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On split clique graphs*

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Abstract

A *complete set* of a graph G is a subset of V_G whose elements are pairwise adjacent. A *clique* is a maximal complete set. The *clique graph* of G, denoted by $K(G)$, is the intersection graph of the family of cliques of G. The clique graph recognition problem asks whether a given graph is a clique graph. This problem was classified recently as NP-complete after being open for 30 years. The complexity of this decision problem is open for very structured and well studied classes of graphs such as planar graphs and chordal graphs. We propose the study of split clique graphs.

1 Introduction

Let G be a finite, simple and undirected graph; V_G and E_G denote its vertex set and its edge set respectively. A *complete set* of G is a subset of V_G whose elements are pairwise adjacent. A *clique* is a maximal complete set. A *stable set* is a subset of V_G whose elements are pairwise non-adjacent.

The *clique graph* of G, denoted by $K(G)$, is the intersection graph of the family of cliques of G. The graph G is *a clique graph* if there exists a graph H such that $G = K(H)$. The Hajós graph depicted in Figure 1 is the graph with minimum number of vertices and edges that is not clique graph.

A sufficient condition for a graph to be a clique graph was given in [3], and characterizations of clique graphs are given in [4] and more recently in [1].

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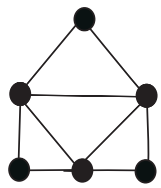

Figure 1: The Hajós graph.

However the time complexity of the problem of recognizing clique graphs was open for 30 years and was established as NP-complete recently in [2]. The complexity of this decision problem is open for very structured and well studied classes of graphs such as planar graphs and chordal graphs [5]. In the present work we propose the study of split clique graphs.

A set family has the *Helly property* if any pairwise intersecting subfamily has non-empty total intersection. A complete set *covers an edge* e if both end vertices of e belong to the set. The following Theorem characterizes clique graphs [4].

Theorem 1.1. *A graph G is a clique graph if and only if there exists a family of complete sets of G which has the Helly property and covers the edges of G.*

2 Split graphs

The graph G is *a split graph* if V_G can be partitioned into a stable set S and a complete set K. The pair (S, K) is called a *split partition* of G. In order to obtain a unique possible split partition, without loss of generality, we ask each vertex of K to be adjacent to some vertex of S; this means $K = \cup_{s \in S} N(s)$, where, as usual, $N(s)$ denotes the open neighborhood of the vertex s. Observe that if $s \in S$ then its closed neighborhood $N[s] = N(s) \cup \{s\}$ is a clique of G. Besides, the complete set K is a clique of G if and only if K is not the open neighborhood of some vertex $s \in S$.

We say that a vertex $x \in K$ is a *private neighbor* of $s \in S$, if s is the only vertex in S adjacent to x. Observe that s has a private neighbor if and only

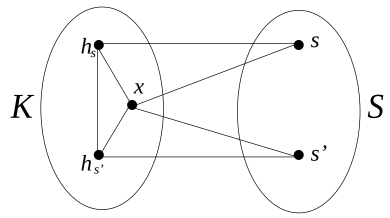

Figure 2: A split graph $G = (V, E)$ with partition (S, K) for V, and partition $(R_s = \{h_s\}, R_{s'} = \{h_{s'}\}, R_1 = \{x\})$ for K.

if $N(s)$ is not contained in the union of $N(s')$ for $s' \in S$, $s' \neq s$.

Theorem 2.1. *Let G be a connected graph with split partition (S, K). If every vertex $s \in S$ has a private neighbor then G is a clique graph.*

Proof. Let G be a graph with split partition (S, K) satisfying every vertex $s \in S$ has a private neighbor h_s. Let x and y be vertices of K. We say that x is a twin of y when $N[x] = N[y]$. Observe this is an equivalence relation, and so the equivalence classes define a partition of K. Let R_s be the class of h_s for $s \in S$; and $R_1, R_2, ..., R_k$ the remaining classes, this means the classes that do not contain any vertex h_s for $s \in S$. We notice that $\left((R_s)_{s\in S}, R_1, R_2, R_3, \dots R_k\right)$ is a partition to K. Since h_s is a private neighbor of s, if $s' \in S$ and $s' \neq s$ then $R_s \neq R_{s'}$. For the convenience of the reader we offer in Figure 2 an example of a graph $G = (V, E)$ with the private neighbors of each vertex of S. We observe that there is only one class R_1 of vertices not twins of private neighbors.
For every $s \in S$, we call I_s to the set $\{i, 1 \leq i \leq k$ such that $R_i \subseteq N(s)\}$.
Let $\mathcal{F}$ be the family of complete sets of G whose members are

- K;
- $F_{s,i} = R_s \cup \{s\} \cup R_i$, for each $s \in S$ such that $I_s \neq \emptyset$ and for each $i \in I_s$;
- $F_s = R_s \cup \{s\}$ for each $s \in S$ such that $I_s = \emptyset$.

We claim that $\mathcal{F}$ satisfies the conditions given by Theorem 1.1; then G is a clique graph. Indeed, $\mathcal{F}$ covers the edges of G and has the Helly property.
To prove the former, let $e \in E_G$. If both end vertices of e are in K then e is covered by the same K which is a member of $\mathcal{F}$. If not, since S is a stable set, then $e = sx$ with $s \in S$ and $x \in K$. If there exists i, $1 \leq i \leq k$ such that $x \in R_i$, since every vertex in R_i is a twin of x and x is adjacent to s, then $R_i \subseteq N(s)$. It follows that $F_{s,i}$ covers $e = sx$.
If such i does not exist, then x belongs to the equivalence class of a private neighbor $h_{s'}$. Since $sx \in E$, $s = s'$ and $x \in R_s$. In this case, e is covered by F_s.
To prove that $\mathcal{F}$ has the Helly property, notice the following facts,

1. F_s is not a member of $\mathcal{F}$ if and only if there exists i such that $F_{i,s}$ is a member of $\mathcal{F}$.

2. if $F_{s,i} \cap F_{s',i'} \neq \emptyset$ then $i = i'$ or $s = s'$.

3. a set F_s has empty intersection with all members of $\mathcal{F}$ except K.

Now, assume $\mathcal{F}'$ is a pairwise intersecting subfamily with at least three members. Consider the members that are not K. By fact 3, all them must be of type $F_{s,i}$. Moreover, by fact 2, there must exists i such that all these members have a same subindex i; or there must exists s such that all them have a same subindex s. In the first case, all members of $\mathcal{F}'$ have the vertices of R_i in common. In the second case, all members of $\mathcal{F}'$ have the vertices of R_s in common. Observe that in this case we can not ensure s is a common vertex since K may be a member of $\mathcal{F}'$. It follows that $\mathcal{F}'$ has non-empty total intersection. This completes the proof. $\square$

Theorem 2.2. *Let G be a graph with split partition (S, K) and $\mid S \mid\leq 3$. The graph G is a clique graph if and only if G is not the Hajós graph in Figure 1.*

Proof. It is well known that if G is a clique graph then G is not the Hajós graph. Let us prove the reciprocal implication. Assume G is a graph with

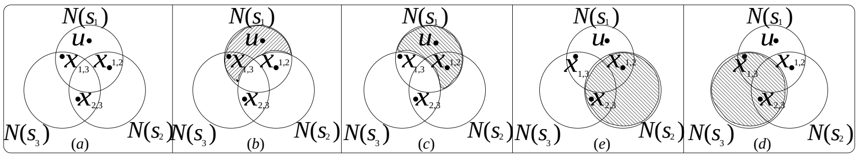

Figure 3: Case in which u is a private neighbor, assumed of s_1.

split partition (S, K), $\mid S \mid \leq 3$ and G is not the Hajós graph. By Theorem 1.1, if the clique family of G has the Helly property then G is a clique graph. If the clique family does not satisfy the Helly property, then there exists a subfamily of cliques pairwise intersecting without a common vertex. It is clear that such subfamily must contain $N[s_1]$, $N[s_2]$ and $N[s_3]$ as members, where s_1, s_2 and s_3 are the vertices in S.

For $1 \leq i < j \leq 3$, let $x_{i,j}$ be three vertices of K such that $x_{i,j} \in N[s_i] \cap N[s_j]$. Since G is not the Hajós graph, then K must contain at least one more vertex.

Let u be that vertex and suppose u is a private neighbor, for instance of s_1, then $u \in N[s_1] \setminus (N[s_2] \cup N[s_3])$. In this case it is easy to check that $\mathcal{F}$ the complete set family

$$N[s_1] \setminus N[s_2], \quad N[s_1] \setminus N[s_3], \quad N[s_2], \quad N[s_3] \quad \text{and} \quad K$$

satisfies the conditions given by Theorem 1.1, then G is a clique graph. We depict in Figure 3 such family. Observe that if $N[s_1] \setminus N[s_2]$ (Figure 3(b)) belongs to a pairwise intersecting family $\mathcal{F}'$, then $N[s_2] \notin \mathcal{F}'$ (Figure 3(e)), since their intersection is empty. The same occurs between $N[s_1] \setminus N[s_3]$ (Figure 3(c)) and $N[s_3]$ (Figure 3(d)). Hence, three intersecting complete sets of $\mathcal{F}'$ have $x_{1,2}$, or $x_{2,3}$, or u as a common element.

If u is not a private neighbor, we can assume $N(s_i) \setminus (N(s_j) \cup N(s_k)) = \emptyset$ for the three different possible subindices. Then u is adjacent to at least two vertices of S; without loss of generality assume $u \in N(s_1) \cap N(s_2)$. In this case it is easy to check that $\mathcal{F}$ the complete set family

$$N[s_1] \setminus (N[s_2] - \{u\}), \quad N[s_1] \setminus N[s_3], \quad N[s_2] \setminus (N[s_1] - \{x_{1,2}\}),$$

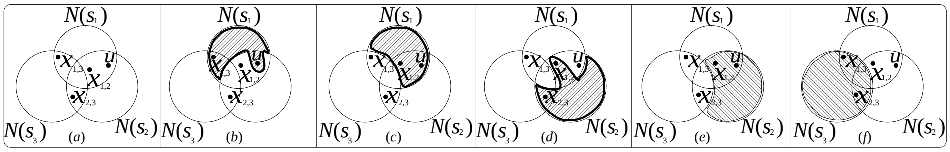

Figure 4: Case in which u is not a private neighbor.

$$N[s_2] \setminus N[s_3], \quad N[s_3] \quad \text{and} \quad K$$

satisfies the conditions given by Theorem 1.1, then G is a clique graph. We depict in Figure 4 such family. Observe that if $N[s_1] \setminus (N[s_2] - \{u\})$ (Figure 4(b)) belongs to a pairwise intersecting family $\mathcal{F}'$, then $N[s_2] \setminus (N[s_1] - \{x_{1,2}\}) \notin \mathcal{F}'$ (Figure 4(d)). The same occurs between $N[s_1] \setminus N[s_3]$ (Figure 4(c)) and $N[s_3]$ (Figure 4(f)). Hence, three intersecting complete sets of $\mathcal{F}'$ have $x_{1,2}$, or $x_{2,3}$, or u as a common element. The proof is complete. $\square$

3 Conclusion

This work presents sufficient conditions for a split graph to be a clique graph. The complexity of recognizing split clique graphs remains open.

References

[1] L. Alcón, M. Gutierrez, A new characterization of Clique Graphs. *Mat. Contemp.*, **25**, (2003), 1–7.

[2] L. Alcón, L. Faria, C. M. H. de Figueiredo, M. Gutierrez, Clique graph recognition is NP-complete. *Proceedings of Graph-theoretic concepts in computer science, WG 2006, Lecture Notes in Comput. Sci.*, vol. 4271, Springer, 2006, pp. 269–277.

[3] R. C. Hamelink, A partial characterization of clique graphs. *J. Combin. Theory Ser. B*, **5**, (1968), 192–197.

[4] F. S. Roberts, J. H. Spencer, A characterization of clique graphs. *J. Combin. Theory Ser. B*, **10**, (1971), 102–108.

[5] J. L. Szwarcfiter, A survey on Clique Graphs. in *Recent Advances in Algorithms and Combinatorics*, C. Linhares-Sales and B. Reed, eds., CMS Books Math./Ouvrages Math. SMC, 11, Springer, New York, 2003, pp. 109–136.

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