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ON THE STRUCTURE OF THE FOCAL LOCUS OF A COMPLEX CURVE.

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0. Introduction

The aim of this paper is to understand the geometry of the focal locus of a complex analytic curve in complex euclidean 2-space C^2. If $X : \Sigma \to C^2$ is a non-singular analytic curve, the focal locus of Σ is the set

$$F_\Sigma = \{X + \rho(X)\xi : X \in \Sigma, \xi \in N_1\Sigma\}$$

where $N_1\Sigma$ is the unit normal sphere bundle of Σ and $\lambda = \rho^{-1}$ is the positive eigenvalue associated to the second fundamental form A^ξ. This eigenvalue λ is independent of the choice of ξ at X. When $\nabla\rho \neq 0$, the focal locus is the union $F_\Sigma = F^3 \cup \Sigma^*$ of a 3-dimensional manifold and a "singular set". In section 1 we conduct a systematic investigation of the geometry of F^3 and proved, among other things, the following result.

Theorem 0.1 *The focal locus of a non-singular analytic curve Σ in C^2 is away from its singular set a strictly convex scalar flat 3-dimensional hypersurface of R^4.*

In section 2 we take a look at the CR-structure on F. In section 3 we suppose locally F given as a graph of a real-valued function f over a domain $\Omega \subset R^3$. We observe that the function f must satisfy the equation

$$\mathrm{trace}(\delta_{ij} + f_i f_j)(f^{ij}) = 0 \qquad (*)$$

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where $\nabla f = (f_1, f_2, f_3)$ is the gradient of f and (f^{ij}) is the inverse of the Hessian (f_{ij}) of f. This is equivalent to following partial differential equation

$$(f_{**}^2 - \sigma_1 f_{**})(\nabla f, \nabla f) + (1 + |\nabla f|^2)\sigma_2 = 0, \quad \det f_{**} \neq 0$$

Here σ_j denotes the j^{th} elementary symmetric function of the eigenvalues of $f_{**} \equiv (f_{ij})$. We may think of this as a transform, i.e., from a complex analytic curve we construct its focal locus that in turn produce a solution of (*). It follows from the Alexandroff - Fenchel - Jessen theorem, (See[1]), that a solution f of (*) on a bounded domain with smooth boundary is completely determined by the values of f and ∇f on the boundary. The non-singular focal locus F of a complex curve is unique in the following sense.

Theorem 0.2 *Let M be a compact hypersurface of R^4 with boundary $\partial M \subset F$ such that*

i) M is strictly convex with scalar curvature $\kappa_M \equiv 0$

ii) The normal vectors of M and F on ∂M are the same.

Then $M \subset F$.

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1. The geometry of the focal locus

In this section we analyse the geometric structure on the focal locus of a non-singular analytic curve Σ in C^2.

Let $\Sigma \subset C^2$ be a non-singular holomorphic curve. We will denote by $<,>$ the standard inner product on C^2 and by $J : C^2 \to C^2$ the multiplication by $\sqrt{-1}$. We set if $z \in C^2$,

$$|z| = <z, z>^{1/2}$$

Let ∇ be the Riemannian conection on C^2. The second fundamental form of Σ is defined by

$$B_{V,W} \equiv (\nabla_V W)^N \tag{1.1}$$

for $V, W \in T_X\Sigma \equiv$ tangent space of Σ at X. Here $(\quad)^N$ denotes projection onto $N_X\Sigma \equiv$ normal space of Σ at X. Given a normal vector $\xi_X \in N_X\Sigma$ we define $A^\xi : T_X\Sigma \to T_X\Sigma$ by

$$A^\xi(V) = -(\nabla_V \xi)^T \tag{1.2}$$

where ξ is an arbitrary vector field in C^2 with the property that ξ is normal to Σ in a neighborhood of X and $(\quad)^T$ denotes projection onto $T_X\Sigma$.

Remark 1 *The $N(\Sigma)$-valued bilinear form B is symmetric and also complex bilinear, i.e., $B_{JV,W} = JB_{V,JW} = B_{V,JW}$. (See Lawson [2]). A and B are related by*

$$< B_{V,W}, \xi > = < A_\xi(V), W > \tag{1.3}$$

In particular A^ξ is self-adjoint. The eigenvalues $\pm\lambda(X,\xi)$ of A^ξ are independent of the choice of ξ at X, and if $B \neq 0$, they vanish only at isolated points. In this paper we will avoid those points.

Given a normal vector field ξ of unit lenght at X on Σ we asociate to ξ the eigendirection of A^ξ with positive eigenvalue λ. There are two eigenvectors of unit length on this "eingen-line", v_ξ and $-v_\xi$. We denote by ξ_t the unit normal vector $\xi cost + (J\xi)sint$. It follows from the above remark that the eigenvalues of A^{ξ_t} do not depend on t. They are given by λ and $-\lambda$ and the eigenline correponding to $-\lambda$ is determined by Jv_{ξ_t}. An easy computation shows that $\pm v_{\xi_t} = v_\xi cos(t/2) + Jv_\xi sin(t/2)$. From now on we will choose the sign of v_ξ so that

Remark 2

$$v_{\xi_t} = e^{it/2} v_\xi, \quad i = \sqrt{-1} \tag{1.4}$$

Definition 1.1 *The focal locus of Σ is the set*

$$F_\Sigma \equiv \{X + \rho(X)\xi \;\vdots\; X \in \Sigma, \xi \in N_1\Sigma\}$$

where

$$\rho(X) \equiv 1/\lambda(X)$$

In order to determine the structure of the focal locus we will consider the mapping $l : \Sigma \times S^1 \to F_\Sigma \subset C^2$ given by

$$l(X,t) = X + \rho(X)e^{it}\nu_X \tag{1.5}$$

where ν is a unit normal vector field on Σ.

One can prove easily that at $(X,t)\epsilon\Sigma \times S^1$ we have

$$l_*v_\nu \wedge l_*Jv_\nu \wedge l_*\partial/\partial t = 2\rho(v_{\nu_t}.\rho)\nu_t \wedge Jv_{\nu_t} \wedge J\nu_t \tag{1.6}$$

This proves the following lemma.

Lemma 1.1 *The mapping $l : \Sigma \times S^1 \to C^2$ given by (1.5) is an immersion at $(X,t)\epsilon\Sigma \times S^1$ if and only if*

$$< \nabla\rho, v_{\nu_t} > \neq 0 \tag{1.7}$$

From now on we will assume that Σ contains no critical points of ρ. In particular $|\nabla\rho| \neq 0$ and we can define the vector fields v_1, v_2 on Σ by

$$Jv_1 = v_2 = \nabla\rho/|\nabla\rho| \tag{1.8}$$

Remark 3 *The vector field ν in (1.5) may be chosen in such a way that $v_\nu = v_1$. This vector field is obviously unique. With this notation we have the following result*

Lemma 1.2 *The focal locus of* Σ *is the union* $F \cup \Sigma^*$ *of a 3-dimensional manifold* F *and a "singular set* Σ^**". Moreover*

$$F = \{X + \rho(X)e^{it}\nu_X : X \in \Sigma, 0 < t < 2\pi\}$$

$$\Sigma^* = \{X + \rho(X)\nu_X : X \in \Sigma\}$$

where ν *is the unique unit normal vector field on* Σ *such that* $v_\nu = v_1$.

Proof: A point $X^{*}{}' \in F_\Sigma$ may be written as $X^* = l(X,t)$ for some $X \in \Sigma$ and $t \in [0, 2\pi)$. We observe now that

$$< \nabla\rho, v_{\nu_t} >= |\nabla\rho| \sin(t/2)$$

The result follows by applying Lemma 1.1

Over F^3 we define a field of orthonormal frames $X^* e_1, e_2, e_3, e_4$ such that for $X^* = X + \rho(X)\xi \in F$ we have

$$\begin{cases} e_1 = Jv_\xi \\ e_2 = \xi \\ e_3 = J\xi \\ e_4 = v_\xi \end{cases} \tag{1.9}$$

The vector field e_4 is obviously normal to F^3 at X^*. We let ω_A , $1 \leq A \leq 4$, be the dual conframe of e_A. To e_A we also associate the conection 1-forms ω_{AB} given by

$$de_A = \sum_{B=1}^{4} \omega_{AB} e_B \quad . \tag{1.10}$$

The Cartan structure equations are

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B \tag{1.11}$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB}, \quad \omega_{AB} + \omega_{BA} = 0 \tag{1.12}$$

Let $T(F)$ and $T^*(F)$ be respectivily the tangent and cotangent bundle of F.

The second fundamental form II of F is a section on $T^*(F) \otimes T(F)$ whose components with respect to the given orthonormal frame e_A are

$$II = (h_{ij}), \quad \omega_{i4} = \sum_{j=1}^{3} h_{ij}\omega_j$$

We have the following result

Lemma 1.3 *At the point $X^* = X + \rho(X)\xi_X \in F$ we have*

$$(v_\xi.\rho^2)II = \begin{bmatrix} -|\nabla\rho|^2/2 & (Jv_\xi).\rho & -v_\xi.\rho \\ & -2 & 0 \\ & & 0 \end{bmatrix} \tag{1.13}$$

Proof: We are going to use the moving frame method. For this we consider the distinguished orthonormal frame field v_A on Σ obtained by making

$$v_2 = Jv_1 = \nabla\rho/|\nabla\rho|, v_3 = \nu, v_4 = J\nu \tag{1.14}$$

where ν is the unique normal vector field such that $v_\nu = v_1$

We then associate to v_A its dual conframe $\theta_A, 1 \leq A \leq 4$ and denote by θ_{AB} the 1-forms on Σ given by

$$dv_A = \sum_{B=1}^{4} \theta_{AB}v_B \tag{1.15}$$

We recall that the focal locus is given by the mapping $l : \Sigma \times S^1 \to C^2$ where

$$l(X,t) = X + \rho e^{it}v_3 \tag{1.16}$$

Taking the differencial of (1.16) gives

$$dl = dX + d\rho e^{it}v_3 + \rho dt e^{it}v_4 + \rho e^{it}dv_3$$

By construction

$$\begin{cases} \rho\theta_{31} = \rho\theta_{42} = -\theta_1 \\ \rho\theta_{32} = \rho\theta_{14} = \theta_2 \end{cases} \tag{1.17}$$

Therefore

$$dl = (1 - e^{it})\theta_1 v_1 + (1 + e^{it})\theta_2 v_2 + d\rho e^{it} v_3 + \rho e^{it}[dt + \theta_{34}] v_4$$

Then

$$dl = 2[-\sin(t/2)\theta_1 + cos(t/2)\theta_2] e_1 + d\rho e_2 + \rho[dt + \theta_{34}] e_3 \tag{1.18}$$

It follows that

$$\begin{cases} l^*\omega_1 = 2[-\sin(t/2)\theta_1 + \cos(t/2)\theta_2] \\ l^*\omega_2 = d\rho = |\nabla\rho|\theta_2 \\ l^*\omega_3 = \rho[dt + \theta_{34}] \end{cases} \tag{1.19}$$

In the following we are going to compute $l^*\omega_{j4}, \;\; j = 1, 2, 3$. We have

$$\begin{aligned} l^*\omega_{14} &= < dJ v_{\nu_t}, v_{\nu_t} >= \left\langle de^{it/2} J v_\nu, e^{it/2} v_\nu \right\rangle \\ &= \left\langle e^{it/2}[dJ v_\nu - 2^{-1} dt v_\nu], e^{it/2} v_\nu \right\rangle \\ &= \theta_{21} - 2^{-1} dt \end{aligned} \tag{1.20}$$

$$\begin{aligned} l^*\omega_{24} &= < d\nu_t, v_{\nu_t} >= \left\langle de^{it}\nu, e^{it/2} v_\nu \right\rangle \\ &= \left\langle e^{it}[d\nu + dt J\nu], e^{it/2} v_\nu \right\rangle \\ &= \left\langle e^{it/2} d\nu, v_\nu \right\rangle \\ &= \cos(t/2)\theta_{31} + sin(t/2)\theta_{41} \\ &= -\rho^{-1}[\cos(t/2)\theta_1 + sin(t/2)\theta_2] \end{aligned} \tag{1.21}$$

Similary we obtain

$$l^*\omega_{34} = -\rho^{-1}[-\sin(t/2)\theta_1 + cos(t/2)\theta_2] \tag{1.22}$$

We are now going to express the 1-forms ω_{j4} in terms of the dual coframe ω_j, $j = 1, 2, 3$. It follows from (1.19),(1.21) and (1.22) that

$$2\rho\omega_{34} = -\omega_1 \tag{1.23}$$

$$2\rho\omega_{24} = \text{cotg}(t/2)\omega_1 - 2(|\nabla\rho|\sin(t/2))^{-1}\omega_2 \tag{1.24}$$

To express ω_{14} in terms of the $\omega_j' s$, $j = 1, 2, 3$, we first observe that

$$0 = l^* d\omega_4 = \sum l^*\omega_j \wedge l^*\omega_{j4}$$

$$= [\sin(t/2)\theta_1 - \cos(t/2)\theta_2] \wedge [2\theta_{12} - \theta_{34} + \rho^{-1}|\nabla\rho|\theta_1]$$

Since this is true for all $t, 0 < t < 2\pi$, it follows that

$$2\theta_{12} - \theta_{34} + \rho^{-1}|\nabla\rho|\theta_1 = 0 \tag{1.25}$$

This allow us to rewrite equation (1.20) as:

$$2\rho l^*\omega_{14} = |\nabla\rho|\theta_1 - \rho(\theta_{34} + dt) \tag{1.26}$$

Using equations (1.19) and (1.26) we obtain

$$2\rho\omega_{14} = \text{cosec}(t/2)[-2^{-1}|\nabla\rho|\omega_1 + \cos(t/2)\omega_2] - \omega_3 \tag{1.27}$$

At the given point $X^* = X + \rho(X)\xi_X \in F$ we write the unit normal vector ξ_X as $\xi_X = e^{it}\nu$ for some $t \in (0, 2\pi)$. Therefore $v_\xi = e^{it/2}v_\nu$. It follows that

$$< \nabla\rho, v_\xi >= v_\xi.\rho = |\nabla\rho|\sin(t/2)$$

$$< \nabla\rho, Jv_\xi >= Jv_\xi.\rho = |\nabla\rho|\cos(t/2)$$

Then

$$\begin{cases} 2\rho(v_\xi.\rho)\omega_{14} = -2^{-1}|\nabla\rho|^2\omega_1 + (Jv_\xi.\rho)\omega_2 - (v_\xi.\rho)\omega_3 \\ 2\rho(v_\xi.\rho)\omega_{24} = (Jv_\xi.\rho)\omega_1 - 2\omega_2 \\ 2\rho\omega_{34} = -\omega_1 \end{cases}$$

This proves the lemma.

We will now prove our main result.

Theorem 1.1 *The focal locus of a non-singular analytic curve Σ in C^2 is away from its singular set a strictly convex scalar flat 3-dimensional hypersurface of R^4.*

Proof: Let $X^* = X + \rho(X)e^{it}\nu \in F = F_\Sigma - \Sigma^*$. The Gauss-Kronecker curvature K of F at X^* is given by the determinant of II. Therefore

$$K = (4\rho^3 v_{\nu_t}.\rho)^{-1}$$

$$= 1/4\rho^3|\nabla\rho|\sin(t/2)$$

for all $t \in (0, 2\pi)$. This shows that F is strictly convex. From lemma 1.3 if follows that

$$\begin{cases} \mathrm{trace} II = -(4 + |\nabla\rho|^2)/2v_\xi\rho^2 \\ \mathrm{trace} II^2 = (4 + |\nabla\rho|^2)^2/4(v_\xi\rho^2)^2 \end{cases}$$

where $\xi = e^{it}\nu$. The scalar curvature κ of F^3 is given by

$$\kappa = (\mathrm{trace} II)^2 - \mathrm{trace} II^2 = 0$$

This completes the proof of the theorem.

Remark 4 *It follows from the above theorem that $H + \rho^2[4 + |\nabla\rho|^2]K = 0$, where ∇ is the gradiente of Σ and $H = \mathrm{trace} II$ is the mean curvature of F.*

Let $F_\Sigma = F^3 \cup \Sigma^*$ be the focal locus of a non-singular analytic curve in C^2. On F_Σ there is a circle action $\Phi : S^1 \times F_\Sigma \to F_\Sigma$ defined by

$$\phi(s, l(X,t)) = l(X, s+t) \tag{1.28}$$

where $l : \Sigma \times S^1 \to F_\Sigma$ is the mapping given by $l(X,t) = X + \rho e^{it}\nu$. We may think of ρ as a function on F_Σ. To make things clear we define $\rho^* : F_\Sigma \to R$ by

$$\rho^*(l(X,t)) = \rho(X) \tag{1.29}$$

The function ρ^* is constant on the orbit os a point $p \in F_\Sigma$. Taking the differential of (1.29) and restricting ρ^* to F we get

$$l^* dp^* = dp = l^* \omega_2 \tag{1.30}$$

The second equality follows from (1.19). Therefore

$$d\rho^* = \omega_2 \tag{1.31}$$

Equivalently

$$\mathrm{grad}\rho^* = e_2 \tag{1.32}$$

where gradρ^* denotes the gradient of F^3. Since $|\mathrm{grad}\rho^*| = 1$ it follows that the integral curves of grad ρ^* are geodesics. The direction back to the complex curve Σ is given by - gradρ. We have the following theorem.

Theorem 1.2 *On the focal locus of a complex curve Σ there is a circle action $\Phi : S^1 \times F_\Sigma \to F_\Sigma$ and a function $\rho^* : F_\Sigma \to R$ satisfying*

(a) ρ^* is constant on the orbit of a point $p \in F_\Sigma$

(b) $|grad\rho^*/F| = 1$ on F

(c) From a point $p \in F^3$ the direction back to the complex curve Σ is given by -grad ρ^*

(d) On the orbit of a point $p \in F^3$ the tangent C-line field H given by $H_p = T_pF \cap JT_pF$, is completely determined by the value of grad ρ^* at p.

Proof: Equation (1.28) defines the circle action Φ. We have already verified conditions (a),(b),(c). To verify (d) we just observe that H is spanned by the vector fields e_2, Je_2 and $e_2 = \text{grad}\rho^*$.

Remark 5 *The tangent C-line field $H \subset TF^3$ defines a CR-structure on the non-singular focal locus F^3. In the next section we will take a look at this structure.*

Example 1: Let $X : C - \{0\} \to C^2$ be the holomorphic curve given by $X(z) = (z, z^n), n \in N$. Since $dX = (1, nz^{n-1})dz$ it follows that $z = x + iy$ are isothermal parameters for $\Sigma_0 = X(C - \{0\})$. The metric is given by $ds^2 = \mu^2|dz|^2$ where $\mu^2 = 1 + n^2|z|^{2(n-1)}, \mu > 0$. Its Gaussian curvature is obtained easily. It is given by

$$K = -2n^2(n-1)^2|z|^{2(n-2)}/\mu^6$$

Observe that Σ_0 is a non-singular holomorphic curve in C^2. In this case the function ρ is given by

$$\rho = (-2/K)^{1/2} = |z|^{2-n}\mu^3/n(n-1)$$

An easy computation shows that $\nabla\rho = 0$ if and only if $z \in C_r$ where $C_r = \{z \in C; |z| = r\}$ and

$$r^{2(n-1)} = (n-2)/(2n-1)n^2$$

For $\Sigma = \Sigma_0 - X(C_r)$ we have that the distinguished unit normal vector field ν is given by

$$\mu|z|^n\nu = (n|z|^{2(n-1)}z, -z^n)$$

Therefore the focal locus of Σ is the union $F_\Sigma = F^3 \cup \Sigma^*$ where

$$\Sigma^* = \{X + \rho(X)\nu_X : X \in \Sigma\}$$

$$F^3 = \{X + \rho(X)e^{i\tau}\nu_X : X \in \Sigma \text{ and } \tau \in (0, 2\pi)\}$$

2. The CR-structure of the non-singular focal locus

We now recall (cf. [3],[5]) the definition of a CR submanifold of a complex m-dimensional Kaehlerian manifold $\overline{M}$. Let J be the almost complex structure of $\overline{M}$ and M a real n-dimensional Riemannian manifold isometrically immersed in $\overline{M}$.

Definition 2.1 *M is called a CR-submanifold of $\overline{M}$ if the holomorphic tangent space to M at x, $H_x(M) = T_x(M) \cap JT_x(M)$, has constant complex dimension. $H(M)$ is called the holomorphic tangent bundle to M. The pair $H(M) \subset T(M)$ is called the CR-structure (Cauchy-Riemann structure) of M.*

Remark 6 *The distribution $H : x \to H_x$ satisfaes the following conditions:*

(a) H is holomorphic, i.e., $JH_x = H_x \forall x \in M$

(b) The complementary orthogonal distribution $H^\perp : x \to H_x^\perp \subset T_x(M)$ is anti-invariant, i.e., $JH_x^\perp \subset T_x(M)^\perp$ for each $x \in M$. If $dimH_x^\perp = dimT_x(M)^\perp$ for any $x \in M$, then the CR Submanifold is called a generic submanifold of $\overline{M}$. When $dimH_x^\perp, dimH_x \neq 0$ for any $x \in M$ then M is said to be non trivial. It is clear that every real hypersurface of a Kaehler manifold is a generic non trivial CR-submanifold.

Let $F_\Sigma = F^3 \cup \Sigma^*$ be focal locus os a non-singular analytic curve in C^2. The tangent C-line field H is given by $H_p = T_pF \cap JT_pF$, $p \in F$. H_p is the maximal complex subspace of T_pC^2 which is contained in T_pF. The pair $H \subset TF^3$ defines a CR-structure on the non singular focal locus F.

Let Aut (F^3) be the local automorphism group of CR-mappings, i.e., C^∞ mappings $f : F \to F$ such that $df/H : H \to H$ is complex linear. By a result of Segre, [4], if the Levi form of F is non degenerate then Aut (F^3) is finite dimensional. We then want to take a look at the Levi form of F^3. For this we let ω be a 1-form on F^3 such that Kern $\omega \equiv H$. We then define the skew symmetric tensor L by

$$L \equiv d\omega/_H \tag{2.1}$$

This skew symmetric tensor is in fact given by $L(V,W) = -\omega([V,W])$ for V,W in H. Note that ω is determined up to multiplication by a positive function. We will choose this function so that

$$L(V,W) = < [V,W], e_1 > \tag{2.2}$$

where e_1 is the vector Jv_ξ translated to $X^* = X + \rho\xi \in F$. L is the so called Levi form of F. We prove the following result.

Theorem 2.1 *The Levi form L defined on the non singular focal locus of a complex curve Σ in C^2 is non degenerate. In particular Aut (F^3) is finite dimensional.*

Proof: Let L be the Levi form given by (2.2). We have to show that the associated hermitian form

$$h_L(V,W) = -L(JV,W) - iL(V,W)$$

is non degenerated. One can easily check that $h_L = a(\ ,\ )$ where $a = L(\xi, J\xi)$ and $(\ ,\ )$ is the standard hermitian form on C^2. With the notation of lemma 1.3 it is sufficient to show that $L(\xi, J\xi) = L(e_2, e_3) \neq 0$. To prove the theorem we observe that

$$L(e_2, e_3) = < \nabla_{e_2} e_3 - \nabla_{e_3} e_2, e_1 >$$

$$= < \nabla_{e_2} Je_3 - \nabla_{e_3} Je_2, Je_1 >$$

$$= < -\nabla_{e_2} e_2 - \nabla_{e_3} e_3, -e_4 >$$

$$= h_{22} + h_{33}$$

Using lemma 1.3 we see that $h_{22} \neq 0$ and $h_{33} = 0$. This completes the proof of the theorem.

3. Final coments

In section 1 we have seen how to construct a scalar flat hypersurface of R^4 from a non singular analytic curve $\sum$ in C^2. Suppose locally F given as a graph of a real-valued function f over a domain $\Omega \subset R^3$. Then locally

$$F = \{(x, f(x) : x = (x_1, x_2, x_3) \in \Omega\} \equiv \Gamma_f$$

We may ask what equation $\epsilon(f) = 0$ must the real function $f : \Omega \to R$ satisfy so that its graph has scalar curvature $\kappa \equiv 0$. We may as well think of this as a transform, i.e., from a complex analytic curve we construct its focal locus that in turn produce a solution of the equation $\epsilon(f) = 0$.

In the induced metric the first and second fundamental forms are given by

$$I = \sum(\delta_{ij} + f_i f_j) dx_i \otimes dx_j \tag{3.1}$$

$$II = -W^{-1} \sum f_{ij} dx_i \otimes dx_j \tag{3.2}$$

where $\nabla f = (f_1, f_2, f_3)$ is the gradient of $f, (f_{ij}) \equiv f_{**}$ is the Hessian of f and $W^2 = 1 + |\nabla f|^2$. A straightforward computation shows that the scalar curvature κ of Γ_f is given by

$$\kappa = 2W^{-4}\epsilon(f)$$

where

$$\epsilon(f) = (1 + |\nabla f|^2)\sigma_2 + (f_{**}^2 - \sigma_1 f_{**})(\nabla f, \nabla f) \tag{3.3}$$

Here σ_j denotes the j^{th} elementary symmetric function of the eigenvalues of $f_{**} \cong (f_{ij})$. The eingenvalues k_1, k_2, k_3 of II relative to I are called the principal curvatures. We recall from theorem 1.1 that Γ_f is strictly convex, i.e., the Gauss-Kronecker curvature K is everywhere > 0. The reciprocals $1/k_1, 1/k_2, 1/k_3$ are called the radii of principal curvatures. They are the roots os the polynomial equation

$$det[W(g_{ij})(f^{ij}) + \lambda I_3] = 0 \qquad (3.4)$$

where $g_{ij} = \delta_{ij} + f_i f_j$, I_3 is the 3×3 identity matrix and (f^{ij}) denotes the inverse of the Hessian of f.

The first elementary symmetric function of $\frac{1}{k_j}, j = 1, 2, 3$ is

$$P_1(\Gamma_f) \equiv \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = \frac{\kappa}{2K} = 0 \qquad (3.5)$$

This shows that the equation trace$(\delta_{ij} + f_i f_j)(f^{ij}) = 0$ is equivalent to

$$\epsilon(f) = 0, \quad \det f_{**} \neq 0 \qquad (3.6)$$

It follows from a generalization of Alexandroff-Fenchel-Jessen Theorem (See Chern,[1]) that if $f, \overline{f} : \Omega \to R$ are solutions of (3.6) in a boundded domain with smooth boundary $\partial\Omega$, and $f = \overline{f}, \nabla f = \nabla \overline{f}$ in $\partial\Omega$ then $f \equiv \overline{f}$. This is a consequence of the fact that the first elementary symmetric functions $P_1(\Gamma_f)$ and $P_1(\Gamma_{\overline{f}})$ of their graphs coincide and their common boundary have the same normal vectors. A more general uniqueness result may be stated in the following way.

Theorem 3.1 *Let M be a compact hypersurface of R^4 with boundary $\partial M \subset F^3$ such that*

i) M is strictly convex with scalar curvature $\kappa_M \equiv 0$

ii) The normal vectors of M and F on ∂M are the same

Then $M \subset F$.

Proof: The manifolds F and M are strictly convex and scalar flat. Therefore $P_1(M) = P_1(F) = 0$. This condition together with condition (ii) implies that $M \subset F$(See Chern [1]).

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