Matemática Contemporânea, Vol. 4, 45–54
http://doi.org/10.21711/231766361993/rmc45

GENERALIZED WILLMORE PROBLEM

Abdênago A. de Barros

1 Introduction

For an immersion $f : M^2 \to R^3$ of a closed surface M into the Euclidean space R^3 we will consider the functional

$$C(f) = \frac{1}{2\pi}\int_M (H^2 - K)dM. \tag{1.1}$$

Here H is the mean curvature, K the Gaussian curvature and dM the element of area of M in the induced metric respectivaly. It is important to point out that $C(f)$ is invariant by change of conformal mappings in R^3.

Now if k_1 and k_2 denote the principal curvatures of the shape operator A_ξ in the normal direction ξ then we may write (1.1) in the form

$$C(f) = \frac{1}{8\pi}\int_M (k_1 - k_2)^2 dM \tag{1.2}$$

The variational problem proposed by T. J. Willmore was to achive the infimum of $C(f)$ among compact embedded surfaces M of a given genus. Of course, $C(f) \geq 0$ and equality occur if and only if $f(M)$ is a round sphere. When M is diffeomorphic to a torus T^2 it was conjectured by Willmore that $C(f) \geq \pi$ and $C(f) = \pi$ if and only if $f(T^2)$ is conformally diffeomorphic to a circular vicinity of a circle whose radii are in the rate $1 : \sqrt{2}$, i.e., $f(T^2)$ is obtained from the Clifford torus on S^3 by stereographic projection.

On the other hand, as M is a closed surface $\int_M H^2 dM \geq 4\pi$, so the Gauss-Bonnet formula implies

$$C(f) \geq \beta_1(M; Z_2), \tag{1.3}$$

where β_1 is the first Betti number of M with Z_2 - coefficient.

Now for an immersion $f : M^n \to R^m$ of a closed manifold M on R^m let $N(f)$ be the unit normal bundle of f. Given $\xi \in N_p(f)$ let A_ξ be the shape operator in the direction ξ and $k_1(\xi), ..., k_n(\xi)$ its eigenvalues. Set

$$\sigma(A_\xi)^2 := \frac{1}{n^2} \sum_{i<j} (k_i - k_j)^2 \tag{1.4}$$

and

$$C_1(f) := \frac{1}{vol(S^{m-1})} \int_{N(f)} (\sigma(A_\xi))^n d\xi \tag{1.5}$$

where $vol(S^{m-1})$ is the canonical volume of S^{m-1} and $d\xi$ the natural volume element of $N(f)$. Of course this agrees with $C(f)$ for surfaces and $C_1(f) \geq 0$ with equality if and only if $f(M)$ is a round sphere.

We observe that this functional appears in the literature in many works. We may cite N. Abe, R. Bryant, B. Y. Chen, U. Pinkall, B. White, J. Weiner among others. But our source here is Pinkall's work [9]. There he proved the following fact which generalizes (1.3)

$$C_1(f) \geq \sum_{k=1}^{n-1} a_k \beta_k, \tag{1.6}$$

where $a_k = [k/(n-k)]^{(n-2k)/2}$ and β_k is the k^{th} Z_2- Betti number of M. For M not homeomorphic to S^n he also showed that if $C_1(M) = \inf(C_1(f))$ and $C_1(n) = \inf(C_1(M))$ then

$$C_1(n) \geq 2a_1 \tag{1.7}$$

and conjectured $C_1(n) = C_1(S^1 \mathrm{x} S^{n-1}) = \frac{4\pi \mathrm{vol}(S^{n-1})}{\mathrm{vol}(S^n)} \sqrt{\frac{(n-1)}{n^n}}$.

Here we present a new conformal invariant $C_p(f)$ which agrees with $C_1(f)$ for $p = 1$ and obtain similar inequalities as above. At first we have

$$C_p(f) \geq \sum_{k=1}^{n-1} a_k^p \beta_k, \tag{1.8}$$

where $a_k^1 = a_k$ and β_k is also the k^{th} Z_2- Betti number of M. So (1.6) is obtained from (1.8) by setting $p = 1$.

On the other hand for M not homeomorphic to S^n there is a constant $\alpha(p)$ such that $\alpha(1) = 2$ and if $C_p(n) = \inf(\inf\ (C_p(f)))$ then

$$C_p(n) \geq \alpha(p)a_1^p, \tag{1.9}$$

therefore (1.7) is a particular case of the last inequality and we also expected that

$$C_p(n) = C_p(S^1 \mathrm{x} S^{n-1}) = \frac{4\pi \mathrm{vol}(S^{n-1})}{\mathrm{vol}(S^n)}\sqrt{\frac{(n-1)^{(n(1-p)/p)+1}}{n^n}}.$$

Next we start with the functional $C_p(f)$.

2 The functional $C_p(f)$

Let M^n be a closed manifold, $f : M^n \to R^m$ an immersion and $N(f)$, ξ, A_ξ, $k_1(\xi), \ldots, k_n(\xi)$ the same quantities as before. Given a positive integer p set

$$(\sigma(A_\xi))^{2p} := \frac{1}{n^{2p}}\sum_{i<j}(k_i - k_j)^{2p} \tag{2.1}$$

The total $p-$conformal curvature, $C_p(f)$, of the immersion f is defined by

$$C_p(f) := \frac{1}{\mathrm{vol}(S^{m-1})}\int_{N(f)}(\sigma(A_\xi))^n d\xi \tag{2.2}$$

At first we note that the functional $C_p(f)$ satisfies:

Proposition 2.1

(i) If $n = 2$ *then* $C_p(f) = C(f) = (1/(2\pi))\int_M(\|H\|^2 - K)dM$

(ii) $C_p(f) \geq 0$ *and equality occurs if and only if* $f(M^n)$ *is a round sphere.*

(iii) If g *is a Mobius transformation in* R^m *then* $C_p(f) = C_p(g \circ f)$.

Proof: The proof of this proposition is standard and we will omit it. For a reference see e.g. [1].

Theorem 2.2 *Let $\beta_k := rank H_k(M; Z_2)$ be the k^{th} Z_2-Betti number of M and $a_k^p := k^{(n-2pk)/2p}(n-k)^{n((1-2p)/2p)+k}$. Then we have*

$$C_p(f) \geq \sum_{k=1}^{n-1} a_k^p \beta_k.$$

Proof: Let $N(f)$ be the unit normal bunble of f and define

$$N_k := \{\xi \in N(f) : A_\xi \text{ has exactly } k \text{ negative eigenvalues}\}$$

From theory of absolute curvature we have (see e.g. [4] or [10])

$$\int_{N(f)} |\det A_\xi| d\xi = \int_{v\in D} \sum_{k=0}^{n} \mu_k(v) dv$$

where $\mu_k(v)$ is the number of critical point of index k of $\langle f,\ v\rangle$ and D is a dense set on S^{m-1}. Therefore it follows from Morse inequality [7] that

$$\int_{N_k} |\det A_\xi| d\xi \geq \beta_k \mathrm{vol}(S^{m-1}).$$

Now the theorem follows from this inequality together with Lemma 2.3 below. In fact, from the lemma

$$(\sigma(A_\xi))^n|_{N_k} \geq a_k^p |\det A_\xi|$$

where $a_k^p = k^{(n-2pk)/2p}(n-k)^{n((1-2p)/2p)+k}$. Therefore,

$$C_p(f) = \frac{1}{\mathrm{vol}(S^{m-1})} \int_{N(f)} (\sigma(A_\xi))^n d\xi \geq \sum_{k=1}^{n-1} a_k^p \beta_k.$$

Lemma 2.3 *Given an integer r, $0 < r < n$, and real numbers $x_1, ..., x_n$ such that $x_1, ..., x_r < 0$, $x_{r+1}, ..., x_n \geq 0$ then, for any positive integer p, we have*

$$\left\{ \frac{1}{n^{2p}} \sum_{i<j} (x_i - x_j)^{2p} \right\}^{2p} \geq a_r^p |x_1 ... x_n|,$$

where $a_r^p = r^{(n-2pr)/2p}(n-r)^{n((1-2p)/2p)+r}$.

Proof: Without loss of generality we may restrict ourselves to the set

$$E := \left\{ (x_1, \ldots, x_n) \in R^n : \sum_{i<j} (x_i - x_j)^{2p} = \rho \right\},$$

where the constant ρ will be chosen later on. Now the subset E_r of E defined by (i) above is bounded, and the function $f(x_1, \ldots, x_n) = |x_1 \ldots x_n|$ vanishes on the boundary of E_r and is differentiable in the interior, $\overset{o}{E}_r$, of E_r. So f attains its maximal value at some point $x_o = (x_1, \ldots, x_n) \in \overset{o}{E}_r$. Therefore there is a Lagrangian multiplier λ such that

$$x_1 \ldots \hat{x}_i \ldots x_n = 2p\lambda \sum_{j=1}^{n} (x_i - x_j)^{2p-1}, \quad i = 1, \ldots, n \tag{2.3}$$

It follows from (2.3) that

$$x_1 p_1 = \ldots = x_n p_n \tag{2.4}$$

Here $p_i = \sum_{j=1}^{n} \ (x_i - x_j)^{2p-1}, \quad i = 1, \ldots, n.$

We now claim that $p_i < 0$ for $i = 1, \ldots, r$, and $p_\alpha > 0$ for $\alpha = r+1, \ldots, n$. Indeed, if $p_1 > 0$ then $q_1 = \sum_{j=1}^{r} (x_1 - x_j)^{2p-1} > 0$, since $\sum_{j=r+1}^{n} (x_1 - x_j)^{2p-1} < 0$. But $p_1 > 0$ implies $p_i > 0$ for $i = 1, \ldots, r$. On the other hand $\sum_{i=1}^{n} q_i = 0$, so $p_i < 0$ and $p_\alpha > 0$. This claim implies $x_1 = \ldots = x_r = t$ and $x_{r+1} = \ldots = x_r = s$. In fact, we suppose $x_1 < x_2$ and $x_{r+1} < x_{r+2}$. This implies $p_1 < p_2$ and $p_{r+1} < p_{r+2}$. Therefore $x_1 p_1 > x_2 p_2$ and $x_{r+1} p_{r+1} < x_{r+2} p_{r+2}$ which contradicts (2.4).

We now choose the constant ρ such that $t = -1$. So (2.4) gives us $s = (n-r)/r$. Since $\left\{ (1/(n^{2p}) \sum_{i<j} \ (x_i - x_j)^{2p} \right\}^{n/2p} = a_r^p |x_1 \ldots x_n|$ and f attains its maximal value at x_0 we conclude the proof of lemma.

3 Generalized Willmore Problem

Let M^n be a closed manifold which is not homeomorphic to S^n and $f : M^n \to R^m$ an immersion. As is [9] we consider the two types of Willmore problems:

(i) Determine or estimate at least

$$C_p(M^n) := inf(C_p(f))$$

(ii) Determine or estimate at least

$$C_p(n) := inf(C_p(M^n)).$$

We treat here problem (*ii*). For $n = 2$ there is a answer for it given by Li and Yau [6] see Theorem 2 of [9]. For $n \geq 3$ we have the following result which generalizes the Theorem 3 in [9].

Theorem 3.1 *Let $f : M^n \to R^m$ be an immersion of a closed manifold which is not homeomorphic to S^n. Then we have*

(i) If $n \geq 4$ then $C_p(n) \geq [(27/16)(16/9)^{1/p} - 1](n-1)^{n((1-2p)/2p)+1}$

(ii) $C_p(3) \geq 2^{(3-2p)/2p}$

To prove this theorem we will need of the following algebraic lemma.

Lemma 3.2 *Set $a_k^p = k^{(n-2pk)/2p}(n-k)^{n((1-2p)/2p)+k}$, where n, k and p are positive integers. If $2 \leq k \leq n-2$ then we have*

$$a_k^p \geq \frac{27}{16}\left[\frac{16}{9}\right]^{1/p} a_1^p \tag{3.1}$$

Proof: By taking the log of both sides of (3.1) we get that this is equivalent to

$$n\left[\log k(n-k) - \log(n-1) - \frac{4}{n}\log\frac{4}{3}\right] +$$

$$2p[(n-1)\log(n-1) - (n-k)log(n-k) - k\log k + \log 16 - \log 27] \geq 0$$

therefore the lemma follows from the following claim:

For $\psi(x,y) = (x-1)\log(x-1) - (x-y)\log(x-y) - y\log y + \log 16 - \log 27$ and $\phi(x,y) = \log y(x-y) - \log(x-1) - \frac{4}{x}\log\frac{4}{3}$ defined in the set $A = \{(x,y) \in R^2 : 2 \leq y \leq x-2\}$ we have $\psi(x,y) \geq 0$ and $\phi(x,y) \geq 0$.

To show this fix y and consider $\varphi_y(x) := \psi(x,y)$. Now given $q = (x,y) \in A$ there exists x' with $4 \le x' \le x$ such that $y = x' - 2$. Since g_y is an increasing function of x we get

$$\psi(x,y) \ge \varphi_{x'-2}(x') = (x'-2)\log\left[\frac{x'-1}{x'-2}\right] + \log\frac{4(x'-1)}{27} \ge 0.$$

In a similar manner we show that $\phi(x,y) \ge 0$ on A. So we conclude the proof of lemma.

Proof of theorem (3.1) Let $f : M^n \to R^m$ be an immersion with M^n closed which is not homeomorphic to S^n. For almost all v in the unit sphere in R^m the height function $h(x) =< f(x),\ v >$ is a Morse function and one of the following is true:

(i) h has at least two critical points of index 1 or $n-1$.

(ii) h has at least one critical point of index r, here $1 < r < n-1$.

(iii) h has only two critical points.

(iv) h or $-h$ has two minima, one critical point of index 1 , one maxima and no other critical point.

From Reeb's theorem [7] case (iii) cannot occurs. By the same argument (iv) is not possible, because by using the cancellation theorem [8] we can construct a Morse function $g : M^n \to R$ having only two critical points. So we have (i) or (ii).

Now we get, by using all previous results, the following:

$$C_p(f) \ge \frac{1}{vol(S^{m-1})}\sum_{k=1}^{n-1}\int_{N_k} a_k^p|\mathrm{det}A_\xi|d\xi \ge$$

$$a_1^p\left\{\sum_{k=1}^{n-1}\mu_k + \left[\frac{27}{16}\left(\frac{16}{9}\right)^{1/p} - 1\right]\sum_{k=2}^{n-2}\mu_k\right\},$$

which implies that $C_p(f) \ge [(27/16)(16/9)^{1/p} - 1]a_1^p$.

Remark 1 *Actually for* $1 \le p \le 3$, $C_p(f) \ge 2a_1^p$ *and for* $p \ge 4$ $C_p(f) \ge (27/16)(16/9)^{1/p}a_1^p$.

Remark 2 *All results obtained here for $p = 1$ were obtained by Pinkall [9].*

Example: Let $x : S^q(r) \rightarrow R^{q+1}$ and $y : S^{n-q}(\sqrt{1-r^2}) \rightarrow R^{n-q+1}$ be standard immersions. Set $f = x + y : M^n \rightarrow R^{n+2}$ where $M^n = S^q(r) \times S^{n-q}(\sqrt{1-r^2})$. Now we consider $(e_0, e_1, ..., e_q, v_0, ..., v_{n-q})$ an adapted orthonormal frame such that $e_1, ..., e_q$ are tangent to S^q , $v_1, ..., v_{n-q}$ are tangent to S^{n-q},$x = re_o$ and $y = \sqrt{1-r^2}v_0$. If $\nu = -\sqrt{1-r^2}e_0 + rv_0$ then

$$A_\nu = -\frac{\sqrt{1-r^2}}{r} I_q \oplus \frac{r^2}{\sqrt{1-r^2}} I_{n-q}$$

Therefore

$$C_p(f) = \frac{2\mathrm{vol}(S^q(1))\mathrm{vol}(S^{n-q}(1))}{\mathrm{vol}(S^n(1))} \cdot \frac{\sqrt{((n-q)q)^{n/p}}}{r^{n-q}\sqrt{(1-r^2)^q}}$$

and its minimum value is for $r = \sqrt{(n-q)/n}$. Observe that f is a minimal immersion if and only if $r = \sqrt{q/n}$.

It was conjectured by Pinkall [9] that for $p = 1$ and $n \geq 3$ $C_1(n) = C_1(S^1 \times S^{n-1})$. Therefore we expect the same with respect to any p, *i.e.*, for $n \geq 3$

$$C_p(n) = C_p(S^1 \times S^{n-1}) = \frac{4\pi \mathrm{vol}(S^{n-1})}{\mathrm{vol}(S^n)} \sqrt{\frac{(n-1)^{n((1-p)/p)+1}}{n^n}}.$$

Moreover if $n \geq 3$ and $H_i(M^n, Z_2) \cong H_i(S^q \times S^{n-q}, Z_2)$ then

$$C_p(n) = C_p(S^q \times S^{n-q}).$$

References

[1] Abe, N.; *On generalized total curvature and conformal mappings*, Hiroshima Math. J., 12, 203-207, (1982).

[2] Bryant, R.; *Surfaces in conformal geometry*, Proceending of Symposium in Pure Math., 48, 227-240, (1988).

[3] Chen, B.Y.; *On a variational problem on hypersurfaces*, J. London Math. Soc., 6, 321-325, (1973).

[4] Kuiper, N.; *Tight imbeddings and maps*, The Chern Symposium 1979, Springer Verlag, 97-144, (1980).

[5] Kusner, R.; *Comparison surfaces for Willmore problem*, Pacific Journal Math., 138, 317-345, (1985).

[6] Li, P & Yau, S.T.; *A new conformal invariant and its applications to Willmore conjecture and the first eigenvalue of compact surfaces*, Invent. Math. 69, 269-291, (1982).

[7] Milnor, J.; *Morse Theory*, Annals of Mathematics Study, 51, Princeton, N.J.; Princeton Univ. Press, 1963.

[8] Milnor, J.; *Lecture on the h-cobordism theorem*, Princeton Univ. Press, 1965.

[9] Pinkall, U.; *Inequalities of Willmore type for submanifolds*, Math. Z., 193, 241-246, (1986).

[10] Rodrigues, L.; *Geometria das Subvariedades*, Ed. IMPA 26, Rio de Janeiro, 1976.

[11] Weiner, J.; *On a problem of Chen, Willmore et al*, Indiana Univ. Math. Journ., 27, 19-35, (1978).

[12] White, J.H.; *A global invariant of conformal mappings in space*, Proc. Amer. Math. Soc., 38, 162-164, (1973).

[13] Willmore, T.J.; *Note on embedded surfaces*, An Stiint. Univ. "Al I Cuza" Iasi, Sect. I-a Math., 11, 493-499, (1965).

[14] Willmore, T.J.; *Total curvature in Riemannian geometry*, Chichester, Wiley, 1982.

Departamento de Matemática
Universidade Federal do Ceará
Campus do Pici
60455-760 Fortaleza-Ce, Brasil