leq n.$$

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