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KÄHLER SUBMANIFOLDS OF REAL AND COMPLEX RIEMANNIAN MANIFOLDS

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The purpose of this article is to present some inequalities involving the type component $\alpha^{(1,1)}$ of the second fundamental form α of a Kähler submanifold and give geometric interpretations to the cases where the equalities hold.

Moreover, since the parallelism of $\alpha^{(1,1)}$ is equivalent, in the case of surfaces, to the parallelism of the mean curvature vector H, this condition is also considered in the case of higher dimensional Kähler submanifolds. Namely, it is given an extension of the Hopf's theorem which characterizes the standard round sphere as the only topological 2-sphere immersed in $\mathbf{R}^3$ with constant mean curvature.

Let $f : \mathbf{M}^{2m} \longrightarrow \mathbf{N}$ be an isometric immersion of a Kähler manifold of real dimension $2m$ into a Riemannian manifold. Let $\mathbf{TM}$, $\mathbf{T}^{\perp}$ M and $\mathbf{J}$ denote respectively the tangent bundle, the normal bundle and the complex structure of $\mathbf{M}$.

The complexified tangent bundle $\mathbf{TM} \otimes \mathbf{C}$ of $\mathbf{M}$ splits in the direct sun of the eighenspaces of $\mathbf{J}$. That is

$$\mathbf{TM} \otimes \mathbf{C} = \mathbf{T}^{(1,0)}\mathbf{M} + \mathbf{T}^{(0,1)}\mathbf{M}$$

where $\mathbf{T}^{(1,0)}\mathbf{M}$ $(\mathbf{T}^{(0,1)}\mathbf{M})$ denotes the eighenspace of $\mathbf{J}$ associated to the eighenvalue $\imath$ $(-\imath)$. let Π_1 (Π_2) denotes the projection on $\mathbf{T}^{(1,0)}\mathbf{M}$ $(\mathbf{T}^{(0,1)}\mathbf{M})$.

The tensor $\alpha^{(1,1)}$ is defined by

$$\alpha^{(1,1)}(x,y) = \frac{1}{2}[\alpha(\Pi_1(x), \Pi_2(y)) + \alpha(\Pi_2(x), \Pi_1(y))]$$

for $x, y \in \mathbf{TM} \otimes \mathbf{C}$, where α denotes the second fundamental form and its complex bilinear extension to $\mathbf{TM} \otimes \mathbf{C} \times \mathbf{TM} \otimes \mathbf{C}$.

The immersion f is called $(1,1)$-geodesic if $\alpha^{(1,1)} \equiv 0$. Geometrically it means that the restriction of f to any complex curve or M is a minimal immersion. It is important to remark that all holomorphic and antiholomorphic immersions are $(1,1)$-geodesic and all $(1,1)$-geodesic immersions are minimal.

The most typical examples of $(1,1)$-geodesic immersions are the minimal surfaces and the holomorphic or antiholomorphic immersions. For non typical examples see [E-T].

It is not easy in general to construct new examples even locally. In fact, for restricted manifolds, for instance $\mathrm{N} = \mathbf{CP}^n$, the typical examples are the only possibles, according to [D-T]. Besides, even weaker restrictions on the tensor $\alpha^{(1,1)}$ may imply strong restrictions to f.

The following theorem generalizes results of [D-G], [D-T] and [F-R-T 1]

Theorem 1. *Let* $f : \mathrm{M}^{2m} \longrightarrow \mathrm{N}$ *be an isometric immersion*

i) *If* $\mathrm{N} = \mathbf{R}^n$ *then* $\|H\| = \frac{1}{\sqrt{2m}} \|\alpha^{(1,1)}\|$

ii) *If* $\mathrm{N} = \mathbf{H}^n$ *(the n-dimensional hyperbolic space) then* $\|H\| \geq \frac{1}{\sqrt{2m}} \|\alpha^{(1,1)}\|$ *and equality holds if and only if* $m = 1$

iii) *If* N *is a* $\frac{1}{4}$*-pinched manifold (that is, any sectional curvature* K *of* N *satisfies* $\frac{1}{4} < K \leq 1$ *) then* $\|H\| \leq \frac{1}{\sqrt{2m}} \|\alpha^{(1,1)}\|$ *and equality holds if and only if* $m = 1$

iv) *If* $\mathrm{N} = G_p(\mathbf{C}^n)$ *(the complex Grassmanian of p dimensional subspaces of* $\mathbf{C}^n$*) then* $\|H\| \leq \frac{1}{\sqrt{2m}} \|\alpha^{(1,1)}\|$ *and if equality holds then either* $m \leq (p-1)(n-p-1)+1$ *or* f *is holomorphic or antiholomorphic.*

In [D-G], [D-T] and [F-R-T 1] the vanishing of $\alpha^{(1,1)}$ is required to obtain the same conclusions of the above theorem. In fact, this hypothesis can be replaced by the weaker assumption of the umbilicity of $\alpha^{(1,1)}$ restricted to real vectors. That is,

$$\alpha^{(1,1)}(x,y) = 2 < x, y > H$$

for $x, y \in \mathrm{TM}$.

Theorem 2. *Let* $f : \mathrm{M}^{2m} \longrightarrow \mathrm{N}$ *be an isometric immersion with umbilical tensor* $\alpha^{(1,1)}$ *restricted to real vectors.*

i) *If* $\mathrm{N} = \mathbf{R}^n$ *then either* $m = 1$ *or* $H \equiv 0$

ii) *If* $\mathrm{N} = \mathbf{H}^n$ *then either* $m = 1$ *or* $\|H\| \equiv 1$

iii) *If* N *is an* $\frac{1}{4}$*-pinched manifold then* $m = 1$

iv) *If* $\mathrm{N} = G_p(\mathbf{C}^n)$ *then either* $m \leq (p-1)(n-p-1)+1$ *or f is holomorphic or antiholomorphic.*

A proof of theorems 1 and 2 can be found in [F-T].

To state the next theorem it is convenient to introduce the following definition:

A Kähler submanifold of $\mathbf{S}^n$ is said to be second order isotropic if $<\alpha^{(2,0)}, \alpha^{(2,0)}>= 0$, where $<\ ,\ >$ denotes the complex bilinear extension of the metric of the normal bundle.

The notation of isotropy, introduced by Calabi in [C], is very important in the study of minimal surfaces of spheres.

Theorem 3. *Let* M *be a compact, connected Kähler manifold with positive first Chern classe and let* $f : \mathrm{M} \longrightarrow \mathbf{R}^n$ *be an isometric immersion with parallel tensor* $\alpha^{(1,1)}$. *Then either* $f(\mathrm{M})$ *is minimal in some sphere and second order isotropic or* $\mathrm{M} = \mathrm{M}_1 \times \cdots \times \mathrm{M}_k$ *is a Riemannian product of Kähler manifolds and* $f = f_1 \times \cdots \times f_k$ *is a product of immersions, where* $f_i : \mathrm{M}_i \longrightarrow \mathbf{R}^{n_i}$ *is minimal in some sphere and second order isotropic.*

Corollary *Let* $f : \mathrm{M} \longrightarrow \mathbf{R}^n$ *be under the conditions of theorem 3*

i) *If the essential codimention is one then* $f(\mathrm{M})$ *is a standard 2-sphere in* $\mathbf{R}^3$.

ii) *If the essential codimention is two then* $f(\mathrm{M})$ *is a product of two standard 2-spheres in* $\mathbf{R}^6$.

The proof of theorem 3 and its corollary can be found in [F-R-T 2].

Remark: If the essential codimention is bigger than two, there are many examples besides product of standard 2-spheres. To see this, consider minimal 2-spheres in $\mathbf{S}^4$. All but the round 2-sphere in $\mathbf{S}^3$ are examples of surfaces in $\mathbf{R}^5$ satisfying the conditions of theorem 3 but they are not standard spheres.

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