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ISOMETRIC IMMERSIONS AND THE GENERALIZED LAPLACE AND SINH-GORDON EQUATIONS

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We consider isometric immersions $f\colon M_c^n \to Q_{\tilde{c}}^N$, $c \neq \tilde{c}$, with codimension $p = N - n \geq 2$ of a connected n-dimensional Riemannian manifold of constant sectional curvature c into a complete and simply connected Riemannian manifold of constant sectional curvature $\tilde{c}$. The study of the case $c < \tilde{c}$ goes back to E. Cartan (**[Ca]**, **[Mo$_1$]**) who showed that $p = n - 1$ is the lowest possible codimension for which case the normal bundle must be flat. Moreover, if M_c^n is simply connected, there exist line of curvature coordinates (principal coordinates) $u = (u_1, \ldots, u_n)$ such that

$$ds^2 = \sum_{j=1}^{n} v_j^2(u)\, du_j^2, \text{ where } \sum_{j=1}^{n} v_j^2(u) = 1/(\tilde{c} - c).$$

When $c > \tilde{c}$, a large set of (at least local) isometric immersions as above can be constructed in any codimension by composing umbilical inclusions of M_c^n into $Q_{\tilde{c}}^{n+1}$ with local isometric immersions of $Q_{\tilde{c}}^{n+1}$ into $Q_{\tilde{c}}^N$. In fact, for codimension $p \leq n - 2$, it was shown recently in **[D-T]** that f must always be such a *composition* if restricted to any connected component of the open and dense subset of points where some regularity conditions are satisfied.

The above result does not hold in codimension $p = n - 1$. But for that case Moore (**[Mo$_2$]**) has shown that at each point the second fundamental form either has the structure corresponding to a composition (weak-umbilical) or the immersion must have flat normal bundle. Based on that fact we obtain the following correspondence between isometric immersions $f\colon M_c^n \to Q_{\tilde{c}}^{2n-1}$, $c > \tilde{c}$, which are nowhere compositions and solutions of a system of PDEs called by us the *Generalized Laplace equations* for $c = 0$ and for $c \neq 0$ the *Generalized Elliptic Sinh-Gordon equations.*

Theorem . *Let the functions* $v = (v_1, \ldots, v_n)$ *and* $h = (h_{ij})$, $1 \leq i \neq j \leq n$, *be a solution on an open and simply connected subset* $U \subset \mathbf{R}^n$ *of the following completely integrable system of PDEs*

$$(I)\begin{cases} i)\ \frac{\partial v_i}{\partial u_j} = h_{ji}v_j, \quad ii)\ \frac{\partial v_i}{\partial u_i} + \sum_j \epsilon_i\epsilon_j h_{ij}v_j = 0, \quad iii)\ \frac{\partial h_{ij}}{\partial u_k} = h_{ik}h_{kj}, \\ iv)\ \frac{\partial h_{ij}}{\partial u_i} + \frac{\partial h_{ji}}{\partial u_j} + \sum_k h_{ki}h_{kj} + cv_iv_j = 0, \\ v)\ \epsilon_i \frac{\partial h_{ij}}{\partial u_j} + \epsilon_j \frac{\partial h_{ji}}{\partial u_i} + \sum_k \epsilon_k h_{ik}h_{jk} = 0, \\ \text{where } i \neq j \neq k,\ \epsilon_1 = -1 \text{ and } \epsilon_j = 1 \text{ for } j \geq 2 \end{cases}$$

with $v_j \neq 0$ *everywhere and* $\sum_j \epsilon_j v_j^2(u^0) = 1/(\tilde{c} - c) < 0$ *at some point* $u^0 \in U$. *Then there exists an immersion* $f: U \to Q_{\tilde{c}}^{2n-1}$ *with induced metric* $ds^2 = \sum_i v_i^2\, du_i^2$ *of constant sectional curvature c which is nowhere a composition. Conversely, any isometric immersion* $f: M_c^n \to Q_{\tilde{c}}^{2n-1}, c > \tilde{c}$, *which is nowhere a composition, gives rise to a solution of the above system verifying* $\sum_j \epsilon_j v_j^2 = 1/(\tilde{c} - c)$ *everywhere.*

For $n = 2$ and taking $v = 1/\sqrt{c - \tilde{c}}(\cosh\phi, \sinh\phi)$, the above system reduces to the following differential equation which justifies our names.

$$\Delta\phi + \frac{c}{c - \tilde{c}} \cosh\phi \sinh\phi = 0.$$

The above theorem holds similarly for $c < \tilde{c}$ if we just take $\epsilon_1 = 1$ and delete the reference on not being a composition. This is due to Aminov ([**Am**$_1$], [**Am**$_2$]) when $\tilde{c} = 0$. In this case and for $n = 2$, system (I) reduces to either the Wave or the Sine-Gordon equation.

System (I) for $\epsilon_1 = 1$ was extensively studied by Bianchi (see [**Bi**] p. 239) but only from an intrinsic point of view. He considers what he calls a n^{th}-*orthogonal system of Guichard-Darboux*, which is an orthogonal system of coordinates $ds^2 = \sum_{j=1}^n v_j^2\, du_j^2$, which satisfies a quadratic condition $\sum_{j=1}^n v_j^2 =$ *constant*, and has constant sectional curvature c. Then, if the h_{ij} are defined by i), equations iii) and iv express the curvature assumption while equation *ii*) turns out to be equivalent to the quadratic condition. Finally, equation *v*) is a consequence of the quadratic condition and the other equations (cf. Theorem 1 in [**B-T**]). Among many other things, Bianchi proved that any real analytic solution of (I) is completely determined by $n(n-1)$ arbitrary real analytic functions in one variable and n nonzero constants.

For the proof of the above result first we show the existence of principal coordinates for a Riemannian manifold M_c^n isometrically immersed into $\mathbf{O}_c^N$ with flat normal bundle and vanishing index of relative nullity. Here and throughout this paper, $\mathbf{O}_c^N$ denotes a geodesically complete and simply connected N-dimensional manifold with a metric of constant sectional curvature c whose signature is either Riemannian or Lorentzian. Making use of the principal coordinates, we then establish a correspondence between isometric immersions $g\colon M_c^n \to \mathbf{O}_c^N$ of arbitrary codimension with flat normal bundle and vanishing index of relative nullity and the solutions of a certain completely integrable system of PDEs. This allows us to characterize the solutions which correspond to immersions whose images lie in an umbilical hypersurface $Q_{\tilde{c}}^{N-1}$ of $\mathbf{O}_c^N$. By the use of two results we can now establish a correspondence between isometric immersions with flat normal bundle of arbitrary codimension $f\colon M_c^n \to Q_{\tilde{c}}^N, c \neq \tilde{c}$ which are not compositions if $c > \tilde{c}$ and a new completely integrable system of PDEs which reduces to system (I) when the codimension is $n-1$.

We say that a solution of system (I) is *stationary with respect to* u_j if it does not depend on the variable u_j. The solution is called *l-stationary* if it is stationary with respect to l of the coordinates $(u_1, \dots, u_n)$. The purpose of our next result is to geometrically characterize the isometric immersions corresponding to l-stationary solutions as certain types of multi-rotational submanifolds of constant sectional curvature whose profiles satisfy the c-helix property. Isometric immersions $f\colon M_c^n \to \mathbf{R}^{2n-1}$, $c < 0$, associated to 1-stationary solutions of system (I) (here $\epsilon_1 = 1$), were first studied in [**Am**$_2$]. Recently, $(n-1)$-stationary solutions of system (I) have been classified (for $c \leq 0$ and $\epsilon_1 = 1$) by Rabelo and Tenenblat ([**R-T**]) and (for $c \geq 0$ and $\epsilon_1 = -1$) Campos ([**Cam**]) who also provided explicit parametrizations for some cases.

Let $f: U \subset Q_c^m \to \mathbf{O}_0^N$ be an isometric immersion. That f satisfies the c-helix property in the direction of $w \in \mathbf{O}_0^N$ implies, geometrically, that any geodesic $\alpha: I \subset \mathbf{R} \to Q_c^m$ is mapped by f onto a c-helix in the direction of w. This means that $\bar{\alpha} = f \circ \alpha$ verifies along I that

$$\langle \bar{\alpha}'' + c\bar{\alpha}, w \rangle = 0.$$

Multi-rotational submanifolds have been introduced in **[D-N]** and its definition relies on the *warped product representations*

$$\psi: N_0 \times_{\sigma_1} N_1 \times_{\sigma_2} \ldots \times_{\sigma_l} N_l \to Q_{\tilde{c}}^N$$

of $Q_{\tilde{c}}^N$, where the submanifolds $N_0, \ldots, N_l$ through a point $\bar{x} \in Q_{\tilde{c}}^N$ are orthogonal, being N_0 totally geodesic and the N_j, $1 \leq j \leq l$, totally umbilical with orthogonal mean curvature vectors $-a_1, \ldots, -a_l$ in $\mathbf{O}_0 \supset Q_{\tilde{c}}^N$ at $\bar{x}$. The warping functions σ_i: $N_0 \to \mathbf{R}$ are given by

$$\sigma_i(x) = \begin{cases} 1 + \langle a_i, x - \bar{x} \rangle, & \text{if } \tilde{c} = 0, \\ \langle a_i, x \rangle, & \text{if } \tilde{c} \neq 0. \end{cases}$$

In the above and the sequel, when we write $Q_{\tilde{c}}^N \subset \mathbf{O}_0$ we mean that $\mathbf{O}_0 = \mathbf{R}^N$ if $\tilde{c} = 0$ and $\mathbf{O}_0 = \mathbf{O}_0^{N+1}$ if $\tilde{c} \neq 0$.

Denote by G_i the isometry group of N_i suitably embedded as a rotational subgroup into the isometry group of the ambient space $Q_{\tilde{c}}^N$. Let $f_0: V \to N_0$ be an isometric immersion and set $\rho_i = \sigma_i \circ f_0$. By the *multi-rotational submanifold* determined by ψ with *profile* f_0 we mean the $(G_1 \times \ldots \times G_l)$-equivariant isometric immersion $f: V \times_{\rho_1} N_1 \times_{\rho_2} \ldots \times_{\rho_l} N_l \to Q_{\tilde{c}}^N$ given by

$$f(x_0, x_1, \ldots, x_l) = \psi(f_0(x_0), x_1, \ldots, x_l).$$

Theorem *Let $f\colon M_c^n \to Q_{\tilde c}^{2n-1}$, $c \neq \tilde c$, be an isometric immersion associated to an l-stationary solution of system (I). Then f is locally a multi-rotational submanifold*

$$V \times_{\rho_1} S_1 \times_{\rho_2} \ldots \times_{\rho_l} S_l \to Q_{\tilde c}^{2n-1} \approx N_0 \times_{\sigma_1} S_1 \times_{\sigma_2} \ldots \times_{\sigma_l} S_l,$$

where $S_1, \ldots, S_l$ are circles in $Q_{\tilde c}^{2n-1}$ with curvature vectors $-a_1, \ldots, -a_l$ in $\mathbf{O}_0 \supset Q_{\tilde c}^{2n-1}$ respectively, and the profile $f_0\colon V \subset Q_c^{n-l} \to N_0$ satisfies the c-helix property with respect to $a_1, \ldots, a_l$ and is nowhere a composition whenever $c > \tilde c$. Conversely, any multi-rotational submanifold as above is associated to an l-stationary solution of system (I).

For the proof or the understanding of the above result we have to solve several problems of independent interest. First we provide a nice parametric description for any isometric immersion $f\colon U \subset Q_c^m \to \mathbf{O}_0^N$ which satisfies the c-helix property where no assumption on the normal bundle is made. Then we characterize multi-rotational submanifolds of constant sectional curvature and flat normal bundle.

Submanifolds corresponding to $(n-1)$-stationary solutions are multi-rotational submanifolds with flat normal bundle and constant sectional curvature of space forms having curves as profiles. In this case, we are able to provide a complete parametric description based on the fact that curves satisfying the c-helix property can be completely described in a parametric form. This extends results in [**R-T**] and [**Cam**]).

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