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Orientations of graphs

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Abstract

We provide here a short survey on orientations of graphs. We concentrate on problems with connectivity properties.

1 Introduction

Mr. Orient, the Mayor of the city called "The Edges", having wanted to make the main street a one way street, unfortunately made a mistake by ordering the "one way" sign and received 100 signs, as many as the number of streets in the city. To be justified, he decides to use all the signs, i.e. *to make all the streets of the city one way.* Having finished his plan, he realizes that it does not enable him to go home. He thus goes back to work while keeping in mind that he must be able, *from the City Hall, to reach any point of the city.* After one moment of reflexion, he realizes that he must be able, *from any point of the city, to reach all the others.* Being proud of himself, he presents his project to his assistant, a well-balanced man, who reminds him that during summer, some streets of the city may be blocked by floods, they thus try to conceive a plan where *blocking any street does not make a district inaccessible.* But they are still not satisfied; examining their plan, they see that there are far too many paths from the downtown to the shopping center and not enough in the other direction. They try an ultimate improvement: to place the "one way" signs so that *the orientation of the streets be well-balanced.*

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Since then, the city was renamed "The Arcs".

2 Definitions

Given an undirected graph $G = (V, E)$ and $X \subset V$, $d_G(X)$ denotes the number of edges of G entering X and $i_G(X)$ denotes the number of edges of G in X. The **local-edge-connectivity**, $\lambda_G(u, v)$, in G from u to v is defined to be the maximum number of edge-disjoint paths from u to v in G. The graph G is said to be k**-edge-connected** if $\lambda_G(u, v) \geq k \ \forall u, v \in V$. By Menger's theorem [12], the maximum number of edge-disjoint paths from u to v is equal to the minimum cardinality of a cut separating u and v, that is $\lambda_G(u, v) = \min\{d_G(X) : v \in X, u \notin X\}$. Consequently, G is k-edge-connected if and only if $d_G(X) \geq k \ \ \forall \emptyset \neq X \subset V$.

Given a directed graph $D = (V, A)$ and $X \subset V$, $d_D^-(X)$ denotes the **in-degree of** X, that is the number of arcs of D entering X and $d_D^+(X)$ denotes the **out-degree of** X, that is the number of arcs of D leaving X. The **local-arc-connectivity**, $\lambda_D(u, v)$, in D from u to v is defined to be the maximum number of arc-disjoint paths from u to v in D. The graph D is said to be **root-connected at** s, where s is a given vertex if $\lambda_D(s, v) \geq 1 \ \forall v \in V - s$, and it is k**-root-connected at** s if $\lambda_D(s, v) \geq k \ \forall v \in V - s$. We say that D is **strongly connected** if $\lambda_D(u, v) \geq 1 \ \forall (u, v) \in V \times V$, and it is k**-arc-connected** if $\lambda_D(u, v) \geq k \ \forall (u, v) \in V \times V$. By Menger's theorem [12], the maximum number of arc-disjoint paths from u to v is equal to the minimum in-degree of a vertex set containg v but not u, that is $\lambda_D(u, v) = \min\{d_D^-(X) : v \in X, u \notin X\}$. Consequently, D is root-connected at s if and only if $d_D^-(X) \geq 1 \ \ \forall \emptyset \neq X \subseteq V - s$, D is k-root-connected at s if and only if $d_D^-(X) \geq k \ \ \forall \emptyset \neq X \subseteq V - s$, D is strongly connected if and only if $d_D^-(X) \geq 1 \ \ \forall \emptyset \neq X \subset V$ and D is k-arc-connected if and only if $d_D^-(X) \geq k \ \ \forall \emptyset \neq X \subset V$.

3 Warming up

In this section we are given an undirected graph G and we want to find an orientation of G that satisfies one of the above defined connectivity properties.

The first property to be considered is *root-connectivity.* The following result is an easy exercice.

Theorem 3.1. *Given an undirected graph G and a vertex s of G, there exists a root-connected orientation of G at s if and only if G is connected.*

Indeed, such an orientation exists if and only if there exists an orientation of G containing an s-arborescence. This is equivalent to the existence of a spanning tree of G, that is to the connectivity of G.

The next connectivity property is *k-root-connectivity.* For a partition $\mathcal{P} = \{V_1, \ldots, V_t\}$ of V, $E(\mathcal{P})$ denotes the set of edges of G between the different members of $\mathcal{P}$ and $|\mathcal{P}| = t$. Note that Theorem 3.2 generalizes Theorem 3.1.

Theorem 3.2 (Frank)**.** *Given an undirected graph $G = (V, E)$, a vertex s of G and an integer $k \geq 1$, there exists an orientation of G that is k-root-connected at s if and only if*

$$|E(\mathcal{P})| \geq k(|\mathcal{P}| - 1) \quad \textit{for every partition } \mathcal{P} \textit{ of } V. \tag{3.1}$$

By Menger's theorem [12], such an orientation exists if and only if there exists an orientation $\vec{G}$ of G such that $d^-_{\vec{G}}(X) \geq k \ \ \forall \, \emptyset \neq X \subseteq V - s$. This is equivalent, by Edmonds'theorem [3], to the existence of an orientation of G containing k arc-disjoint s-arborescences. Trivially, such an orientation exists if and only if there exist k edge-disjoint spanning trees in G. By Nash-Williams' theorem [13], this is equivalent to condition (3.1).

Now we continue with *strong connectivity.* The following result is due to Robbins [15].

Theorem 3.3 (Robbins)**.** *Given an undirected graph G, there exists a strongly connected orientation of G if and only if G is 2-edge-connected.*

Indeed, by Robbins' theorem [15], an orientation $\vec{G}$ of G is strongly connected if and only if $\vec{G}$ has a directed ear-decomposition that is $\vec{G}$ can be constructed from a vertex by adding at each step a directed path whose end-vertices belong to the graph constructed so far but the inner vertices do not. This is equivalent to the existence of an ear-decomposition of G (the undirected counterpart), and hence to the 2-edge-connectivity of G.

Let us finish this section with *k-arc-connectivity*. The following weak-orientation theorem of Nash-Williams [14] generalizes Theorem 3.3.

Theorem 3.4 (Nash-Williams)**.** *Given an undirected graph G, there exists a k-arc-connected orientation of G if and only if G is $2k$-edge-connected.*

To see the necessity of the condition let $\vec{G}$ be a k-arc-connected orientation of G. By Menger's theorem [12], for any vertex set X, there are k arcs entering X and k arcs leaving X in $\vec{G}$, that is there are $2k$ edges entering X in G, so by Menger's theorem [12], G is $2k$-edge-connected.

To see the sufficiency we may get a minimally $2k$-edge-connected graph G' by deleting some edges of G. Then, by Mader's theorem [11], there exists a vertex s of degree $2k$ in G'. By Lovász' theorem [10], there exists a $2k$-admissible complete splitting off, that is we can replace pairs of edges incident to s by edges between the two other end-vertices to get a $2k$-edge-connected graph G'' on the vertex set $V - s$. Then, by induction, there exists a k-arc-connected orientation $\vec{G}''$ of G''. By hooking up the arcs split off before, $\vec{G}''$ provides an orientation $\vec{G}'$ of G'. Since the in- and out-degree of the vertex s will be k, $\vec{G}'$ is k-arc-connected. By adding the deleted edges with arbitrary orientations to $\vec{G}'$, we get a k-arc-connected orientation $\vec{G}$ of G.

4 Degree constrained versions

We say that the vector m on V is the **in-degree vector** of D if $m(v) = d_D^-(v)\ \forall v \in V$. We recall that $d_D^-(X)$ is the in-degree function of D. Note that

$$m(X) - i_D(X) = d_D^-(X). \tag{4.1}$$

Indeed, in the sum $m(X)$ each arc of $i_D(X)$ and each arc entering X is counted once.

By (4.1), the in-degree vector characterizes the in-degree function of D. Moreover, as we have seen, the in-degree function characterizes the connectivity properties of D. In other words, if we know only the underlying undirected graph of D and the in-degree vector of D, then we know everything on the connectivity properties of D.

The following result [8] characterizes graphs having an orientation with a given in-degree vector.

Theorem 4.1 (Hakimi)**.** *Given an undirected graph $G = (V, E)$ and a vector $m : V \to \mathbb{Z}_+$, there exists an orientation of G with in-degree vector m if and only if*

$$m(X) \geq i_G(X)\ \forall X \subseteq V, \tag{4.2}$$

$$m(V) = |E|. \tag{4.3}$$

The necessity follows from (4.1). The sufficiency can be proved by taking an arbitrary orientation of G and if its in-degree vector is not m, then by repeating the following : reorient a path from a vertex whose in-degree is too small to a vertex whose in-degree is too big.

We mention that Theorem 4.1 has nice applications: Eulerian orientation of an undirected graph (Euler), Eulerian orientation of a mixed graph (Ford-Fulkerson), perfect matching in a bipartite graph (Hall, Frobenius), f-factor in a bipartite graph (Ore, Tutte).

Now let us see the problems of Section 3 with degree constraint.

Theorem 4.2 (Frank)**.** *Given an undirected graph* $G = (V, E)$ *and a vector* $m\colon V \to \mathbb{Z}_+$*, there exists an orientation of* G *with in-degree vector* m *that is*

(a) root-connected if and only if $m(X) - i_G(X) \geq 1 \ \ \forall\, \emptyset \neq X \subseteq V - s$ *and* $m(V) = |E|$.

(b) k-root-connected if and only if $m(X) - i_G(X) \geq k \ \ \forall\, \emptyset \neq X \subseteq V - s$ *and* $m(V) = |E|$.

(c) strongly connected if and only if $m(X) - i_G(X) \geq 1 \ \ \forall\, \emptyset \neq X \subset V$ *and* $m(V) = |E|$.

(d) k-arc-connected if and only if $m(X) - i_G(X) \geq k \ \ \forall\, \emptyset \neq X \subset V$ *and* $m(V) = |E|$.

Note that each of these conditions implies, by Theorem 4.1, that there exists an orientation $\vec{G}$ of G with in-degree vector m and, by (4.1), $\vec{G}$ will have the desired connectivity property for free.

5 Well-balanced orientation

To introduce well-balanced orientations, we start with an easy remark on Eulerian orientations.

Claim 1. *If* G *is an Eulerian graph then there exists an (Eulerian) orientation* $\vec{G}$ *of* G *such that* $d^-_{\vec{G}}(v) - d^+_{\vec{G}}(v) = 0 \ \forall v \in V$ *and* $\lambda_{\vec{G}}(u,v) = \frac{1}{2}\lambda_G(u,v) \ \forall (u,v) \in V \times V$.

An orientation $\vec{G}$ of a graph G is called **smooth** if (5.1) is satisfied, it is **well-balanced** if (5.2) is satisfied. If both conditions are satisfied then we say that the orientation is **best-balanced**.

$$|d^-_{\vec{G}}(v) - d^+_{\vec{G}}(v)| \leq 1 \qquad \forall v \in V, \tag{5.1}$$

$$\lambda_{\vec{G}}(u,v) \geq \lfloor \frac{1}{2}\lambda_G(u,v) \rfloor \qquad \forall (u,v) \in V \times V. \tag{5.2}$$

The best we may hope for an arbitrary graph is to find a best-balanced orientation. The strong orientation theorem of Nash-Williams [14] says that this hope can be satisfied.

Theorem 5.1 (Nash-Williams)**.** *Every graph G admits a best-balanced orientation.*

Note that of course the strong orientation theorem implies the weak orientation theorem.

Let us denote by T_G the set of odd degree vertices of G. A **pairing** of T_G (or of G) is a new edge set M on T_G such that exactly one edge of M is incident to each vertex of T_G. Let us reformulate the strong orientation theorem as follows.

Theorem 5.2 (Nash-Williams)**.** *There exists a pairing M of G and there exists an eulerian orientation $\vec{G}+\vec{M}$ of $G+M$ such that $\vec{G}$ is well-balanced.*

In fact Nash-Williams proved the following stronger result called the pairing theorem [14].

Theorem 5.3 (Nash-Williams)**.** *There exists a pairing M of G such that for every eulerian orientation $\vec{G}+\vec{M}$ of $G+M$, $\vec{G}$ is well-balanced.*

Let us continue with some generalizations of the strong orientation theorem.

Theorem 5.4 (Nash-Williams)**.** *For every subgraph H of a graph G, there exists an orientation $\vec{G}$ of G such that $\vec{G}$ and $\vec{G}(H)$ are best-balanced orientations of G and H.*

In [9], we extended this result:

Theorem 5.5 (Király-Szigeti)**.** *For every partition $\{E_1, \dots E_k\}$ of $E(G)$, there exists an orientation $\vec{G}$ of G such that $\vec{G}$ and $\vec{G}(E_i)$ $\forall i$ are best-balanced orientations of the corresponding graphs.*

We provided in [9] another extension of the strong orientation theorem. We denote by $\vec{G}/\overline{X}$ the directed graph obtained from $\vec{G}$ by contracting the complement of X.

Theorem 5.6 (Király-Szigeti)**.** *For every partition $\{V_1, \dots V_k\}$ of $V(G)$, there exists an orientation $\vec{G}$ of G such that $\vec{G}$ and $\vec{G}/\overline{V}_i$ $\forall i$, are best-balanced orientations of the corresponding graphs.*

We mention that the last two results can be proved using the pairing theorem, see [9].

6 Polyhedral aspects

In this section we consider the polyhedral aspects of k-arc-connected orientations and then that of well-balanced orientations.

Let us define the following polyhedra:

$$P_G^k := \{m : \mathbb{R}^V : m(X) \geq i_G(X) + k \ \forall X \subset V, m(V) = |E|\}.$$

By Theorem 4.2, the integer points of P_G^k are exactly the in-degree vectors of k-arc-connected orientations of G. Frank and Tardos [7] showed that P_G^k is an integer polyhedra: the function $p(X) = i_G(X) + k$ if $X \neq \emptyset, V$ and 0 otherwise is crossing supermodular so P_G^k is a base polyhedra and hence P_G^k is an integer polyhedra. It follows that P_G^k is the convex hull of in-degree vectors of k-arc-connected orientations of G. Frank [6] showed that the minimum cost k-arc-connected orientation problem can be solved in polynomial time.

Let us define now the following polyhedra:

$$P_G^w := \{m : \mathbb{R}^V : m(X) \geq i_G(X) + R_G(X) \ \forall X \subset V, m(V) = |E|\},$$

where $R_G(X) = \max\{\lfloor \frac{1}{2}\lambda_G(u,v)\rfloor : u \in X, v \in V - X\}$.

It is easy to see that the integer points of P_G^w are exactly the in-degree vectors of well-balanced orientations of G. In [2], we gave an example that shows that P_G^w is not an integer polyhedra in general. In the same paper we have also shown that the minimum cost well-balanced orientation problem is NP-complete.

7 k-vertex-connected orientations

A directed graph $D = (V, A)$ with $|V| > k$ is **k-vertex-connected** if $D - X$ is strongly connected for all $X \subset V$ with $|X| = k - 1$.

The following conjecture says that the natural necessary conditions to have a k-vertex-connected orientation are sufficient.

Conjecture 1 (Frank, Thomassen). *Given an undirected graph $G = (V, E)$ with $|V| > k$, there exists a k-vertex-connected orientation of G if and only if $G - X$ is $(2k - 2|X|)$-edge-connected for all $X \subseteq V$ with $|X| < k$.*

This conjecture is open even in the special case when $k = 2$. Let us formulate this case.

Conjecture 2 (Frank, Thomassen). *Given an undirected graph $G = (V, E)$ with $|V| > 2$, there exists a 2-vertex-connected orientation of G if and only if G is 4-edge-connected and $G - v$ is 2-edge-connected for all $v \in V$.*

It is known that this is true for Eulerian graphs [1]:

Theorem 7.1 (Berg-Jordán). *Given an Eulerian graph $G = (V, E)$ with $|V| > 2$, there exists a 2-vertex-connected (Eulerian) orientation of G if and only if $G - v$ is 2-edge-connected for all $v \in V$.*

This last theorem can be generalized as follows and it can be proved by the pairing theorem, see [9].

Theorem 7.2 (Király-Szigeti). *Given an Eulerian graph $G = (V, E)$ with $|V| > k + 1$, there exists an Eulerian orientation $\vec{G}$ of G such that $\vec{G} - v$ is k-arc-connected $\forall\ v \in V$ if and only if $G - v$ is $2k$-edge-connected for all $v \in V$.*

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