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Multiplicity of Nontrivial Solutions to a Problem Involving the Weighted p-Biharmonic Operator

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Dedicated to Professor J. V. Gonçalves on the occasion of his 60th birthday

Abstract

In this paper we prove the existence of three solutions to a problem involving the weighted p-biharmonic operator. The first and second solutions are obtained as local minima using the Ekeland's Variational Principle and the third one is obtained by a variant of the Mountain Pass Theorem.

1 Introduction

In this paper we study the following class of quasilinear elliptic problems involving the p-biharmonic operator

$$\begin{cases} \Delta(\rho(x)|\Delta u|^{p-2}\Delta u) + g(x,u) = \lambda_1 h(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 = \Delta u & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary, $\rho \in \mathcal{C}(\bar{\Omega}, \mathbb{R})$ with $\inf_{\bar{\Omega}} \rho(x) > 0$. We also use the assumptions

(G_1)

$g : \Omega \times \mathbb{R} \to \mathbb{R}$ is bounded continuous function satisfying $g(x, 0) = 0$,

and its primitive denoted by

$$(G_2) \qquad G(x,s) = \int_0^s g(x,t)dt \text{ is assumed to be bounded.}$$

Let $X \equiv W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ be a Sobolev space endowed with the norm given by

$$\|u\| \equiv \left\{ \int_\Omega \rho |\Delta u|^p dx \right\}^{\frac{1}{p}}.$$

We define

$$\lambda_1 = \inf_N \left\{ \int_\Omega \rho |\Delta u|^p dx \right\},$$

where

$$N = \left\{ u \in X \colon \int_\Omega h|u|^p dx = 1 \right\},$$

the first eigenvalue of the following weighted eigenvalue problem

$$\begin{cases} \Delta(\rho(x)|\Delta u|^{p-2}\Delta u) = \lambda_1 h(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 = \Delta u \qquad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where

(h)

$h \in \mathcal{C}(\bar{\Omega}, \mathbb{R})$, $h \geq 0$ and $h > 0$ on a subset of Ω with positive measure.

We recall that by using a result by Talbi and Tsouli [18] (see also Drábek and Ôtani [8]), we know that the first eigenvalue λ_1 is simple, isolated and positive. Moreover every eigenfunction ϕ_1 associated with λ_1 can be chosen positive.

Here $\Delta(\rho(x)|\Delta u|^{p-2}\Delta)$ denotes the operator of fourth order called the p-biharmonic operator with weight. For $p = 2$ and $\rho = 1$, the operator becomes the iterated Laplacian which have been studied by many authors.

For example, Lazer and McKenna [13] have pointed out that this type of nonlinearity furnishes a model for studying travelling waves in suspension bridges. Since then, more nonlinear biharmonic equations, including the p-biharmonic equations, have been studied. (See [14, 19].)

More exactly, this type of problem appears, for instance, in the study of Hooke's law of nonlinear elasticity. (See [4, 6] and references therein.) While the p-biharmonic operator can be used to study a semilinear hamiltonian system of the form

$$\begin{cases} -\Delta u = v^p \quad \text{in} \quad \Omega, & -\Delta v = u^q \quad \text{in} \quad \Omega, \\ \quad u, v > 0 \quad \text{in} \quad \Omega, & \quad u, v = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is smooth bounded domain and $p, q \geq 1$.

Formally, from the first equation we have

$$v = (-\Delta u)^{1/p}$$

and substituting on the second equation, we get

$$\begin{aligned} -\Delta(|-\Delta u|^{1/p-1}(-\Delta u)) = -\Delta(-\Delta u)^{1/p} = u^q, \qquad & x \in \Omega \\ u = \Delta u = 0, \qquad & x \in \partial\Omega. \end{aligned}$$

In this case, we are looking for solution in the Sobolev space $W^{2,(p+1)}(\Omega)$. (See [7, 11]).

We define the energy functional $I : X \longrightarrow \mathbb{R}$ associated to problem (1.1) by

$$I(u) \equiv \frac{1}{p} \int_\Omega \rho |\Delta u|^p dx + \int_\Omega G(x, u) dx - \frac{\lambda_1}{p} \int_\Omega h |u|^p dx. \tag{1.3}$$

Under assumptions G_1 and G_2, the functional $I \in \mathcal{C}^1(\Omega, \mathbb{R})$ and its Fréchet derivative is given by

$$I'(u) \cdot v = \int_\Omega \rho |\Delta u|^{p-2} \Delta u \Delta v dx + \int_\Omega g(x, u) v dx - \lambda_1 \int_\Omega h |u|^{p-2} u v dx. \tag{1.4}$$

The main goal of this paper is to show the existence of multiple solutions for problem (1.1). We were inspired by Gonçalves and Miyagaki [10] and also by Alves, Carrião and Miyagaki [3], in which problems involving the laplacian and p-laplacian operators are studied, respectively. See also Ma and Sanches [15].

We define

$$V = \langle \phi_1 \rangle \qquad \text{and} \qquad Z = \left\{ u \in X \colon \int_{\mathbb{R}} hu|\phi_1|^{p-2}\phi_1 = 0 \right\}.$$

Note that Z is a closed complementary subspace of V and therefore we have the direct sum

$$X = V \oplus Z.$$

We define

$$\lambda_2 = \inf_Z \left\{ \int_\Omega \rho|\Delta u|^p dx \colon \int_\Omega h|u|^p dx = 1 \right\}, \tag{1.5}$$

which satisfies $0 < \lambda_1 < \lambda_2$, and it follows that

$$\int_\Omega h|w|^p dx \leq \frac{1}{\lambda_2} \int_\Omega \rho|\Delta w|^p dx, \ \text{ for all } w \in Z. \tag{1.6}$$

We impose the following

(G_3) $$g(x,t) \to 0 \text{ as } |t| \to \infty, \text{ for all } x \in \Omega.$$

(G_4) $$G(x,t) \geq \frac{\lambda_1 - \lambda_2}{p} h(x)|t|^p, \text{ for all } x \in \Omega \text{ and for all } t \in \mathbb{R}.$$

(G_5) There exist $\delta > 0$ and $0 < m < \lambda_1$ such that

$$G(x,t) \geq \frac{m}{p} h(x)|t|^p, \text{ for all } x \in \Omega \text{ and for all } |t| < \delta.$$

We define

$$T(x) = \liminf_{|t|\to\infty} G(x,t) \text{ and } S(x) = \limsup_{|t|\to\infty} G(x,t) \text{ for all } x \in \Omega.$$

(G_6) There exist $t^-, t^+ \in \mathbb{R}$ with $t^- < 0 < t^+$ such that

$$\int_\Omega G(x,t^\pm)\phi_1 dx \leq \int_\Omega T(x) dx < 0$$

and

$$(G_7) \qquad \int_\Omega S(x)dx \le 0.$$

Define the following subsets

$$C^+ = \{t\phi_1 + z : \; t \ge 0 \text{ and } z \in Z\} \text{ and } C^- = \{t\phi_1 + z : \; t \le 0 \text{ and } z \in Z\}.$$

We remark that $\partial C^+ = \partial C^- = Z$.

Now we state our main result.

Theorem 1.

(i) Under assumptions (h), (G_1), (G_2), (G_4) and (G_6), there exist $u \in C^+$ and $v \in C^-$ solutions of problem (1.1) such that $I(u) < 0$ and $I(v) < 0$.

(ii) Under assumptions (h), (G_1)–(G_3), (G_5)–(G_7), problem (1.1) has a solution w such that $I(w) > 0$.

The first and second solutions are obtained as local mimima of the energy functional I. To do this, we use the Ekeland's variational principle in each of the subsets C^+ and C^-. The third solution is obtained by using a variant of the Mountain Pass Theorem. In the last section we give an example for Theorem 1.

2 Preliminary results

We begin by recalling that $I : X \to \mathbb{R}$ is said to satisfy the Palais-Smale condition at the level $c \in \mathbb{R}$ ($(PS)_c$ in short), if any sequence $\{u_n\} \subset X$ such that

$$I(u_n) \to c \qquad \text{and} \qquad I'(u_n) \to 0 \text{ as } n \to \infty,$$

has a convergent subsequence in X.

Our first lemma is proved by adapting some arguments used by Anane and Gossez [1] and by Alves, Carrião and Miyagaki [3].

Lemma 2. *Assume the conditions (h), (G_1) and (G_2). Then the functional I satisfies the $(PS)_c$ condition for all $c < \int_\Omega T(x)dx$.*

Proof. We will prove that the sequence $\{u_n\} \subset X$ is bounded. Suppose, on the contrary, that it is unbounded. Then, up to subsequence, we have

$$\|u_n\| \to \infty \text{ as } n \to \infty.$$

Define

$$v_n = \frac{u_n}{\|u_n\|}. \tag{2.1}$$

Clearly $\|v_n\| = 1$ and the sequence $\{v_n\} \subset X$ is bounded. Taking a subsequence if necessary (still denoted in the same way) we obtain

$$v_n \rightharpoonup v \text{ weakly in } X \quad \text{as } n \to \infty$$

and

$$v_n \to v \text{ in } L^s(\mathbb{R}), \text{ as } n \to \infty, \text{ for } 1 \leq s < p^\star = \frac{np}{n-2p}, \tag{2.2}$$
$$\text{and } p^\star = +\infty, \text{ if } \; n \leq 2p.$$

We will show that $v \neq 0$ and that there exists $\mu \in \mathbb{R}$ such that

$$v(x) = \mu\phi_1(x) \text{ for all } x \in \Omega.$$

We are going to consider only the case $n > 2p$, the other case is easier. By definition of I and by the fact that $\Delta u_n = \Delta v_n \|u_n\|$ we have

$$\begin{aligned} I'(u_n) \cdot u_n &= \int_\Omega \rho|\Delta u_n|^p dx + \int_\Omega g(x, u_n)u_n dx - \lambda_1 \int_\Omega h|u_n|^p dx \\ &= \|u_n\|^p \int_\Omega \rho|\Delta v_n|^p dx + \int_\Omega g(x, u_n)u_n dx - \lambda_1 \|u_n\|^p \int_\Omega h|v_n|^p dx. \end{aligned}$$

Choosing $t_n = \|u_n\|$, it follows that

$$\frac{I'(u_n) \cdot u_n}{t_n^p} = \int_\Omega \rho|\Delta v_n|^p dx + \frac{1}{t_n^p} \int_\Omega g(x, u_n)u_n dx - \lambda_1 \int_\Omega h|v_n|^p dx. \tag{2.3}$$

We will denote the terms of the equality (2.3) by I_j $(j = 1, 2, 3, 4)$, respectively.

Claim 3.

(a) $\lim_{n\to\infty} I_1 = 0$,

(b) $\lim_{n\to\infty} I_3 = 0$,

(c) $\lim_{n\to\infty} I_4 = \lambda_1 \int_\Omega h|v|^p dx$.

Proof. (a) From the fact that $\lim_{n\to\infty} I'(u_n) = 0$ and since $\{u_n\} \subset X$ is unbounded we have the inequality

$$\left|\frac{I'(u_n)\cdot u_n}{t_n^p}\right| \le \epsilon \frac{\|u_n\|}{\|u_n\|^p} = \epsilon \|u_n\|^{1-p}.$$

This implies that $\lim_{n\to\infty} I_1 = 0$.

(b) By the condition (G_1), the Hölder's inequality, and (2.2) we get

$$\begin{aligned}\left|\frac{1}{t_n^p}\int_\Omega g(x,u_n)u_n dx\right| &\le \frac{C}{t_n^p}\int_\Omega |u_n| dx \le \frac{C}{t_n^p}\left[\int_\Omega |u_n|^p dx\right]^{\frac{1}{p}}\left[\int_\Omega 1^{\frac{p}{p-1}} dx\right]^{\frac{p-1}{p}} \\ &\le \frac{M}{t_n^p}\left[\int_\Omega |v_n|^p |t_n|^p dx\right]^{\frac{1}{p}} = M t_n^{1-p}\left[\int_\Omega |v_n|^p dx\right]^{\frac{1}{p}},\end{aligned}$$

where C and M are positive constants. This implies that $\lim_{n\to\infty} I_3 = 0$.

(c) Follows immediately from (2.2).

□

Using Claim 3 and (2.1) we obtain that $v \ne 0$ because

$$\lim_{n\to\infty}\left[\int_\Omega \rho|\Delta v_n|^p dx - \lambda_1\int_\Omega h|v_n|^p dx\right] = 1 - \lambda_1 \int_\Omega h|v|^p dx = 0.$$

Since $v_n \rightharpoonup v$ weakly in X, as $n \to \infty$, we have $\|v\| \le \liminf_{n\to\infty} \|v_n\| = 1$. Therefore

$$\|v\| \le 1 \tag{2.4}$$

and we conclude that v is an eigenfunction associated to the simple eigenvalue λ_1. Hence, there exists $\mu \in \mathbb{R}$, $\mu \ne 0$, such that

$$v(x) = \mu\phi_1(x) \text{ for all } x \in \Omega. \tag{2.5}$$

In particular, by (2.1) we conclude that

$$\lim_{n\to\infty} v_n(x) = \lim_{n\to\infty}\frac{u_n}{\|u_n\|} = v(x) = \mu\phi_1(x), \text{ for all } x \in \Omega.$$

But $\mu\phi_1(x) \neq 0$, then $v_n(x) \neq 0$ and this implies that

$$\lim_{n\to\infty} |u_n(x)| = \lim_{n\to\infty} \|u_n(x)\| v_n(x) = \infty, \text{ for all } x \in \Omega. \tag{2.6}$$

Using Fatou's Lemma, we have that

$$\liminf_{n\to\infty} \int_\Omega G(x, u_n(x))dx \geq \int_\Omega \liminf_{n\to\infty} G(x, u_n(x))dx \geq \int_\Omega T(x)dx. \tag{2.7}$$

By definition of λ_1 we conclude that

$$\int_\Omega \rho|\Delta u_n|^p dx - \lambda_1 \int_\Omega h|u_n|^p dx \geq 0 \tag{2.8}$$

and hence

$$c + o_n(1) = I(u_n) \geq \int_\Omega G(x, u_n(x))dx. \tag{2.9}$$

Since $\lim_{n\to\infty} |u_n(x)| = \infty$, by (G_2) it follows that

$$c \geq \int_\Omega T(x)dx,$$

which contradicts the hypothesis of the Lemma. Hence the sequence $\{u_n\} \subset X$ is bounded.

We claim that $\lim_{n\to\infty} u_n = u \in X$. In fact, consider

$$\begin{aligned} I'(u_n)\cdot(u_n - u) &= \int_\Omega \rho|\Delta u_n|^{p-2}\Delta u_n \Delta(u_n - u)dx + \int_\Omega g(x, u_n)(u_n - u)dx \\ &\quad -\lambda_1 \int_\Omega h|u_n|^{p-2}u_n(u_n - u)dx. \end{aligned}$$

Since the sequence $\{u_n - u\} \subset X$ is bounded and $\lim_{n\to\infty} I'(u_n) = 0$, we have

$$\lim_{n\to\infty} I'(u_n)\cdot(u_n - u) = 0. \tag{2.10}$$

Using (G_2), the facts that $u_n \to u$ in $L^s(\mathbb{R})$ (for $1 \leq s < p^\star$) and that $u_n \to u$ a. e. on Ω as $n \to \infty$, as well as the Dominated Convergence Theorem we obtain

$$\lim_{n\to\infty} \int_\Omega g(x, u_n)(u_n - u)dx = 0. \tag{2.11}$$

and

$$\lim_{n\to\infty} \lambda_1 \int_\Omega h|u_n|^{p-2}u_n(u_n - u)dx = 0. \tag{2.12}$$

It follows from (2.10), (2.11) and (2.12) that

$$0 = \lim_{n\to\infty} \left[\int_\Omega \rho|\Delta u_n|^{p-2}\Delta u_n \Delta(u_n - u)dx \right]. \tag{2.13}$$

Since $|\Delta u|^{p-2}\Delta u \in L^{\frac{p}{p-1}}(\mathbb{R})$, $\rho\Delta(u_n - u) \in L^p(\mathbb{R})$, by a result in [12, Theorem 13.44] we conclude that

$$\lim_{n\to\infty} \int_\Omega \rho|\Delta u|^{p-2}\Delta u \Delta(u_n - u)dx = 0, \tag{2.14}$$

where we are assuming that

$$\Delta u_n \longrightarrow \Delta u, \text{ a.e., as } n \to \infty.$$

The above affirmative can be proved arguing as in [5] (see also Alves, Carrião and Miyagaki in[2] for the case in dimension 1), together with the inequalities

$$\left[|x|^{p-2}x - |y|^{p-2}y\right](x-y) \geq \begin{cases} C_p \dfrac{|x-y|^2}{(|x|+|y|)^{2-p}} & \text{if} \quad 1 < p < 2 \\ \\ C_p|x-y|^p & \text{if} \quad p \geq 2, \ \forall x, y \in \mathbb{R}^N, \end{cases}$$

(for the proof, see [16, 17]).

Now, by using again the above inequality, we obtain by (2.13) and (2.14)

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int_\Omega \left[|\Delta u_n|^{p-2}\Delta u_n - |\Delta u|^{p-2}\Delta u\right] \rho\Delta(u_n - u)dx \\ &\geq \begin{cases} C_p \lim\limits_{n\to\infty} \displaystyle\int_\Omega \rho \frac{|\Delta u_n - \Delta u|^2}{(|\Delta u_n| + |\Delta u|)^{2-p}}dx & \text{if} \quad 1 < p < 2 \\ C_p \lim\limits_{n\to\infty} \displaystyle\int_\Omega \rho|\Delta u_n - \Delta u|^p dx & \text{if} \quad p \geq 2. \end{cases} \end{aligned} \tag{2.15}$$

If $p \geq 2$, we have that

$$\lim_{n\to\infty} \int_\Omega \rho|\Delta u_n - \Delta u|^p dx \leq 0.$$

If $1 < p < 2$, by Hölder's inequality it follows that

$$\begin{aligned}
&\int_\Omega \rho|\Delta u_n - \Delta u|^p dx \\
&\qquad \le \left[\int_\Omega \rho \frac{|\Delta u_n - \Delta u|^2}{(|\Delta u_n| + |\Delta u|)^{2-p}} dx\right]^{\frac{p}{2}} \left[\int_\Omega \rho(|\Delta u_n| + |\Delta u|)^p dx\right]^{\frac{2-p}{2}} \\
&\qquad \le C \left[\int_\Omega \rho \frac{|\Delta u_n - \Delta u|^2}{(|\Delta u_n| + |\Delta u|)^{2-p}} dx\right]^{\frac{p}{2}}.
\end{aligned}$$

By (2.15) and the previous inequality it follows that

$$0 \ge C_p \lim_{n\to\infty} \int_\Omega \rho \frac{|\Delta u_n - \Delta u|^2}{(|\Delta u_n| + |\Delta u|)^{2-p}} dx \ge C_p C^{-1} \left[\lim_{n\to\infty} \int_\Omega \rho|\Delta u_n - \Delta u|^p dx\right]^{\frac{2}{p}}.$$

Therefore, in both cases we have

$$\lim_{n\to\infty} \|u_n - u\| = 0 \text{ in } X$$

and this concludes the proof of the Lemma.

□

Lemma 4. *Assume the conditions* (h), (G_2) *and* (G_6). *Then the functional* I *is bounded from below on* X *and* $\inf_{C^\pm} I$ *is negative on* C^+ *and on* C^-.

Proof. Let $u \in X$; by condition G_2, we have $\left|\int_\Omega G(x,u)dx\right| \le C$. Hence, by the definition of λ_1 we get

$$|I(u)| \ge \int_\Omega G(x,u)dx \ge -C$$

and I is bounded from below on X.

Using condition (G_6) and the eigenfunction ϕ_1 associated to the eigenvalue λ_1 we obtain

$$I(t^\pm\phi_1) = \int_\Omega G(x, t^\pm\phi_1)dx \le \int_\Omega T(x) < 0. \tag{2.16}$$

If $u \in C^+$, we have that $I(u) = I(t\phi_1 + z)$. In particular, consider $t = t^+$ and $z = 0$; by inequality (2.16), we obtain that $I(t^+\phi_1) < 0$. Similarly, we have $I(t^{-}\phi_1) < 0$. Hence $\inf_{C^{\pm}} I(t^{\pm}\phi_1) < 0$. This concludes the proof of the lemma.

□

Now we show that the energy functional I verifies the geometry of the Mountain Pass Theorem.

Lemma 5. *The energy functional I verifies the following properties.*

(a) $I(0) = 0$.

(b) *There exist positive constants ρ and R such that $I(u) \geq \rho > 0$ if $\|u\| = R$.*

(c) *There exists $z \in X$ such that $I(z) < 0 = I(0)$ if $\|z\| > R$.*

Proof. The proof of the item (a) is immediate.

Since G is bounded and continuous, there exist $\theta \in \mathbb{R}$ (with $p < \theta < p^*$) and a constant C such that

$$G(x,t) \geq \frac{m}{p}h(x)|t|^p - C|t|^\theta, \text{ for all } |t| > \delta,$$

where δ is given by (G_5). Therefore, by (G_5) we conclude that

$$G(x,t) \geq \frac{m}{p}h(x)|t|^p - C|t|^\theta, \text{ for all } |t| \in R,\ p < \theta < p^\star \text{ and for all } x \in \Omega. \tag{2.17}$$

By the previous inequality we have

$$I(u) \geq \frac{1}{p}\int_\Omega \rho|\Delta u|^p dx + \int_\Omega \left[\frac{m}{p}h(x)|u|^p - C|u|^\theta\right] dx - \frac{\lambda_1}{p}\int_\Omega h|u|^p dx.$$

We recall that the embedding $W^{1,p}(\mathbb{R}) \hookrightarrow L^s(\mathbb{R})$ is continuous for $1 < p < s \leq p^\star$ and compact for $s < p^\star$ and

$$\lambda_1 \leq \frac{\displaystyle\int_\Omega \rho|\Delta u|^p dx}{\displaystyle\int_\Omega h|u|^p dx}.$$

Then, for $p < \theta < p^*$ we have

$$\begin{aligned} I(u) &\geq \frac{1}{p}\int_\Omega \rho|\Delta u|^p dx - \left[\frac{\lambda_1 - m}{p}\right]\frac{1}{\lambda_1}\int_\Omega \rho|\Delta u|^p dx - C\int_\Omega |u|^\theta dx \\ &\geq \frac{m}{p\lambda_1}\|u\| - \mu\|u\|^\theta. \end{aligned}$$

Since
$$I(u) \geq \frac{m}{p\lambda_1}\|u\| + o(\|u\|), \text{ as } \|u\| \to \infty,$$
we can find $R > 0$ small enough and $\rho > 0$ such that if $\|u\| \leq R$, then $I(u) \geq 0$ and if $\|u\| = R$, then $I(u) \geq \rho > 0$. As a result, item (b) is proved.

To prove item (c), it is sufficient to remark that by (G_6) we conclude that $I(t^{\pm}\phi_1) < 0$. Then we define $z \equiv t^{+}\phi_1$ and we get $\|z\| = \int_\Omega \rho|\Delta(t^{+}\phi_1)|^p dx = t^{+}\|\phi_1\| \equiv R_1$. Note that $R_1 > R$ and it follows that $\|z\| > R$ and $I(z) < 0$. This concludes the proof of item (c).

□

3 Proof of Theorem 1

To prove item (i) we use inequality (2.16) to obtain
$$\inf_{C^{\pm}} I(u) \leq I(t^{\pm}\phi_1) = \int_\Omega G(x, t^{\pm}\phi_1)dx \leq \int_\Omega T(x) < 0.$$
If $\inf_{C^{\pm}} I(u) = I(t^{\pm}\phi_1)$, then it is enough to take $u = t^{+}\phi_1$ and $v = t^{-}\phi_1$ to get two solutions such that $I(u) < 0$ and $I(v) < 0$.

Otherwise, if $\inf_{C^{\pm}} I(u) < I(t^{\pm}\phi_1)$ then we have
$$\inf_{C^{\pm}} I(u) < \int_\Omega T(x). \tag{3.1}$$

By Lemma 4, the functional I is bounded from below on X and it is easy to prove that I is lower semicontinuous in X. Hence, the Ekeland's Variational Principle guarantees the existence of two sequences $u_n \subset C^{+}$ and $v_n \subset C^{-}$ satisfying
$$I(u_n) \to \inf_{C^{+}} I(u) \qquad \text{and} \qquad I'(u_n) \to 0,$$
and
$$I(v_n) \to \inf_{C^{+}} I(v) \qquad \text{and} \qquad I'(v_n) \to 0.$$
as $n \to \infty$. By (3.1) and by Lemma 2, there exist u and v such that
$$u_n \to u \qquad \text{and} \qquad v_n \to v \text{ in } X$$

as $n \to \infty$. Therefore, u and v are solutions of problem 1.1 verifying

$$I(u) = \inf_{C^+} I(z) < 0 \text{ and } I(v) = \inf_{C^-} I(z) < 0.$$

Moreover, it follows from assumption (G_4) and from inequality (1.6) that

$$I(z) \geq \frac{1}{p}\int_\Omega \rho|\Delta u|^p dx - \frac{\lambda_2}{p}\int_\Omega h|z|^p dx \geq 0, \text{ for all } z \in Z.$$

Then $I(z) \geq 0$ for all $z \in Z$ and Lemma 2 implies that the infimum of I on $C^\pm$ is achieved in $C^\pm \backslash Z$. Therefore $u \in C^+$ and $v \in C^-$.

To prove item (ii) we use Lemma 5 and a variant of the Mountain Pass Theorem without the Palais-Smale condition. (See [9, Theorem 6].) Then there exists a sequence $\{w_n\} \subset X$ such that

$$I(w_n) \to c_1 > \rho > 0 \text{ and } \|I'(w_n)\|_{X^*}(1 + \|w_n\|) \to 0 \text{ in } X^* \text{ as } n \to \infty. \quad (3.2)$$

Arguing as in the prove of Lemma 2, we choose $t_n = \|w_n\|$ to obtain

$$\left|\frac{I'(u_n)\cdot w_n}{t_n^p}\right| \leq \frac{\|I'(w_n)\|_{X^*}(1 + \|w_n\|)}{t_n^p} \to 0, \text{ as } n \to \infty.$$

If the sequence $\{w_n\} \subset X$ is unbounded, then

$$|w_n(x)| \to 0, \text{ as } n \to \infty, \text{ for all } x \in \Omega.$$

Since

$$\left|I'(w_n)\cdot w_n\right| \leq \|I'(w_n)\|_{X^*}(1 + \|w_n\|),$$

by (3.2) we obtain that

$$\left|I'(w_n)\cdot w_n\right| \to 0 \text{ as } n \to \infty$$

and hence

$$o(1) = I'(w_n)\cdot w_n = \|w_n\|^p + \int_\Omega g(x, w_n)w_n dx - \lambda_1 \int_\Omega h|w_n|^p dx.$$

By (2.8) we conclude that

$$0 \leq \|w_n\|^p - \lambda_1 \int_\Omega h|w_n|^p dx = -\int_\Omega g(x, w_n)w_n dx + o(1) \leq \left|\int_\Omega g(x, w_n)w_n dx\right| + o(1)$$

By (G_1) and (G_3), the function $g(x, w_n(x))w_n(x)$ is bounded for all $x \in \Omega$ and for all n. By (2.6), $g(x, w_n(x))w_n(x) \to 0$ as $n \to \infty$ a. e. on Ω. Using the Dominated Convergence Theorem we obtain $\int_\Omega g(x, w_n)w_n dx \to 0$, as $n \to \infty$.
Then

$$\|w_n\|^p - \lambda_1 \int_\Omega h|w_n|^p dx \to 0, \text{ as } n \to \infty.$$

Since

$$c_1 + o(1) = I(w_n) = \frac{1}{p}\left[\|w_n\|^p - \frac{\lambda_1}{p}\int_\Omega h|w_n|^p dx\right] + \int_\Omega G(x, w_n)dx,$$

using Fatou's Lemma, together with (2.6) and (G_7), we obtain

$$c_1 \leq \limsup_{n\to\infty} \int_\Omega G(x, w_n)dx \leq \int_\Omega S(x)dx \leq 0,$$

which contradicts (3.2). Hence the sequence $\{w_n\} \subset X$ is bounded and, passing to a subsequence if necessary (still denoted in the same way), there exists $w \in X$ such that

$$w_n \rightharpoonup w \quad \text{in } X, \quad \text{ as } n \to \infty.$$

We also have $\|I'(w_n)\|_{X^*}(1 + \|w_n\|) \to 0$ as $n \to \infty$, it follows that $\|I'(w_n)\|_{X^*} \to 0$, in X^* as $n \to \infty$ and by a similar argument as that of Lemma 2 we conclude that

$$w_n \to w, \quad \text{in} \quad X \quad \text{ as } n \to \infty$$

and the Theorem is proved.

□

4 Example

In this section, inspired by [3], we will define a function g that satisfies the assumptions $(G_1) - (G_7)$.

Consider $\Omega = (0, 1)$, $p = 2$ and $h = 1$. In this case, the function $\phi_1(x) = \sin(\pi x)$ is an eigenfunction associated to the first eigenvalue $\lambda_1 = \pi^4$ of problem (1.1). We remark that ϕ_1 is symmetric with respect to $x = \frac{1}{2}$. Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x, s) = R(x)g_1(x)$$

where $R(x) = 1$ and $g_1 : \mathbb{R} \to \mathbb{R}$ is given by

$$g_1(s) = \begin{cases} s, \ \text{for} \ \ 0 \le s \le 1, \\ \\ 2 - s, \ \text{for} \ \ 1 < s \le 5, \\ \\ s - 8, \ \text{for} \ \ 5 < s \le 8 + \frac{\sqrt{30}}{2}, \\ \\ 8 + \sqrt{30} - s, \ \text{for} \ \ 8 + \frac{\sqrt{30}}{2} < s \le 8 + \sqrt{30}, \\ \\ 0, \ \text{for} \ \ s \ge 8 + \sqrt{30}, \\ \\ -g(-s), \ \text{for} \ \ s \le 0, \\ \end{cases}$$

Defining $G_1(s) = \displaystyle\int_0^s g_1(t)dt)$ we have

$$G(x, s) = \int_0^s g(x, t)dt = R(x)G_1(s) \qquad \text{and} \qquad S(x) = T(x) = -\frac{R(x)}{2}.$$

Choosing $\delta < 1$ it is easy to see that g verifies the assumptions $(G_1)-(G_5)$ and (G_7).

Now we have to prove that g also verifies (G_6), for $t^+ = 8$ and $t^- = -8$. Since ϕ_1 is symmetric with respect to $x = \frac{1}{2}$, the same is true for G. Then we have $G(x, 8\phi_1(x)) = G(1 - x, 8\phi_1(1 - x))$ and

$$\begin{aligned} \int_0^1 G(x, 8\phi_1(x))dx &= 2\int_0^{\frac{1}{2}} G(x, 8\phi_1(x))dx = \int_0^{\frac{1}{2}} G_1(8\phi_1(x))dx \\ &= 2\left[\int_0^{\frac{1}{6}} G_1(8\phi_1(x))dx + \int_{\frac{1}{6}}^{\frac{1}{3}} G_1(8\phi_1(x))dx + \int_{\frac{1}{3}}^{\frac{1}{2}} G_1(8\phi_1(x))dx\right]. \end{aligned}$$

Note that for $0 \leq x \leq \frac{1}{6}$ we have that $0 \leq 8\sin(\pi x) \leq 4$. Then we have

$$\max_{x\in[0,\frac{1}{6}]} G_1(8\phi_1(x)) = \max_{y\in[0,4]} G_1(y) = G_1(2).$$

Similarly,

$$\max_{x\in[\frac{1}{6},\frac{1}{3}]} G_1(8\phi_1(x)) = \max_{y\in[4,4\sqrt{3}]} G_1(y) = G_1(4)$$

and

$$\max_{x\in[\frac{1}{3},\frac{1}{2}]} G_1(8\phi_1(x)) = \max_{y\in[4\sqrt{3},8]} G_1(y) = G_1(4\sqrt{3}) < G_1(6).$$

Therefore,

$$\begin{aligned}\int_0^1 G(x,8\phi_1(x))dx &= 2\int_0^{\frac{1}{2}} G(x,8\phi_1(x))dx = \int_0^{\frac{1}{2}} G_1(8\phi_1(x))dx \\ &\leq 2\left[\int_0^{\frac{1}{6}} G_1(2)dx + \int_{\frac{1}{6}}^{\frac{1}{3}} G_1(4)dx + \int_{\frac{1}{3}}^{\frac{1}{2}} G_1(6)dx\right] < \int_0^1 T(x)dx < 0.\end{aligned}$$

Similarly, we can prove that G satisfies (G_6) for $t^- = -8$.

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