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A SHARP ESTIMATE FOR THE INDEX OF RELATIVE NULLITY

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1. Introduction

For an isometric immersion between Riemannian manifolds $f : M^n \to N^{n+p}$, the Gauss equation says that the (sectional) *extrinsic curvature* of M^n in N^{n+p} at $x \in M^n$ for a plane $\sigma \subset T_xM$, $K_f(\sigma) := K_M(\sigma) - K_N(\sigma)$, is given by

$$K_f(\sigma) = \langle \alpha(X,X), \alpha(Y,Y) \rangle - \|\alpha(X,Y)\|^2,$$

where α is the second fundamental form of the immersion and $\{X, Y\}$ any orthonormal basis of σ. Superscripts always means dimension.

Chern and Kuiper ([**2**]) have shown that $\nu \geq n - p$ at the points where the extrinsic curvature vanishes. Here $\nu(x)$ is the dimension of the subspace

$$\Delta(x) = Ker\,\alpha(x) = \{X \in T_xM : \alpha(X,Y) = 0,\ \forall\ Y \in T_xM\}$$

and is called the *index of relative nullity* of f at x. It is a well known fact that the positiveness of the index of relative nullity imposes strong conditions on the metric of the submanifold and on the structure of the immersion. Therefore, it is a natural question to ask what happens if the extrinsic curvature is merely nonpositive.

In that direction, Borisenko had shown in [**1**] that at points where $K_f \leq 0$, the index of relative nullity verifies $\nu \geq n - p^2 - p$. The main purpose of this paper is to show the following improvement of Borisenko's result.

Theorem 1. *Let $f : M^n \to N^{n+p}$ be an isometric immersion between Riemannian manifolds. Suppose that at $x_0 \in M^n$ we have $K_f(x_0) \leq 0$. Then $\nu(x_0) \geq n - 2p$.*

The following example shows that our estimate in Theorem 1 is sharp.

Example. Let $U^2 \subset R^3$ be a surface in the euclidean space with negative Gaussian curvature at $x_0 \in U^2$. Then the product immersion of p factors $U^2 \times \cdots \times U^2 \to R^{3p}$ satisfies $\nu(x_0, \ldots, x_0) = n - 2p = 0$.

The strong restrictions that $\nu > 0$ imposes on an isometric immersion allow us to find several applications of Theorem 1. The following corollary is an improvement of Theorem 3 in [1], where a much stronger quadratic hypothesis for the codimension is needed.

Corollary 2. *Let $f : M^n \to S_c^{n+p}$ be an isometric immersion of a complete Riemanian manifold into the euclidean sphere of constant sectional curvature c. If $K_M \le c$ and $2p < n - \nu_n$, then f is totally geodesic.*

In the above statement ν_n is defined as $\nu_n = \max\{k : \rho(n-k) \ge k+1\}$, where $\rho(n)$ is given by $\rho((\mathrm{odd})2^{4d+b}) = 8d + 2^b$, with d being any nonnegative integer and $b = 0, 1, 2, 3$. Some values of ν_n are: $\nu_n = n -$ (highest power of $2 \le n$) for $n \le 24$, $\nu_n \le 8d - 1$ for $n < 16^d$ and $\nu_{2^d} = 0$.

At least for some dimensions, the hypothesis in the codimension in the above corollary cannot be improved to $2p < n$. For example, the simplest of Cartan's isoparametric hypersurfaces, i.e., the unit normal bundle of the Veronesse surface in S_1^4, is a compact non totally geodesic submanifold of S_1^4 with curvature less or equal than one.

We have the following for isometric immersions of Riemannian products.

Corollary 3. *Let $M^n = N_1^{n_1} \times N_2^{n_2}$ be the product of two Riemannian manifolds. Suppose that there exists $(x, x') \in M^n$ such that $K_{N_1}(x), K_{N_2}(x') \le c$. Then, there is no isometric immersion of M^n into S_c^{n+p} for $2p < n$.*

Corollaries 2 and 3 also hold if we replace the ambient space by any manifold of constant sectional curvature c.

By Q_c^n (resp. CQ_c^n) we denote the standard real (resp. complex) simply connected space form of constant sectional (resp. holomorphic) curvature c and real (resp. complex) dimension n. Dajczer and Rodríguez ([3]) have shown that any isometric immersion of a Kähler manifold with $\nu > 0$ everywhere into CQ_c^N, $c \neq 0$, must be holomorphic. From the proof of that theorem and our main result we conclude the following statement.

Corollary 4. *Let M^{2n} be a Kähler manifold and $x_0 \in M^{2n}$ such that $K_M(x_0) \leq c \neq 0$. If $p < n$, then there exists no isometric immersion of M^{2n} into Q_c^{2n+p}.*

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The proofs will appear in Mathematische Annalen.

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