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HARMONIC MAPS INTO SURFACES WITH TWO-DIMENSIONAL CONE METRICS

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1. Introduction

The study of harmonic maps into spaces with singularities has not sufficiently been considered. In [1] Gromov and Schoen developped the theory of harmonic maps into certain singular spaces of nonpositive curvature. Another contribuition was given by Leite in [3], where she proved the existence of harmonic mappings with respect to some degenerate metrics.

Let (Σ, z) and (S, w) be closed Riemann surfaces of the same genus $p \geq 2$, where z and w are local complex coordinates, with $z = x + iy$. If $ds^2 = \rho^2(w)|dw|^2$ is a smooth Riemannian metric of nonpositive curvature, then in any homotopy class α of degree one mappings from Σ into S, there exists a map $u \in C^\infty(\Sigma, S)$ that minimizes the Dirichlet energy

$$E(u) = \int_\Sigma 2\rho^2(u)(|u_z|^2 + |u_{\bar{z}}|^2)dxdy.$$

Moreover, it was proved in [4] that u is a (harmonic) diffeomorphism. In [3] Leite considered the same problem for a singular metric

$$ds^2 = |\eta| = |h(w)||dw|^2,$$

where η is a holomorphic quadratic differential in S. Approximating $|\eta|$ by smooth negative curvature metrics she proved that the corresponding harmonic

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diffeomorphisms subconverge uniformly to a surjective, energy minimizing map in a homotopy class of maps with finite energy.

Let $\mathcal{C}$ denote the set of cone points, i.e., $\mathcal{C} = \{q \in S : \eta(q) = 0\}$, and $\mathcal{P}$ the preimage of such points, $\mathcal{P} = u^{-1}(\mathcal{C})$, and denote by

$$\phi = |h(u)| u_z \bar{u}_z dz^2$$

the holomorphic Hopf differential of u. Recall that if $\phi \not\equiv 0$ then through any $p \in \Sigma$ there passes a leaf of the foliation by negative real trajectories ϕ (i.e., maximal smooth curves γ along which $\phi(\gamma', \gamma') < 0$, [5]). The main purpose of this paper is to study the singular set $\mathcal{P}$ and other properties of the map u.

We assume that $u \in C^o(\Sigma, S)$ is a map as constructed by Leite in [3]. In particular, u is uniformly approximated by orientation preserving diffeomorphisms and the associated Hopf differential ϕ is holomorphic. Moreover, u is $\frac{2}{m+2}$ - Hölder continuous where m is the maximal order of a zero of η. Thus we prove in Theorem 1 (section 3) that if $\phi \not\equiv 0$ then for any $q \in \mathcal{C}$ the singular set $u^{-1}\{q\}$ is an isolated point or a union of arcs contained in the negative real trajectories of ϕ. Moreover, outside the singular set, $u : \Sigma \backslash \mathcal{P} \to S \backslash \mathcal{C}$ is a diffeomorphism. Also in the last section we show that u is the unique minimizer map in its homotopy class α. When $\mathcal{P}$ is a finite set of points, we prove in Theorem 1 that if u is injective then it is the Teichmüller map (H^1- orientation preserving homeomorphism with constant dilatation).

A weaker version of the last statement was already obtained in [3], where Leite assumed the additional condition that u has dilatation bounded away from one.

Our study of the set $\mathcal{P}$ was based on some recent results on the vanishing order and the local behavior of harmonic maps into singular spaces which are due to Gromov and Schoen [1]. We use that a neighbourhood of any $q \in \mathcal{P}$ is metrically a cone with cone angle $(m+2)\pi$, where m is the vanishing order of η. In particular, it can be isometrically embedded as a geometric cone $X \subset \mathbf{R}^3$ with a singularity at its vertex. We then prove that whenever $u \in H^1(\Sigma, X)$ is energy minimizing, the preimage of the cone point can be written locally as a graph over the negative real line of the Hopf differential ϕ around any point p with $\phi(p) \neq 0$. These remarks are contained in Lemmas 1,2 and Proposition 1 in section 2.

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2. The Local Structure of The Singular Set

Let $D = \{z = x + iy \in \mathbf{C} : |z| < 1\}$ and let $X \subset \mathbf{R}^N$ be a two- dimensional cone with vertex $0 \in \mathbf{R}^N$. We assume that the generating curve $X \cap S^{N-1}$ is a (piecewise) smooth, simply closed curve of length $\ell > 2\pi$, so that X (with the induced inner metric) is a realization of the cone metric

$$ds^2 = \mu^2 |z|^{2(\mu-1)} |dz|^2,$$

where $z \in \mathbf{C}, \mu = \ell/2\pi > 1$. In particular, X has nonpositive curvature in the sense of [1]. Let $U \in H^1(D, X) = \{V \in H^1(D, \mathbf{R}^N) : V(z) \in X \ a.e\}$ be a local minimizer for the standard Dirichlet energy

$$E(U) = \frac{1}{2} \int_D (|U_x|^2 + |U_y|^2) dx dy$$

in $H^1(D, X)$. It is shown in [1] that if U has bounded image, then it is locally Lipschitz continuous. We begin with the study of the set $\mathcal{P} = U^{-1}\{0\}$ in the infinitesimal level.

Lemma 1. *Let $U \in H^1_{loc}(\mathbf{R}^2, X)$ be a homogeneous degree one minimizing map. Then $\varphi = |U_x|^2 - |U_y|^2 - 2i < U_x, U_y > \equiv a \in \mathbf{C}^*$ and there exists $z_o \in S^1$ with $a z_o^2 > 0$, such that U is independent of the direction iz_o and $U^{-1}\{0\} = \mathbf{R} iz_o$*

Proof. Since dU is homogeneous of degree zero, $\varphi \equiv a \in \mathbf{C}$ by Liouville's theorem. Now by [1], proposition 3.1, we have $U = J \circ V$ where $V : \mathbf{C} \to \mathbf{R}^m$ is a homogeneous degree one harmonic map and $J : \mathbf{R}^m \to X$ is an isometric, totally geodesic embedding. Since the vertex of X is singular, we must have $m = 1$ (if $m = 2$, then J would have to map an euclidean circle of radius $\rho > 0$ to a ρ-distance circle in X of the same length). Therefore there exist $z_o \in S^1, \mu > 0$ such that $V(z) = \mu < z, z_o >$. It follows that U is invariant in the direction iz_o. Now $\frac{1}{2}\phi(w,w) = |dU \cdot w|^2 - \frac{1}{2}|dU|^2|w|^2 - i < dU \cdot w, dU \cdot iw >$, so that $az_o^2 = \phi(z_o, z_o) = -\phi(iz_o, iz_o) = |dU|^2 > 0$. Moreover we have that the curve $U(tz_o) = J(\mu t)$ is a minimizing geodesic with speed $|dU \cdot z_o|^2 = |a|$. $\square$

Let now $U \in H^1(D, X)$ be an energy minimizing map. We want to study the set $\mathcal{P} = U^{-1}\{0\}$ in the neighbourhood of a point $z_o \in D$ at which the Hopf differential does not vanish. As this is a local problem and as the energy is conformally invariant, we may assume that $z \in D$ is a natural parameter for ϕ, i.e.

$$|U_x|^2 - |U_y|^2 - 2i < U_x, U_y > \equiv 1. \tag{1}$$

Following [1],we introduce the notation (for $z_o \in D, 0 < \sigma < 1 - |z_o|$)

$$E(z_o, \sigma) = \int_{D_\sigma(z_o)} (|U_x|^2 + |U_y|^2) dx dy,$$

$$I(z_o, \sigma) = \int_{\partial D_\sigma(z_o)} d^2(U(z), U(z_o)) ds,$$

$$ord(z_o, \sigma) = \frac{\sigma E(z_o, \sigma)}{I(z_o, \sigma)}.$$

Also for $\lambda > 0$, $z_o \in D$ with $U(z_o) = 0$ and $|z_o| \le 1/2$ we define

$$U_{z_o,\lambda} : D_{\frac{1}{2\lambda}}(0) \to X, \; U_{z_o,\lambda}(z) = \frac{1}{\lambda} U(z_o + \lambda z).$$

Lemma 2. *Let $U \in H^1(D, X)$ be an energy minimizing map with $z \in D$ a natural parameter for ϕ and $d_o = diam\ U(D) < \infty$. If $U(z_i) = 0$, $|z_i| \le 1/2$, $\lambda_i \downarrow 0$ then the sequence U_{z_i,λ_i} contains a subsequence that converges locally uniformly and locally in H^1 to a homogeneous degree one, nonconstant minimizing map $U_* : \mathbf{C} \to X$ with $U_*^{-1}\{0\} = i\mathbf{R}$.* **Proof.** Let $|z_o| \le 1/2$, $0 < \sigma < 1/2$. We begin by deriving upper and lower bounds for the quantities $E(z_o,\sigma), I(z_o,\sigma)$ and the Lipschitz constant of U. We clearly have from (1) that

$$E(z_o,\sigma) \geq \pi\sigma^2.$$

Also $ord(z_o,\sigma)$ is nondecreasing in σ and converges to $ord(z_o) \in [1,\infty)$ as $\sigma \downarrow 0$, because U is Lipschitz continuous (see [1], p. 41). Setting $\alpha = ord(z_o)$, we infer as in [1] that $\sigma^{-1-2\alpha} I(z_o,\sigma)$ is a nondecreasing function (there is no error term because we work in the standard metric). Now for σ small we have

$$2\alpha \geq ord(z_o,\sigma) = \frac{\sigma E(z_o,\sigma)}{I(z_o,\sigma)} \geq \frac{\pi\sigma^3}{I(z_o,\sigma)} \geq \sigma^{2(1-\alpha)} \frac{\pi\rho^{1+2\alpha}}{I(z_o,\rho)}$$

for fixed $\rho > \sigma$. By taking σ small enough we conclude that $\alpha = 1$. Therefore, if $\sigma \in (0,\frac{1}{2})$, we have

$$\begin{aligned}\pi &\leq \liminf_{r\to 0} \frac{E(z_o,r)}{r^2 ord(z_o,r)}\\ &= \liminf_{r\to 0} \frac{I(z_o,r)}{r^3} \leq \frac{I(z_o,\sigma)}{\sigma^3} \leq \frac{I(z_o,1/2)}{(1/2)^3} \leq K,\end{aligned}$$

with K depending only on d_o. Thus

$$\pi\sigma^3 \leq I(z_0,\sigma) \leq K\sigma^3. \tag{2}$$

Using the subharmonicity of $d^2(U(\cdot),U(z_o))$ it now follows that

$$|dU(z_o)| \leq K,$$

$$E(z_o,\sigma) \leq K\sigma^2 \; for \; \sigma \in (0,1/4).$$

Summing up, we have uniform local Lipschitz and uniform local energy bounds for the sequence $U_i = U_{z_i,\lambda_i}$. After selection of a subsequence, U_i will converge locally uniformly and weaky in H^1 to a map $U_* : \mathbf{R}^2 \to X$, and (2) gives that U_* is not constant. The argument of [1] proposition 3.3 carries over word by word to show that U_* is locally minimizing and that the energies converge. Also one obtains that U_* is homogeneous of degree one. Now the convergence of the energies implies that $dU_i \to dU_*$ pointwise almost everywhere, and therefore the quadratic differential associated to U_* is identically equal to dz^2. Then we conclude from lemma 1 that $U_*^{-1}\{0\} = i\mathbf{R}$. □

Let now U be as in lemma 2. For any $z \in D, \alpha \in (0, \pi/2)$ let $C_\alpha(z) \subset \mathbf{R}^2$ be the cone of opening angle α around the y-direction through z,

$$C_\alpha(z) = \{\zeta = \xi + i\eta \in \mathbf{C} : \frac{|\xi - x|}{|\eta - y|} < tg\ \alpha\}.$$

Let $Q_\epsilon(z) = \{\zeta \in \mathbf{C} : |\xi - x| \le \epsilon,\ |\eta - y| \le \epsilon\}$. We claim that for any $\alpha \in (0, \pi/2)$, there exists an $\epsilon > 0$ such that for all $z \in D,\ |z| \le 1/4,\ z \in \mathcal{P} = U^{-1}\{0\}$, we have

$$\mathcal{P} \cap Q_\epsilon(z) \subset C_\alpha(z). \tag{3}$$

If this were not true, we could find a sequence $z_i \in D,\ |z_i| \le 1/4,\ U(z_i) = 0$ and a sequence $z_i' \in \mathcal{P},\ \lambda_i = |z_i' - z_i| \to 0$, such that $z_i' \notin C_\alpha(z_i)$. Considering the blowup sequence U_{z_i,λ_i} we infer from lemma 2 that after selection of a subsequence U_{z_i,λ_i} converges to a limit map U_* with $U_*^{-1}\{0\} = i\mathbf{R}$. However $U_{z_i,\lambda_i}(\frac{1}{\lambda_i}(z_i' - z_i)) = \frac{1}{\lambda_i}U(z_i') = 0$ and $\frac{1}{\lambda_i}(z_i' - z_i) \notin C_\alpha(0)$. By uniform convergence, this gives a contradiction.

Condition (3) shows that any point $z_o \in \mathcal{P}$ has a neighbourhood $Q_\epsilon(z_o)$, such that $\mathcal{P} \cap Q_\epsilon(z_o)$ can be written as a graph over a closed subset of the vertical line through z_o, and additionally that

$$sup\{\frac{|x - x_o|}{|z - z_o|} : z = x + iy \in \mathcal{P}\backslash\{z_o\},\ |z - z_o| \le \rho\} \to 0 \ \ as\ \ \rho \to 0. \tag{4}$$

We can summarize these results as follows:

Proposition 1. *Let $U \in H^1(D, X)$ be energy minimizing, $\mathcal{P} = U^{-1}\{0\}$. Assume that $z_o \in \mathcal{P}$ is a point at which the Hopf differential ϕ does not vanish. Let $\zeta = \xi + i\eta$ be a natural parameter for ϕ at z_o, so that $\zeta(z_o) = 0$ and $\phi = d\zeta^2$. In a neighbourhood of z_o, $\mathcal{P}$ can be described as a graph $\xi = f(\eta)$ over a subset $\mathcal{E}$ of the line $\xi = 0$. Moreover, the local Lipschitz constant of the graph function is zero, i.e.*

$$\lim_{\substack{\eta \to \eta_o \\ \eta \in \mathcal{E}}} \frac{|f(\eta) - f(\eta_o)|}{|\eta - \eta_o|} = 0.$$

Remark: The same statement holds if X is a nonpositively curved surface with an isolated singularity, at which the tangent cone is of the type described above.

3. Global Consequences

We are now going to discuss the consequences of the above analysis in the situation mentioned in the introduction.

Let (Σ, z), (S, w) be closed Riemann surfaces of the same genus $p \geq 2$ and let α be a homotopy class of degree one mappings from Σ to S. Throughout this section we assume that $u \in C^0(\Sigma, S)$ is a map as constructed by Leite in [3], which minimizes the energy with respect to the singular metric $|\eta|$ associated to a holomorphic quadratic differential $\eta = h(w)dw^2$, in a homotopy class of maps with finite energy. In particular we assume that u has the following properties proved in [3]:

Proposition 2.

(i) u is uniformly approximated by orientation preserving diffeomorphisms (hence it is surjective);

(ii) If $C = \{q \in S : \eta(q) = 0\}$, $\mathcal{P} = u^{-1}(C)$, then $u : \Sigma \backslash \mathcal{P} \to S \backslash C$ is a smooth harmonic map with nonnegative jacobian;

(iii) $\phi = |h(u)| u_z \bar{u}_z dz^2$ is a holomorphic quadratic differential.

With those assumptions for the map u we are able to state the main result of this paper.

Theorem 1. *For u as in proposition 2, the following statements hold:*

(i) If $\phi \equiv 0$ then u is a biholomorphic map;

(ii) If $\phi \not\equiv 0$ then :

(a) For any $q \in \mathcal{C}$, $u^{-1}\{q\}$ is a single point or it is a compact and simply connected union of arcs contained in the negative real trajectories of ϕ;

(b) $u : \Sigma \backslash \mathcal{P} \to S \backslash \mathcal{C}$ is a diffeomorphism;

(iii) If u is injective then it is the Teichmüller map.

Proof. The proof of this theorem will be given in several steps. First of all we observe that in a natural parameter, in a neighbourhood of a zero of order m, η has the form

$$\eta = (\frac{m+2}{2})^2 w^m dw^2.$$

Thus a neighbourhood of any point $q \in \mathcal{C}$ can be isometrically embedded as a cone in $\mathbf{R}^3$ over a curve in S^2 with length $(m+2)\pi$. Composing this embedding with u we obtain a map U to which the results of section 1 can be applied.

Step 1. For any $q \in \mathcal{C}$ the set $\mathcal{P}_q = u^{-1}\{q\}$ is connected and can not contain a closed curve that is homotopically nontrivial in Σ.

These are consequences of the fact that u is uniformly approximated by orientation preserving diffeomorphisms.

Step 2. If $\phi \equiv 0$ then u is a biholomorphic map.

We start by proving that u is injective. Assume that for some $q \in S$, $card\, u^{-1}\{q\} > 1$. Let

$$U : D_r(0) \subset \mathbf{C} \to X \subset \mathbf{R}^3 \ , \ U = J \circ u \circ z^{-1},$$

where z is a local parameter and J is an isometric embedding of a neighbourhood of q as a cone in $\mathbf{R}^3$. Then $U \in H^1(D_r, X)$ is energy minimizing and from step 1 we have $u^{-1}\{0\} \cap \partial D_\rho(0) \neq \emptyset$ for all $\rho \in (0, r)$, if r is small enough. Also U is a conformal map, i.e.

$$|U_x|^2 - |U_y|^2 - 2i < U_x, U_y > \equiv 0.$$

Now from [1],proposition 3.3, we infer that U has a homogeneous degree α approximating map $U_* : D \to X$ which is not constant, for some $\alpha \in [1, \infty)$.

As any rescaled map $U_{\lambda,\mu}(z) = \frac{1}{\mu}U(\lambda z)$ is conformally parametrized and U_* is approximated locally in H^1 by such maps, we infer that U_* is also conformal. Introducing polar coordinates $z = re^{i\theta}$ on D, we compute (recall that U_* is locally Lipschitz) that

$$U_*(rz) = r^\alpha U_*(z).$$

Then

$$\frac{\partial U_*}{\partial r} = \frac{\alpha}{r}U_*(z)$$

and also

$$\frac{1}{2}\frac{\partial}{\partial\theta}|U_*|^2 =< U_*, \frac{\partial U_*}{\partial\theta} >= \frac{r}{\alpha} < \frac{\partial U_*}{\partial r}, \frac{\partial U_*}{\partial\theta} >\equiv 0.$$

As U_* is not constant we conclude that $U_*^{-1}\{0\} = \{0\}$. On the other hand, if λ is small enough we have that for any rescaled map $U_{\lambda,\mu}$ and for any $r \in (0, \frac{1}{2})$,

$$U_{\lambda,\mu}^{-1}\{0\} \cap \partial D_r(0) \neq \emptyset.$$

As U_* is the uniform limit of such a blowup sequence, we obtain a contradiction. This proves that u is a homeomorphism. But the jacobian of u is nonnegative on $\Omega = \Sigma\backslash\mathcal{P}$, so that u is holomorphic on Ω and thus it is holomorphic on Σ because $\mathcal{P}$ is a finite set. This proves the first statement of the theorem.

From now on we assume that $\phi \not\equiv 0$, so that the trajectory structure of ϕ is available.

Step 3. Let $z_o \in \mathcal{P}$ be a point with $\phi(z_o) \neq 0$. Then one of the following alternatives holds:

(a) z_o is an isolated point of $\mathcal{P}$;

(b) near z_o, $\mathcal{P}$ is an interval in the negative real trajectory of ϕ through z_o, containing z_o as an interior or as an end point.

Suppose that the first alternative fails. Let $\zeta = \xi + i\eta \in [-\epsilon, \epsilon] \times [-\epsilon, \epsilon]$ be a natural parameter around z_o, so that in the domain Q_ϵ of ζ we have a graph representation

$$f : \mathcal{E} \to \mathbf{R},\ \mathcal{E} \subset [-\epsilon, \epsilon], \xi = f(\eta)$$

of $\mathcal{P}$ (according to section 1). Taking ϵ small enough we may assume that $|f(\eta)| \leq \frac{\epsilon}{2}$ for $\eta \in \mathcal{E}$. Now if for some $0 < \eta_1 < \eta_2 \leq \epsilon$ we have $\eta_i \notin \mathcal{E}$ $(i = 1, 2)$, then it follows by the connectivity of $\mathcal{P}$ that $[\eta_1, \eta_2] \cap \mathcal{E} = \emptyset$, because otherwise the decompositon $\Omega_1 = (\eta_1, \eta_2) \times (-\epsilon, \epsilon)$, $\Omega_2 = \Sigma \backslash \overline{\Omega_1}$ disconnects $\mathcal{P}$. Assuming that z_o is an accumulation point from above (i.e. $\eta > 0$), we see that we can assume $[0, \epsilon] \subset \mathcal{E}$. But then from section 1 (Proposition 1) we conclude that $f'(\eta) \equiv 0$ on $[0, \epsilon]$, so that $f|_{[0,\epsilon]} \equiv 0$. If necessary we repeat the argument for negative η, so that the statement is proved.

Step 4. $\mathcal{P}$ cannot contain a closed loop and $\Omega = \Sigma \backslash \mathcal{P}$ is connected .

We have already excluded homotopically nontrivial loops in $\mathcal{P}$. But the metric $|\phi|$ does not admit contractible closed geodesics (see [5]),so that the claim follows from step 3.

Step 5. $u : \Sigma \backslash \mathcal{P} \to S \backslash C$ is a diffeomorphism.

On $\Omega = \Sigma \backslash \mathcal{P}$, u is a smooth solution of the harmonic map equation

$$h(u)u_{z\bar{z}} + \frac{1}{2}h'(u)u_z u_{\bar{z}} = 0,$$

where $\eta = h(w)dw^2$. This implies that

$$\phi_1 = h(u)u_z^2 dz^2 \quad \phi_2 = \overline{h(u)u_{\bar{z}}^2 dz^2}$$

are holomorphic quadratic differentials on Ω. As the jacobian $\mathcal{J}u$ is nonnegative, i.e. $|u_{\bar{z}}|^2 \leq |u_z|^2$, we conclude that

$$g := \frac{\phi_2}{\phi_1}$$

has a holomorphic extension to Ω with $|g| \leq 1$. ($\phi_1 \equiv 0$ is not possible because then u would have to be constant).

Now a zero z of the jacobian must also be a zero of the Hopf differential, because otherwise $|g(z)| = 1$, which implies that $|g| \equiv 1$ (actually $g =$ constant) on Ω, by the maximum principle (here we use that Ω is connected). But then the jacobian would vanish identically, which contradicts the fact that u is surjective. Thus $\{z \in \Omega : \mathcal{J}u(z) = 0\}$ is a finite set.

We proceed to show that $u|_\Omega$ is injective. If this were not the case, i.e. $card\ u^{-1}\{q\} > 1$ for some $q \in S\backslash\mathcal{C}$, then we could use the connectivity of $u^{-1}\{q\}$ as in step 2 to conclude that any $z \in u^{-1}\{q\}$ is an accumulation point of $u^{-1}\{q\}$, so that the zeroes of $\mathcal{J}u$ are not isolated. The proof is finished by the argument of Heinz [2] which says that a univalent harmonic map has nonvanishing jacobian.

Step 6. If u is injective then it is the Teichmüller map.

Let $g : \Sigma\backslash\mathcal{P} \to \mathbf{C}$ be the holomorphic function considered in step 5. Since $|g| < 1$ and $\mathcal{P}$ is finite, g extends to all of Σ and is therefore constant. Now

$$|g(z)| = |\frac{u_{\bar{z}}}{u_z}|^2$$

is the square of the dilatation of the map u and the Teichmüller map is characterized as being the unique homeomorphism with constant dilatation in its homotopy class. This completes the proof of the theorem. □

4. Uniqueness

In this section we are going to prove that the map $u_o : \Sigma \to S$, constructed in section 2 is the unique minimizer in its homotopy class. Assume that $u_1 : \Sigma \to S$ is another minimizer homotopic to u_o. For any $p \in \Sigma$ we can replace the curve joining $u_o(p)$ to $u_1(p)$ by the unique constant speed geodesic homotopic to that curve, with the same end points, thus obtaining a geodesic homotopy

$$u : [0,1] \times \Sigma \to S, \ \ u(t,p) = u_t(p)$$

connecting u_o to u_1. Let $\mathcal{P}_t = u_t^{-1}\{\mathcal{C}\}$, $\Omega_t = \Sigma\backslash\mathcal{P}_t$ and let

$$Z_t : \Omega_t \to TS, \ \ Z_t(p) = \frac{\partial u_t}{\partial t}(t,p)$$

denote the tangent field of the homotopy along u_t. Observe that for any $\Omega \subset\subset \Omega_t$ there exists an $\epsilon > 0$ such that $u|_{(t-\epsilon,t+\epsilon)\times\Omega} \to S$ is a smooth map so that Z_t is well defined and smooth. Now recall from [1], Theorem 4.1, that for any $\Omega \subset \Sigma$,

$E_\Omega(u_t)$ is a continuous, convex function satisfying the differential inequality

$$\frac{d^2}{dt^2}E_\Omega(u_t) \geq \int_\Omega |\nabla d(u_o, u_1)|^2 dxdy,$$

in the weak sense. As $E_\Omega(u_t)$ is constant, we have that $d(u_o, u_1) \equiv L > 0$. Now let $\Omega \subset\subset \Omega_t$, then we have by a standard computation that

$$0 = \frac{d^2}{dt^2}E_\Omega(u_t) = 2\int_\Omega |DZ_t(p)|^2 dxdy,$$

where D is the covariant derivative along u_t. We obtain that Z_t is parallel along $u_t|_{\Omega_t}$. In particular we have a parallel field Y_o on $S\backslash\mathcal{C}$ by setting

$$Y_o(q) := Z_o(u_o^{-1}q) \;\; for\; q \in S\backslash\mathcal{C}$$

(recall that $u_o : \Omega_o \to S\backslash\mathcal{C}$ is a diffeomorphism) with $ds^2(Y_o) \equiv L$. Also since Y_o is parallel then Y_o rotates around the origin by the angle $-m\pi \leq 0$, i.e.,

$$index(Y_o, q) = -\frac{m}{2},$$

for any $q \in \mathcal{C}$ which is a zero of η with order m.

Now observe that $\mathcal{P}_o \subset \Omega_t$ for any $t > 0$, t small enough ($d(u_o, u_t) = Lt$). Let $\beta(s)$ be a smooth arc in $\mathcal{P}_l$. It follows that

$$u_o(\beta(s)) \equiv q \in \mathcal{C}$$

$$u_t(\beta(s)) \subset \{q' : \; dist(q, q') = Lt\} = \partial B(q, Lt).$$

For any fixed s_o, $u(\cdot, \beta(s_o))$ is a geodesic ray emanating from q. Therefore, $Z_t(\beta(s))$ is proportional (with a constant factor) to the normal to $\partial B(q, Lt)$ at the point $u_t(\beta(s))$. As Z_t is parallel along u_t this proves that $u_t(\beta(s))$ has to be constant. Thus for t small enough, u_t maps each component of $\mathcal{P}_o$ to a single point $q(t)$ with $dist\ (q(0), q(t)) = Lt$, and the curve $q(t)$ is a geodesic ray emanating from $q(0)$.

Now assume $\{p_i\} \subset \Omega_o$, and p_i converges to $p_o \in \mathcal{P}_o$. Let

$$q_i = u_o(p_i) \to u_o(p_o) = q_o,$$

$$q_i' = u_1(p_i) \to u_1(p_o) = q_o'.$$

Let $\gamma_i(t) = u_t(p_i)$, $0 \leq t \leq 1$. It follows that $\gamma_i(t)$ converges to the geodesic segment $q(t)$, $0 \leq t \leq 1$. Thus if we take an euclidean coordinate background metric, we must have that $Y(q_i) \to q'(0)L$. As this is true for any such sequence $\{p_i\}$ we have shown that Y_o has a continuous extension into the singular point q. But this is not possible because *index* $(Y_o, q) = -\frac{m}{2} \leq -1$ (the argument also works in the case where $u_o^{-1}\{q\}$ degenerates to a point). It follows now that $Y_o \equiv 0$, and thus $u_1 \equiv u_o$ which means that u_o is the unique minimizer in its homotopy class.

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