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THE MAXIMUM PRICIPLE AT NON-FLAT POINTS FOR ZERO SCALAR CURVATURE HYPERSURFACES IN $\mathbf{R}^4$.

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A hypersurface in $\mathbf{R}^4$ with scalar curvature identically zero *(scalar-flat hypersurface)* is locally parametrized as the graph $x_4 = u(x_1, x_2, x_3)$ of a C^∞ function whose derivatives u_i and u_{ij} verify the equation

$$\begin{aligned}&(1+{u_1}^2)(u_{22}u_{33}-u_{23}^2)+(1+{u_2}^2)(u_{11}u_{33}-u_{13}^2)+\\&+(1+{u_3}^2)(u_{11}u_{22}-u_{12}^2)+2[u_1u_2(u_{13}u_{23}-u_{12}u_{33})+\\&+u_1u_3(u_{12}u_{32}-u_{13}u_{22})+u_2u_3(u_{21}u_{31}-u_{23}u_{11})]=0.\end{aligned} \tag{1}$$

This quadratic second order partial differential equation is known to be non-elliptic, in contrast to the equations for constant mean curvature and for positive scalar curvature. The elliptic ones satisfy maximum principles, from which many geometrical results have been derived (see [1],[2],[4],[7] and [9], among several authors).

We know from Differential Geometry that flat graphs in $\mathbf{R}^4$ are particular examples of scalar-flat hypersurfaces. The corresponding flat solutions, e.g. functions depending on only one variable, vanish each quadratic term on equation (1), since their Hessian matrices have rank at most 1.

Our main result is that ellipticity of (1) fails precisely for flat solutions. We also prove that Hopf's Maximum Principle can be applied to the linearized equation for a convex family of non-flat solutions (theorem 1.1). Non-flat solutions are described in Section 3.

The geometric version of theorem 1.1 is the content of theorem 2.1, according to which two scalar-flat hypersurfaces tangent at a non-flat point either intersect

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or coincide locally, provided their spherical images have same orientation with respect to the upward normal.

1. Ellipticity

Recall from [8, p.150] that a non-linear PDE, written in the form $G(\mathbf{D}u, \mathbf{D}^2 u) = 0$, is elliptic with respect to a function z at a point $\mathbf{x}$ if the matrix $[\partial G/\partial u_{ij}]$ is positive definite, when the values $z_i(\mathbf{x})$ and $z_{ij}(\mathbf{x})$ are inserted.

Theorem 1.1 *The equation (1), or the equation obtained from (1) by multiplying it by -1, is elliptic for a solution u at a critical point $\mathbf{x}$ if and only if the Hessian matrix $[u_{ij}(\mathbf{x})]$ has rank > 1. Moreover, it is convex the set of functions z for which (1) is elliptic at $\mathbf{x}$, with $\mathbf{D}z(\mathbf{x}) = 0$.*

Lemma 1.1 *Let us consider the second symmetric function*

$$F(\mathbf{A}) = A_{11}A_{22} - A_{12}^2 + A_{11}A_{33} - A_{13}^2 + A_{22}A_{33} - A_{23}^2$$

of a 3×3 real symmetric matrix $\mathbf{A} = [A_{ij}]$ and let us denote by $[\partial F/\partial \mathbf{A}]$ the symmetric matrix of derivatives $[\partial F/\partial A_{ij}]$.

i) *If $F(\mathbf{A}) = 0$, then rank $\mathbf{A} = 0, 1$ or 3, but never 2.*

ii) *If $F(\mathbf{A}) = 0$, then $[\partial F/\partial \mathbf{A}]$ is singular if and only if rank $\mathbf{A} \leq 1$.*

iii) *If $F(\mathbf{A}) = 0$ and rank $\mathbf{A} = 3$, then $[\partial F/\partial \mathbf{A}]$ is positive (or negative)*

iv) *The matrices for which $[\partial F/\partial \mathbf{A}]$ is positive definite form a convex set V.*

Proof: We know that $F(\mathbf{A}) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$. If $F(\mathbf{A}) = 0$ and one of the eigenvalues, say λ_1, is zero, then $\lambda_2\lambda_3 = 0$ also, and rank $\mathbf{A}$ is at most 1. Otherwise, all eigenvalues of $\mathbf{A}$ are different from zero and rank $\mathbf{A} = 3$, so (i) is proved.

Differentiation of $F(\mathbf{A})$ gives us $\partial F/\partial A_{11} = A_{22} + A_{33}$, $\partial F/\partial A_{12} = -A_{12}$, etc., so

$$[\partial F/\partial \mathbf{A}] = (tr\mathbf{A})\mathbf{I} - \mathbf{A}.$$

The characteristic polynomial of $\mathbf{A}$ annihilates $\mathbf{A}$, therefore

$$\mathbf{A}^3 - (tr\mathbf{A})\mathbf{A}^2 + F(\mathbf{A})\mathbf{A} - (det\mathbf{A})\mathbf{I} = 0.$$

If $F(\mathbf{A}) = 0$, it follows that

$$[\partial F/\partial \mathbf{A}] \times \mathbf{A}^2 = (tr\mathbf{A})\mathbf{A}^2 - \mathbf{A}^3 = -(det\mathbf{A})\mathbf{I} \quad .$$

When rank $\mathbf{A} \leq 1$, $\mathbf{A}$ has two eigenvalues zero, say λ_1 and λ_2, hence $[\partial F/\partial \mathbf{A}]$ is singular, for it has two eigenvalues equal to λ_3 and the third one equal to zero.

Let us suppose that rank $\mathbf{A} = 3$. In this case $\mathbf{A}$ has an inverse so that $[\partial F/\partial \mathbf{A}] = -(det\mathbf{A}) \times \mathbf{A}^{-2}$ is positive (or negative) definite, depending on $\det \mathbf{A} < 0$ (or $det\mathbf{A} > 0$), which proves ii) in the other direction and also iii). We observe that $(tr\mathbf{A})\mathbf{I} - \mathbf{A}$ is a linear function of $\mathbf{A}$. Given $\mathbf{A}_1, \mathbf{A}_2 \in V$, one has that

$$[\partial F/\partial \mathbf{A}]\,(s\mathbf{A}_1 + (1-s)\mathbf{A}_2) = s\,[\partial F/\partial \mathbf{A}_1] + (1-s)\,[\partial F/\partial \mathbf{A}_2]$$

is a segment between positive matrices, as $s \in [0,1]$. Besides, the set of positive matrices is convex, since

$$< (s\mathbf{P}_1 + (1-s)\mathbf{P}_2)(\xi), \xi >= s < \mathbf{P}_1(\xi), \xi > +(1-s) < \mathbf{P}_2(\xi), \xi >,$$

which is positive for all $\xi \neq \mathbf{0}$, as $\mathbf{P}_1$ and $\mathbf{P}_2$ are positive matrices. Thus iv) is proved.

Proof of Theorem 1.1: Let u be a solution of (1) written as $G(\mathbf{D}u, \mathbf{D}^2u) = 0$. At a critical point $\mathbf{x}$, one has that

$$[\partial G/\partial u_{ij}]\,(\mathbf{x}) = [\partial F/\partial \mathbf{A}]\,(\mathbf{D}^2u(\mathbf{x})),$$

since $G(\mathbf{0}, \mathbf{A}) = F(\mathbf{A})$, with F as in Lemma 1.1.

A direct application of Lemma 1.1 gives us that $[\partial G/\partial u_{ij}]\,(\mathbf{x})$, or $[\partial(-G)/\partial u_{ij}]\,(\mathbf{x})$, evaluated at $(\mathbf{D}u(\mathbf{x}), \mathbf{D}^2u(\mathbf{x}))$, is positive definite if and only if $[u_{ij}(\mathbf{x})]$ has rank > 1.

To prove the second statement of Theorem 1.1, suppose that the equation (1) is elliptic with respect to the functions z and Z at a point where their gradients vanish. That is, the matrices $[\partial G/\partial u_{ij}]$ evaluated at $(\mathbf{0}, \mathbf{D}^2 z(\mathbf{x}))$ and $(\mathbf{0}, \mathbf{D}^2 Z(\mathbf{x}))$ are positive definite. It follows from part (iv) of Lemma 1.1 that the equation (1) is elliptic at $\mathbf{x}$ with respect to the function $sz + (1-s)Z$, $\forall s \in [0,1]$, so the theorem is proved.

Remark 1.1 As pointed out by the referee, the ellipticity argument (first part of theorem 1.1) can be generalized, in a simple way, to hypersurfaces in $\mathbf{R}^{n+1}$, $n > 3$, of zero scalar curvature, whose Gauss Kronecker curvature never vanishes. Indeed, the ellipticity matrix of the linearized equation at a critical point $\mathbf{x}$ is again $(tr\mathbf{A})\mathbf{I} - \mathbf{A}$, where $\mathbf{A} = [u_{ij}(\mathbf{x})]$. Denoting by $\lambda_1, ..., \lambda_n$ the eigenvalues of $\mathbf{A}$, one has that $(\lambda_1 + \lambda_2 + ... + \lambda_n)^2 = \lambda_1^2 + \lambda_2^2 + ... + \lambda_n^2$, since $F(\mathbf{A}) = 0$. On the other hand, the Gauss-Kronecker curvature condition means that $det\mathbf{A} \neq 0$, thus $(tr\mathbf{A})^2 > \lambda_j^2 > 0, \forall j$, yielding that the eigenvalues $tr\mathbf{A} - \lambda_j$ are all positive when $tr\mathbf{A} > 0$, and negative otherwise, which proves ellipticity.

Let us point out that theorems 1.1 and 2.1 have been generalized in [3] to hypersurfaces in $\mathbf{R}^{n+1}$ with zero k-th curvature function, for any $k < n$, under the hypothesis that the Gauss map has rank $> k-1$. Symmetry results are also derived there.

2. The Tangency Principle for Scalar-flat Hypersurfaces

Let us recall the geometry of a graph. The scalar curvature of a Riemannian manifold is an average of curvatures on orthogonal planar sections, thus flat manifolds are scalar-flat.

In the case of a hypersurface in $\mathbf{R}^4$, the equation of Gauss yields that the curvature of a section with orthonormal basis $\{X, Y\}$ is $K(X \wedge Y) = II(X,X)II(Y,Y) - II(X,Y)II(X,Y)$, where II is the second quadratic form associated to the normal map differential. As a consequence, the hypersurface scalar curvature is the second symmetric function of II and a point $\mathbf{p}$ is flat if and only if rank $II_{\mathbf{p}} \leq 1$. The eigenvalues k_1, k_2 and k_3 of II are the principal curvatures of the hypersurface and the determinant $k_1 k_2 k_3$ its Gauss-Kronecker curvature. With respect to the basis $\{X_i = (\mathbf{e}_i, u_i), i = 1, 2, 3\}$ of a graph, the matrix of II is a positive multiple of the Hessian matrix $[u_{ij}]$, therefore the rank, the nullity of the second symmetric function and the determinant sign of $II(\mathbf{x}, u(\mathbf{x}))$ and of $[u_{ij}(\mathbf{x})]$ are equally verified.

Theorem 2.1 *Let M_1 and M_2 be scalar-flat hypersurfaces in $\mathbf{R}^4$ tangent at a non-flat point $\mathbf{p}$ and equally oriented at $\mathbf{p}$ with normal $(0,0,0,1)$. Suppose that the Gauss-Kronecker curvatures of M_1 and M_2 at $\mathbf{p}$, which are different from zero as $\mathbf{p}$ is non-flat, have the same sign. Then it can not happen that one hypersurface is above the other near the tangengy point, unless they coincide in a neighborhood of $\mathbf{p}$.*

Proof: We may assume that both Gauss-Kronecker curvatures at $\mathbf{p}$ are negative, by performing a reflection of the hypersurfaces around their tangent space at $\mathbf{p}$ if necessary. We parametrize M_1 and M_2 as $(0,0,0,0) \in \mathbf{R}^4$, so we get two elliptic solutions u and w of the equation (1) such that $u(\mathbf{0}) = w(\mathbf{0})$ and $\mathbf{D}u(\mathbf{0}) = \mathbf{D}w(\mathbf{0}) = \mathbf{0}$. By continuity, we assume that the Hessian determinants of u and w are negative in a neighborhood of $\mathbf{0}$.

The mean value theorem in the integral form implies that the following pointwise equation holds (compare with [9]):

$$0 = G(\mathbf{D}u, \mathbf{D}^2 u) - G(\mathbf{D}w, \mathbf{D}^2 w) = \int_0^1 \left[\frac{dG}{dt}(\mathbf{D}v^t, \mathbf{D}^2 v^t) \right] dt,$$

where $v^t = tu + (1-t)w$. Clearly $\mathbf{D}v^t = t\mathbf{D}u + (1-t)\mathbf{D}w$ and $\mathbf{D}^2 v^t = t\mathbf{D}^2 u + (1-t)\mathbf{D}^2 w$, thus implying that the above integrand is equal to

$$\sum_{k=1}^{3} \frac{\partial G}{\partial u_k}(\mathbf{D}v^t, \mathbf{D}^2 v^t) v_k + \sum_{i,j=1}^{3} \frac{\partial G}{\partial u_{ij}}(\mathbf{D}v^t, \mathbf{D}^2 v^t) v_{ij},$$

where $v = u - w$. After integration we arrive at a linear EDP for v of the form

$$\sum_{k=1}^{3} b_k v_k + \sum_{i,j=1}^{3} c_{ij} v_{ij} = 0, \tag{2}$$

where $c_{ij} = \int_0^1 \frac{\partial G}{\partial u_{ij}}(\mathrm{D}v^t, \mathrm{D}^2 v^t)dt$. The coefficients b_k and c_{ij} of equation (2) are clearly continuous.

If $u \leq w$ near $\mathbf{0}$, then the difference v attains the maximum value 0 at the interior point $\mathbf{0}$, so $v_k(\mathbf{0}) = 0$ and $[v_{ij}(\mathbf{0})]$ is negative semi-definite. It follows from Theorem 1.1 that (1) is elliptic with respect to v_t at the point $\mathbf{0}$, for all $t \in [0,1]$, so $c_{ij}(\mathbf{0})$ is positive definite. A continuity argument shows that the linear equation (2) is uniformly elliptic near $\mathbf{0}$, so Hopf's Maximum Principle can be applied (see [8]). Thus, the solution v can not attain an interior maximum, unlesss it is constant equal to zero. That is, M_2 can not be above M_1 near $\mathbf{p}$, unless they coincide.

3. Non-flat solutions

a) $u(\mathbf{x}) = 2\sqrt{\|\mathbf{x}\| - 1}$, $\|\mathbf{x}\| > 1$. This graph is the upperhalf of a rotational hypersurface with axis of revolution in the vertical direction and profile given by a parabola ([5]).

b) $u(\mathbf{x}) = \log\left[\cos^2(\frac{x_3}{2})/(\cos x_1 \cos x_2)\right]$, $\mathbf{x} \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\pi, \pi)$. This function is decomposed as a sum $g(x_1) + h(x_2) + l(x_3)$ (see [6]).

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