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THE LAWSON CORRESPONDENCE FOR BRYANT SURFACES IN EXPLICIT COORDINATES

J. A. Gálvez **P. Mira**

Dedicated to Professor Manfredo do Carmo on the occasion of his 80th birthday

Abstract

We use a complex analysis trick to provide a description in explicit coordinates of some fundamental dualities appearing in Bryant surface theory. In particular, given a Bryant surface, we construct in explicit coordinates the minimal surface in $\mathbb{R}^3$ associated to it via the Lawson correspondence. We also give in explicit coordinates, for any simply connected surface (Σ, g) of constant curvature κ, the canonical isometric immersion of Σ into the model space $\mathbb{Q}^2(\kappa)$ in terms of a solution to the Liouville equation.

1 Introduction

Integrability theorems such as the Frobenius theorem and its modifications constitute a fundamental tool in surface theory, since they provide

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in many cases existence of surfaces with prescribed geometric data, or correspondences between different classes of surfaces. In this sense, we may quote for instance the fundamental theorem of surface theory, or the correspondence between surfaces of constant negative curvature in $\mathbb{R}^3$ and solutions to the sine-Gordon equation.

A limitation of these integrability techniques is that they do not provide explicit coordinates for the surfaces that they prove to exist. Hence, it is natural to analyze if some of these correspondences can be made explicit in an alternative way.

In this note we will discuss the previous problem for the class of *Bryant surfaces*. Let us recall that a Bryant surface is an immersed surface of constant mean curvature $H = 1$ in the hyperbolic 3-space $\mathbb{H}^3$. These surfaces are special among constant mean curvature (CMC) surfaces in many aspects. For instance, they are connected to minimal surfaces in $\mathbb{R}^3$ by the so-called *Lawson correspondence*: if (I, II) denote the first and second fundamental forms of a simply connected Bryant surface, then there exists a minimal surface in $\mathbb{R}^3$ whose first and second fundamental forms are $(I, II - I)$. In particular, both surfaces are locally isometric.

The term *Bryant surface* comes from the celebrated paper by R.L. Bryant [Bry], in where a conformal representation for this type of surfaces was obtained. This representation constitutes the basic tool in the global study of Bryant surfaces, and is basically a correspondence between such surfaces and the class of holomorphic null curves in $\mathbf{SL}(2,\mathbb{C})$.

The Bryant representation uses the *Hermitian model* for $\mathbb{H}^3$ (see Section 3), and tells the following:

Theorem 1 ([Bry]). *Let $F : \Sigma \to \mathbf{SL}(2,\mathbb{C})$ be a holomorphic immersion from a Riemann surface Σ, and suppose that F is* null, *i.e.* $\det(dF) = 0$. *Then*

$$f := FF^* : \Sigma \to \mathbb{H}^3 \tag{1.1}$$

is a Bryant surface.

Conversely, any simply connected Bryant surface in $\mathbb{H}^3$ can be expressed as (1.1) *for some holomorphic null immersion $F : \Sigma \to \mathbf{SL}(2,\mathbb{C})$.*

These two fundamental results, the Bryant representation and the Lawson correspondence, rely on the Frobenius theorem or some of its variants. Thus, none of them is explicit at a first sight (although they can actually be reformulated only in terms of first order data on the surfaces).

Our aim in this note is to show how, by means of a very simple classical trick of complex analysis (see Section 2), we can make both correspondences explicit. As a corollary, we will also describe in explicit coordinates other useful dualities of Bryant surface theory, due to Umehara and Yamada, and MartÃn, Umehara and Yamada, respectively. This will be done in Section 3.

Besides, in Section 4, we will provide explicit coordinates for the canonical isometric immersion of a simply connected surface with constant curvature surface κ into the 2-dimensional model space $\mathbb{Q}^2(\kappa)$. This is another basic result of surface theory that relies on auxiliary integrability results. We make this existence result explicit by using complex analysis and the connection of the problem with Liouville's equation $\Delta u + ae^u = 0$.

It is a pleasure for us to dedicate this paper to Prof. Manfredo do Carmo, from whom we have learned so much through his books, articles, conferences and personal conversations.

2 The extension operation

Let $a(s,t) : \Omega \subset \mathbb{R}^2 \to \mathbb{C}$ denote a real analytic function with complex values, where Ω is a simply connected domain. We shall identify $\mathbb{R}^2 \equiv \mathbb{C}$ by means of $(s,t) \mapsto s + it$.

By real analyticity, we may *extend* $a(s,t)$ to a complex function

$$a(w_1, w_2) : \widetilde{\Omega} \subset \mathbb{C}^2 \to \mathbb{C},$$

where $a(w_1, \cdot), a(\cdot, w_2)$ are holomorphic functions on their corresponding domains.

Let us remark that the complex function $a(w_1, w_2)$ *extends* the original real analytic function $a(s,t)$, in the sense that

$$\Omega \subset \{(w_1, w_2) \in \widetilde{\Omega} : \operatorname{Im} w_1 = 0 = \operatorname{Im} w_2\}.$$

So, formally, the extension is performed just by replacing the real variable s (resp. t) by the complex variable w_1 (resp. w_2) in the expression $a(s,t)$.

In order to simplify our reasoning, we will assume $(0,0) \in \Omega$. Thus, the image of the complex curve

$$\Gamma : \Omega \longrightarrow \mathbb{C} \times \mathbb{C}$$
$$z \mapsto \left(\frac{z}{2}, \frac{z}{2i}\right).$$

lies on $\widetilde{\Omega}$ for $|z|$ small enough.

Therefore, sufficiently close to the origin, the holomorphic function

$$a\left(\frac{z}{2}, \frac{z}{2i}\right) \tag{2.1}$$

is well defined. Formally, (2.1) is obtained just by making the substitutions $s \mapsto z/2$, $t \mapsto z/(2i)$ on $a(s,t)$.

At this point, it is important to observe that if $a(s,t)$ is holomorphic and we denote $z = s + it$, then the extension (2.1) is actually the proper function $a(s+it)$. Contrastingly, if $a(s,t)$ is antiholomorphic, i.e. $a(s,t) = f(s - it)$ where f is holomorphic, then

$$a\left(\frac{z}{2}, \frac{z}{2i}\right) = f(0) = \text{const}.$$

Thus, the idea behind the above extension operation is that (2.1) preserves the holomorphic parts of $a(s,t)$ and *kills* the antiholomorphic parts, turning them into constants.

3 Bryant surfaces

Let $\mathbb{L}^4$ be the Minkowski 4-space with canonical coordinates (x_0, x_1, x_2, x_3) and the Lorentzian metric $\langle , \rangle = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$. The Hermitian model for $\mathbb{L}^4$ identifies $\mathbb{L}^4 \equiv \mathrm{Herm}(2)$ as

$$(x_0, x_1, x_2, x_3) \in \mathbb{L}^4 \longleftrightarrow \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \in \mathrm{Herm}(2).$$

The metric $\langle , \rangle$ on this model is determined by $\langle m, m \rangle = -\det(m)$ for all $m \in \mathrm{Herm}(2)$. In addition, the complex Lie group $\mathbf{SL}(2,\mathbb{C})$ acts on $\mathbb{L}^4$ through the isometric and orientation-preserving action

$$\Phi \in \mathbf{SL}(2,\mathbb{C}) \mapsto \Phi \cdot m = \Phi m \Phi^*, \quad m \in \mathrm{Herm}(2), \;\; \Phi^* = \bar{\Phi}^{\mathrm{t}}.$$

This implies that the hyperbolic 3-space $\mathbb{H}^3 = \{x \in \mathbb{L}^4 : \langle x, x \rangle = -1, x_0 > 0\}$ may be regarded as $\mathbb{H}^3 = \{\Phi\Phi^* : \Phi \in \mathbf{SL}(2,\mathbb{C})\}$, where in this decomposition Φ is unique up to right multiplication by an element of $\mathrm{SU}(2)$.

The next result recovers the holomorphic null immersion F in the Bryant representation from the explicit coordinates of the surface, using the extension procedure explained in the previous section.

Theorem 2. *Let $f(s,t) : \Omega \subset \mathbb{C} \to \mathbb{H}^3$ denote a simply connected Bryant surface, where $z = s + it$ is a conformal parameter of the surface. Assume without loss of generality that $(0,0) \in \Omega$ and that $f(0,0) = \mathrm{Id}_2$.*

Then, the holomorphic null immersion $F : \Omega \to \mathbf{SL}(2,\mathbb{C})$ such that $F(0) = \mathrm{Id}_2$ given by the Bryant representation can be explicitly obtained from f by the formula

$$F(z) = f\left(\frac{z}{2}, \frac{z}{2i}\right). \tag{3.1}$$

Proof: Define $\widehat{F}(z) := F^*(\bar{z})$, which is a holomorphic curve in $\mathbf{SL}(2,\mathbb{C})$. Then, from the Bryant representation, and using the real parameters (s,t) with $z = s + it$, we have

$$f(s,t) = F(s+it)\,\widehat{F}(s-it).$$

Now, we are in the conditions to apply the extension technique of Section 2, from which we get

$$f\left(\frac{z}{2}, \frac{z}{2i}\right) = F(z)\widehat{F}(0).$$

Finally, using that $f(0,0) = \mathrm{Id}_2$ and the SU(2) ambiguity of F, we can assume that $F(0) = \mathrm{Id}_2$, and hence we obtain (3.1).

$\square$

Remark: It is interesting to observe that for establishing (3.1) we did not use that F is null, or that $z = s + it$ is conformal for the first fundamental form of the surface. In other words, Theorem 2 is also true for any class of surfaces in $\mathbb{H}^3$ for which a representation formula of the type $f = FF^*$ holds ($F : \Sigma \to \mathbf{SL}(2,\mathbb{C})$ being holomorphic). This is the case, for instance, of flat surfaces [GMM1] and, more generally, of linear Weingarten surfaces of Bryant type [GMM2].

Example 3. *An example of a Bryant surface invariant by hyperbolic translations is*

$$f(s+it) = \begin{pmatrix} e^s \cosh t & -ie^{it} \sinh t \\ ie^{-it} \sinh t & e^{-s} \cosh t \end{pmatrix} : \mathbb{C} \to \mathbb{H}^3 \subset \mathrm{Herm}(2). \qquad (3.2)$$

The characteristic property of this example is that it contains the axis of the hyperbolic translation group, see Figure 3.

By applying Theorem 2 we get directly that its associated null immersion is

$$F(z) = f\left(\frac{z}{2}, \frac{z}{2i}\right) = \begin{pmatrix} e^{z/2} \cos(z/2) & -e^{z/2} \sin(z/2) \\ e^{-z/2} \sin(z/2) & e^{-z/2} \cos(z/2) \end{pmatrix} : \mathbb{C} \to \mathbf{SL}(2,\mathbb{C}).$$

Indeed, a straightforward computation yields that F is null with $f = FF^*$.

Application: the Lawson correspondence.

Any holomorphic null immersion $F : \Sigma \to \mathbf{SL}(2,\mathbb{C})$ satisfies (see [UY1])

$$F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega, \qquad (3.3)$$

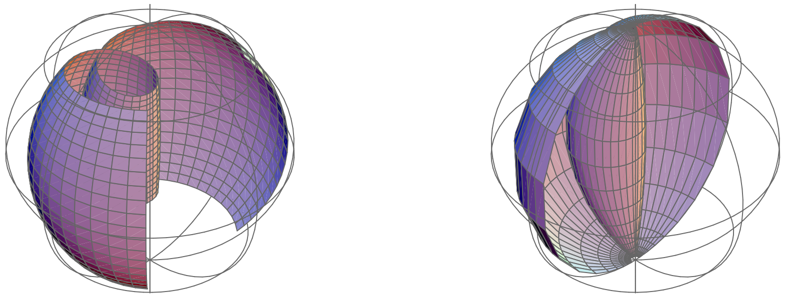

Figure 1: Hyperbolic invariant Bryant surfaces in the Poincaré model containing the axis of the hyperbolic translation

where g is meromorphic and ω is a holomorphic 1-form. As F is an immersion, the quantity $(1+|g|^2)^2|\omega|^2$ is a well defined Riemannian metric, and (g, ω) are the Weierstrass data of a minimal surface $\psi : \Sigma \to \mathbb{R}^3$ given by

$$\psi(z) = \mathrm{Re} \int_{z_0}^{z} \left((1-g^2)\omega, i(1+g^2)\omega, 2g\omega\right), \tag{3.4}$$

provided that Σ is simply connected. In this situation, ψ is the *cousin* surface of $f = FF^*$, i.e. ψ and f are connected by the Lawson correspondence.

Once here, it comes clear from Theorem 2 and (3.4) that the cousin surface $\psi : \Sigma \to \mathbb{R}^3$ can be obtained from the coordinates of $f : \Sigma \to \mathbb{H}^3$, just by performing an integration. Specifically, if

$$f(s,t) = \begin{pmatrix} a(s,t) & b(s,t) \\ \bar{b}(s,t) & c(s,t) \end{pmatrix} : \Sigma \to \mathbb{H}^3, \tag{3.5}$$

we have that

$$\psi(z) = \mathrm{Re} \int_{z_0}^{z} \begin{pmatrix} cb_z - bc_z + a\bar{b}_z - \bar{b}a_z \\ i(bc_z - cb_z + a\bar{b}_z - \bar{b}a_z) \\ 2(ca_z - b\bar{b}_z) \end{pmatrix} \left(\frac{w}{2}, \frac{w}{2i}\right) dw. \qquad (3.6)$$

Here, by definition,

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial s} - i\frac{\partial}{\partial t}\right).$$

Example 4. *Let us explain how formula* (3.6) *works if we start with the specific Bryant surface of Example 3. In this case, the coordinates* (3.5) *are given by* (3.2)*, and so we have*

$$\begin{pmatrix} cb_z - bc_z + a\bar{b}_z - \bar{b}a_z \\ i(bc_z - cb_z + a\bar{b}_z - \bar{b}a_z) \\ 2(ca_z - b\bar{b}_z) \end{pmatrix} (s,t) = \begin{pmatrix} \sinh(s-it) - i\cosh(s-it)\sinh(2t) \\ i\cosh(s-it) + \sinh(s-it)\sinh(2t) \\ \cosh(2t). \end{pmatrix}.$$

Applying now the substitution in (3.6) *and integrating yields*

$$\psi(s,t) = (\cos s \cosh t, -t, \sin s \cosh t) : \mathbb{C} \to \mathbb{R}^3,$$

i.e. the standard conformal parametrization of (the universal covering of) an Euclidean catenoid.

Application: dual Bryant surfaces.

An extremely useful notion in Bryant surface theory is the following duality introduced in [UY2]: if $F : \Sigma \to \mathbf{SL}(2,\mathbb{C})$ is a null holomorphic immersion, then so is F^{-1}. By applying the Bryant representation to F^{-1}, and if Σ is simply connected, we obtain a new Bryant surface $f^\sharp : \Sigma \to \mathbb{H}^3$ that is called the *dual* of the Bryant surface $f = FF^*$. This duality switches the roles played by the meromorphic function g and the hyperbolic Gauss map G of the surface, and one is complete if and only if the other one is complete, see [Yu].

With this, it is immediate from Theorem 2 that if $f : \Sigma \to \mathbb{H}^3$ is a Bryant surface given in coordinates by (3.5), then its dual surface $f^\sharp$ is explicitly given by

$$f^\sharp(z) = \begin{pmatrix} c\left(\frac{z}{2}, \frac{z}{2i}\right) & -b\left(\frac{z}{2}, \frac{z}{2i}\right) \\ -\bar{b}\left(\frac{z}{2}, \frac{z}{2i}\right) & a\left(\frac{z}{2}, \frac{z}{2i}\right) \end{pmatrix} \overline{\begin{pmatrix} c\left(\frac{z}{2}, \frac{z}{2i}\right) & -\bar{b}\left(\frac{z}{2}, \frac{z}{2i}\right) \\ -b\left(\frac{z}{2}, \frac{z}{2i}\right) & a\left(\frac{z}{2}, \frac{z}{2i}\right) \end{pmatrix}} : \Sigma \to \mathbb{H}^3.$$

Application: a correspondence for null curves in $\mathbb{C}^3$ and $\mathbf{SL}(2,\mathbb{C})$.

In [MUY], the following correspondence was used to prove the existence of complete bounded Bryant surfaces *Ã la Nadirashvili*:

$$\mathcal{T} : \{(x_1, x_2, x_3) \in \mathbb{C}^3 : x_3 \neq 0\} \leftrightarrow \{(y_{ij}) \in \mathbf{SL}(2,\mathbb{C}) : y_{11} \neq 0\},$$

$$\mathcal{T}(x_1, x_2, x_3) = \frac{1}{x_3} \begin{pmatrix} 1 & x_1 + ix_2 \\ x_1 - ix_2 & x_1^2 + x_2^2 + x_3^2 \end{pmatrix}.$$

This correspondence takes holomorphic null immersions in $\mathbb{C}^3$ into holomorphic null immersions in $\mathbf{SL}(2,\mathbb{C})$. As every minimal surface in $\mathbb{R}^3$ is the real part of a null holomorphic immersion in $\mathbb{C}^3$, we can associate to any Bryant surface $f : \Sigma \to \mathbb{H}^3$ a new minimal surface $\psi^\flat : \Sigma \to \mathbb{R}^3$ by the formula

$$\psi^\flat = \mathrm{Re}(\mathcal{T}^{-1} \circ F),$$

where F is the null $\mathbf{SL}(2,\mathbb{C})$ immersion associated to f of Bryant's representation.

Thus, again, we can recover $\psi^\flat$ explicitly from the coordinates of f, by means of Theorem 2. We omit the final formula, as the process is clear.

4 The Liouville equation

The *Liouville equation* is the quasilinear elliptic PDE

$$\Delta u + ae^u = 0, \tag{4.1}$$

where $a \in \mathbb{R}$ is a constant that can be assumed to be $a = 2\varepsilon$, $\varepsilon \in \{-1, 0, 1\}$, up to a change of the form $u \mapsto u + c$, $c \in \mathbb{R}$. This equation has a geometrical nature. Indeed, on a given planar domain Ω, the conformal metric $e^u(ds^2+dt^2)$ has constant curvature $a/2$ if and only if u is a solution to (4.1). This tells in particular that (4.1) is conformally invariant.

The Liouville equation admits a resolution in terms of holomorphic data, mainly due to Liouville [Lio] (see also [Bry, ChWa, GaMi1, GaMi2]).

Theorem 5. *Let $u:\Omega\subset\mathbb{R}^2\equiv\mathbb{C}\to\mathbb{R}$ denote a solution to $\Delta u+2\varepsilon\, e^u=0$ in a simply connected domain Ω. Then there exists a locally univalent meromorphic function g (holomorphic with $1+\varepsilon|g|^2>0$ if $\varepsilon\le 0$) in Ω such that*

$$u=\log\frac{4|g'|^2}{(1+\varepsilon|g|^2)^2}. \tag{4.2}$$

Conversely, if g is a locally univalent meromorphic function (holomorphic with $1+\varepsilon|g|^2>0$ if $\varepsilon\le 0$) in Ω, then (4.2) is a solution to $\Delta u+2\varepsilon e^u=0$ in Ω.

The function g in the above theorem is called the *developing map*, and is unique up to a transformation of the form

$$g\mapsto\frac{\alpha g-\bar\beta}{\varepsilon\beta g+\bar\alpha},\qquad |\alpha|^2-\varepsilon|\beta|^2=1. \tag{4.3}$$

Consider now $d\sigma^2=e^u(ds^2+dt^2)=e^u|dz|^2$ a Riemannian metric of constant curvature $\varepsilon\in\{-1,0,1\}$ defined on a simply connected complex domain $\Omega\subset\mathbb{C}$. Let g denote the developing map of u. Then $g:(\Omega,d\sigma^2)\to\mathbb{Q}^2(\varepsilon)$ is an isometric immersion, where $\mathbb{Q}^2(\varepsilon)$ is the 2-dimensional space form of constant curvature ε:

$$\begin{aligned}\mathbb{Q}^2(1)&=(\mathbb{C}\cup\{\infty\},\tfrac{4|dw|^2}{(1+|w|^2)^2}),\qquad \mathbb{Q}^2(0)=(\mathbb{C},4|dw|^2),\\ \mathbb{Q}^2(-1)&=(\mathbb{D},\tfrac{4|dw|^2}{(1-|w|^2)^2}).\end{aligned}$$

Observe that the change (4.3) amounts to compose g with an isometry of $\mathbb{Q}^2(\varepsilon)$, i.e. (4.3) is the natural ambiguity of the isometric immersion problem.

Thus, any simply connected surface of constant curvature ε can be isometrically immersed into $\mathbb{Q}^2(\varepsilon)$, and the problem that we address here is: *can this canonical isometric immersion be explicitly described?* For that, we only need to find the developing map g explicitly from $d\sigma^2 = e^u|dz|^2$.

Theorem 6. *Let $d\sigma^2 = e^u|dz|^2$ denote Riemannian metric of constant curvature $\varepsilon \in \{-1, 0, 1\}$ defined on a simply connected domain $\Omega \subset \mathbb{C}$. Assume without loss of generality that $0 \in \Omega$, and that its developing map g satisfies $g(0) = 0$ and $g'(0) \in \mathbb{R}$. Then g is explicitly given by*

$$g'(z) = \frac{1}{2\exp\left(u(0)/2\right)} \exp\left(u\left(\frac{z}{2}, \frac{z}{2i}\right)\right). \tag{4.4}$$

Proof: Observe first that the conditions on g are not restrictive, by the ambiguity (4.3).

Writing $g^*(z) = \overline{g(\bar{z})}$, by (4.2) we have

$$e^{u(s,t)} = \frac{4|g'(s+it)|^2}{(1+\varepsilon|g(s+it)|^2)^2} = \frac{4g'(s+it)(g^*)'(s-it)}{(1+\varepsilon g(s+it)g^*(s-it))^2}.$$

By the extension operation of Section 2 we have

$$\exp\left(u\left(\frac{z}{2}, \frac{z}{2i}\right)\right) = 4(g^*)'(0)\frac{g'(z)}{(1+\varepsilon g(z)\overline{g(0)})^2}. \tag{4.5}$$

Now, as $g'(0) \in \mathbb{R}$ and $g(0) = 0$, we have

$$(g^*)'(0) = \overline{g'(0)} = \frac{1}{2}\exp(u(0)/2).$$

Thus, we obtain (4.4) from (4.5).

$\square$

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Depto. de Geometría y Topología
Universidad de Granada
E-18071 Granada, Spain
E-mail: jagalvez@ugr.es

Depto. de Mat. Aplicada y Estadística
Universidad Politécnica de Cartagena
E-30203 Cartagena, Murcia, Spain
E-mail: pablo.mira@upct.es