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GAP OF THE FIRST TWO EIGENVALUES OF THE SCHRÖDINGER OPERATOR WITH NONCONVEX POTENTIAL

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Dedicated to Professor Manfredo do Carmo on the occasion of his 80th birthday

In this essay, I will extend my previous work [9] on operators whose potential is nonconvex. In particular, the results given here can be applied to the double well potential. I define an invariant associated to the potential in S4. It defines a distance between the point where $\frac{u_2}{u_1}$ achieves its maximum, $\sup \frac{u_2}{u_1}$, to the point where $\frac{u_2}{u_1} = \varepsilon \sup \frac{u_2}{u_1}$. Here u_i are the eigenfunctions of the Schrödinger operator. I will show how the gap $\lambda_2 - \lambda_1$ can be estimated from below in terms of this distance. Theorem 6.1 is the main theorem of this essay. It is a very interesting problem to locate the maximum of $\frac{u_2}{u_1}$ and its zeroes. The upper bound of $\lambda_2 - \lambda_1$ depends on the choice of a good trial function, and I shall come back to this question in the future.

This line of research on gradient estimates started from my work on bounded harmonic functions [8] and the method was used by Peter Li [2] and Li-Yau [3] for the Laplacian of a manifold. Li-Yau [3] also applied it to the Schrödinger operator where a distance function similar to the one used here was introduced.

The Li-Yau type distance function was also used by Perelman in his famous work [6].

In the Li-Yau's approach of estimating the first eigenvalue of the Laplacian, it was conjectured by Li-Yau and proved by Zhong-Yang [10] that if d is the diameter of a manifold with nonnegative Ricci curvature, then $\lambda_1 d^2$ has an universal lower bound which is achieved when the manifold is a circle.

Convex domain and convex potential can be considered as an analogue of manifold with nonnegative curvature. In Singer-Wong-Yau-Yau [7], we improved on the log concavity result of Brascamp-Lieb [1] and proved that $(\lambda_2 - \lambda_1)d^2$ has a universal lower bound. It is natural for us to expect that the interval will give this optimal lower estimate.

I would like to dedicate this work to my friend Manfredo do Carmo whose works on minimal surfaces are very original and influential.

1 Generalized log concavity of the first eigenfunction

In [7, 9], I used method of continuity to generalize the log concavity result of Brascamp and Lieb [1] when the potential is convex. I generalize it further in this section.

Let u_1 be the first eigenfunction of the operator $-\Delta + V$ on a domain Ω_1 with zero boundary valued data. Let $\varphi = -\log u_1$. Then we have the following theorem:

Theorem 1.1. *When Ω is strictly convex, the Hessian of φ has eigenvalue greater than $g(x)$ where*

$$g(x) = \sup\Big\{ f(x) \mid \begin{array}{l} f \text{ is a smooth function defined on } \Omega, |f|\, d(x,\partial\Omega) \\ \text{is bounded by the principle curvature of } \partial\Omega, \text{ and} \\ \text{the lowesteigenvalue of the Hessian of } V \text{ plus } \Delta f \\ \text{is greater than } f^2 - 2\sum_j \varphi_j f_j \end{array} \Big\}$$

Proof: Differentiating the equation

$$\Delta\varphi = |\nabla\varphi|^2 - V + \lambda_1 \tag{1.1}$$

we obtain

$$\Delta(\varphi_{ii} - f) = \sum \varphi_{ji}^2 - V_{ii} - \Delta f + 2\sum_j \varphi_j(\varphi_{ii})_j \tag{1.2}$$

where $\varphi_{ji} = \dfrac{\partial^2 u}{\partial x_j\, \partial x_i}$ and $V_{ii} = \dfrac{\partial^2 V}{\partial x_i^2}$.

Note that φ_{ii} goes to infinity in a neighborhood of $\partial\Omega$. Therefore $\varphi_{ii} - f$ has a minimum in the interior of Ω where we have

$$\begin{aligned} \varphi_{ji} &= 0 \qquad \text{for} \qquad j \neq i && (1.3)\\ \Delta(\varphi_{ii} - f) &\geq 0 && (1.4)\\ \nabla(\varphi_{ii} - f) &= 0 && (1.5) \end{aligned}$$

Hence at such a point,

$$\varphi_{ii}^2 \geq V_{ii} + \Delta f - 2\sum_j \varphi_j f_j \tag{1.6}$$

Note that the continuity argument was introduced by me and discussed in [7]. (I had lectured on this in 1979 as was noted in [4].) It can be applied in the following way.

Replace the potential V by $V_t = ||x||^2 + tV$. When $t = 0$, the theorem is obviously true. We have to prove the theorem for all $t > 0$.

Suppose the theorem is true for $t < t_0$ and the theorem fails for $t > t_0$. Then at $t = t_0$, there is a function f, depending on t, so that $\varphi_{ii} \geq f$, $(V_t)_{ii} + \Delta f > f^2 + 2\sum_j \varphi_j f_j$ and at some point, $\varphi_{ii} = f$. By (1.5), we obtain

$$f^2 \geq (V_t)_{ii} + \Delta f - 2\sum_j \varphi_j f_j \tag{1.7}$$

This contradict the choice of f.

Hence t_0 is arbitrarily large and we conclude that Theorem 1.1 holds.

Remark 1.1. *The argument of Theorem 1.1 can be generalized to manifolds with negative curvature.*

In order to apply Theorem 1.1, we shall give a bound on $\sum \varphi_j d_j$ where d is the distance function to $\partial\Omega$.

For simplicity, we restrict our attention to the important case where Ω is a ball of radius R. We shall consider the function

$$\begin{aligned} F &= (R^2 - \textstyle\sum_i x_i^2) \sum_i \varphi_i x_i \\ F_j &= -2x_j \textstyle\sum \varphi_i x_i \; + (R^2 - \sum x_i^2)(\varphi_j + \sum x_i \varphi_{ij}) \\ \Delta F &= -(2n+4) \textstyle\sum \varphi_i x_i \; - 2\sum x_i \varphi_{ij} x_j \\ &= \quad +(R^2 - \textstyle\sum_i x_i^2)\,(2\Delta\varphi + \sum_i x_i (\Delta\varphi)_i) \\ &= -(2n+4) \textstyle\sum_i \varphi_i x_i \; - 2\sum x_i x_j \varphi_{ij} \\ &\quad +(R^2 - \textstyle\sum_i x_i^2)(-2V + 2\lambda + 2|\nabla\varphi|^2 - \sum x_i V_i + 2\sum x_i \varphi_{ij}\varphi_j) \end{aligned} \tag{1.8}$$

When F achieves its maximum, $F_j = 0$ and $\Delta F \leq 0$. Hence if we multiply the last equation of (1.8) by $(R^2 - \sum_i x_i^2)^2$, we find

$$\begin{aligned} 0 \;\geq\; & -(2n+4)(R^2 - \|x\|^2)F \; - 4\|x\|^2 F + 2(R^2 - \|x\|^2)F \\ & +(R^2 - \|x\|^2)^3(-2\,V + 2\lambda - \textstyle\sum x_i V_i) + 4\,F^2 \end{aligned} \tag{1.9}$$

Proposition 1.1 *Let* $\Omega = \{x \,|\, \|x\| < R\}$. *Then either,*

$$\sup\left\{(R^2 - \|x\|^2)\sum_i x_i\varphi_i\right\} \leq \sup\left\{(R^2 - \|x\|^2)^{3/2}(-V + \lambda - \frac{1}{2}\sum x_i V_i)^{1/2}\right\} \tag{1.10}$$

or

$$\sup\left\{(R^2 - \|x\|^2)\sum_i x_i\varphi_i\right\} \leq \left(\frac{n}{2} + 1\right) R^2 \tag{1.11}$$

To construct f that satisfies the inequality

$$\Delta f > f^2 + 2\sum \varphi_j f_j - \inf_i V_{ii} \tag{1.12}$$

we can use the ansatz

$$f = h(R^2 - \|x\|^2) \tag{1.13}$$

Then

$$\begin{aligned} \Delta f &= 4\,h''\|x\|^2 - 2\,n\,h' \\ &= -4\,t\,h'' - 2\,n\,h' + 4R^2\,h'' \end{aligned} \tag{1.14}$$

where $t = R^2 - \|x\|^2$ and $\sum \varphi_i f_i = -h' \sum x_i \varphi_i$. If we choose $h' \le 0$, then

$$\begin{aligned} \sum \varphi_i f_i &\le -\left(\tfrac{h'}{t}\right)(R^2 - \|x\|^2) \sum x_i \varphi_i \\ &\le -\left(\tfrac{h'}{t}\right)\left[(\tfrac{n}{2}+1)R^2 + t^{3/2} \sup_{\|x\|^2 = R^2 - t} \left(-V + \lambda - \tfrac{1}{2}\sum x_i V_i\right)^{1/2}\right] \end{aligned} \tag{1.15}$$

Theorem 1.2. *Let $\Omega = \{x \mid \|x\| < R\}$ and f be a function h of $t = R^2 - \|x\|^2$. If h is chosen to satisfy the inequality*

$$\begin{aligned} -4\,t\,h'' - \; & 2nh' + 4R^2 h'' \\ > \; & h^2 - 2\left(\tfrac{h'}{t}\right) \\ & \left[\left(\tfrac{n}{2}+1\right)R^2 + t^{3/2} \sup_{\|x\|^2 = R^2 - t} \left(-V + \lambda - \tfrac{1}{2}\sum x_i V_i\right)^{1/2}\right] \\ & - \inf_i V_{ii} \end{aligned} \tag{1.16}$$

then $f = h(t)$ satisfies the inequality

$$\Delta f > f^2 + 2\sum_i \varphi_i f_i - \inf_i V_{ii} \tag{1.17}$$

2 Gradient estimate of first eigenfunction

Let Ω be a convex subdominant. Then we can choose a smooth nonnegative function ρ with compact support.

Let

$$G = \rho^2 (V + \alpha)^{-1} \; \Delta\varphi \tag{2.1}$$

where α is a constant to be chosen later. Then according to (1.1), we find

$$\nabla G = G(2\nabla \log \rho - \nabla \log(V + \alpha)) + \rho^2 (V + \alpha)^{-1} \; (2\nabla\varphi \cdot \nabla\nabla\varphi - \nabla V) \tag{2.2}$$

$$\begin{aligned}\Delta G &= \nabla G(2\nabla\log\rho - \nabla\log(V+\alpha)) + G(2\Delta\log\rho - \Delta\log(V+\alpha)) \\ &\quad +2\nabla G\cdot\nabla\log\rho - 4|\nabla\log\rho|^2\ G - \nabla G\cdot\nabla\log(V+\alpha) \\ &\quad +4G\nabla\log(V+\alpha)\cdot\nabla\log\rho - G|\nabla\log(V+\alpha)|^2 \\ &\quad +\rho^2\ (V+\alpha)^{-1}\ [2|\nabla\nabla\varphi|^2 + 4\nabla\varphi\cdot\nabla\nabla\varphi\cdot\nabla\varphi - 2\nabla\varphi\cdot\nabla V - \Delta V] \\ &= \nabla G(4\nabla\log\rho - 2\nabla\log(V+\alpha) + 2\nabla\varphi) \\ &\quad +G(2\Delta\log\rho - \Delta\log(V+\alpha) - 4|\nabla\log\rho|^2 - |\nabla\log(V+\alpha)|^2 \\ &\qquad -4\nabla\log\rho\cdot\nabla\varphi + 2\nabla\log(V+\alpha)\cdot\nabla\log\rho) \\ &\quad +\rho^2(V+\alpha)^{-\frac{1}{2}}\ (|\nabla\nabla\varphi|^2 - \Delta V)\end{aligned} \tag{2.3}$$

By (1.1)

$$\begin{aligned}|\nabla\nabla\varphi|^2 &\geq \frac{1}{n}|\Delta\varphi|^2 \\ &= \frac{1}{n}(V+\alpha)^2\ \rho^4\ G^2\end{aligned} \tag{2.4}$$

Hence at the point where G achieves its maximum,

$$\begin{aligned}0 \ \geq\ & \frac{2}{n}G^2 - \rho^4(V+\alpha)^{-2}\ \Delta V \\ & +\rho^2\ (V+\alpha)^{-1}\ G[2\Delta\log\rho - \Delta\log(V+\alpha) - 4|\nabla\log\rho|^2 \\ &\qquad -|\nabla\log(V+\alpha)|^2 - 4\log\rho\cdot\nabla\varphi + 2\nabla\log(V+\alpha)\cdot\nabla\log\rho]\end{aligned} \tag{2.5}$$

Note that

$$\begin{aligned}\rho^2(V+\alpha)^{-1}\ |\nabla\varphi|^2 &\leq \rho^2\ (V+\alpha)^{-1}\ (|\nabla\varphi|^2 - V - \lambda_1) \\ &\quad +\rho^2\ (V+\alpha)^{-1}\ (V-\lambda_1) \\ &= G + \rho^2\ (V+\alpha)^{-1}\ (V-\lambda_1)\end{aligned} \tag{2.6}$$

Hence

$$\begin{aligned}\frac{2}{n}G^2 &\leq \rho^4\ (V+\alpha)^{-2}\ \Delta V \\ &\quad -\rho^2\ (V+\alpha)^{-1}\ G[2\Delta\log\rho - \Delta\log(V+\alpha) - 4|\nabla\log\rho|^2 \\ &\qquad -|\nabla\log(V+\alpha)|^2 + 2\nabla\log(V+\alpha)\cdot\nabla\log\rho] \\ &\quad +4\rho(V+\alpha)^{-\frac{1}{2}}\end{aligned} \tag{2.7}$$

Therefore,
either

$$G \leq \frac{4}{n^2}\ \rho^2 (V+\alpha)^{-1}\ |\nabla \log \rho|^2 \tag{2.8}$$

or

$$G \leq 3\sqrt{\frac{n}{2}}\ \rho^2\ (V+\alpha)^{-1}\ \sqrt{(\Delta V)_+} \tag{2.9}$$

or

$$\begin{aligned} G \quad \leq \quad & \frac{3n}{2}\ \rho^2\ (V+\alpha)^{-1}\ [-2\Delta \log \rho + 4|\nabla \log \rho|^2 + \Delta \log (V+\alpha) \\ & +|\nabla \log (V+\alpha)|^2 - 2\nabla \log (V+\alpha)\cdot \nabla \log \rho + 4(V-\lambda_1)^{\frac{1}{2}}] \end{aligned} \tag{2.10}$$

Theorem 2.1. *Let* u_1 *be a positive solution of* $(\Delta+V)u_1 = \lambda_1 u$ *and* $\varphi = -\log u_1$. *Then*

$$\begin{aligned} & \rho^2\ (V+\alpha)^{-1}(|\nabla\varphi|^2 - V + \lambda_1) \\ & \quad \leq 10n\ \rho^2\ (V+\alpha)^{-1}\ (|\nabla \log \rho|^2 + |\Delta \log \rho|) \\ & \qquad + \tfrac{3n}{2}\ \rho^2\ (V+\alpha)^{-1}\ \left\{(\Delta \log (V+\alpha))_+ + 2|\nabla \log (V+\alpha)|^2 + 4(V-\lambda_1)^{\frac{1}{2}}\right\} \end{aligned} \tag{2.11}$$

In particular,

$$\begin{aligned} \rho^2\ (V+\alpha)^{-1}|\nabla\varphi|^2 \quad \leq \quad & \frac{\sup V - \lambda_1}{\sup V + \alpha} \\ & +10n^2 \sup (V+\alpha)^{-1}\ \rho^2\ (|\nabla \log \rho|^2 + |\Delta \log \rho|) \\ & +3n \sup \left[(V+\alpha)^{-2}\ ((\Delta V)_+) + (V+\alpha)^{-3}\ |\nabla V|^2\right] \\ & +6n \sup (V+\alpha)^{-\frac{1}{2}}\ \left(\frac{\sup V - \lambda_1}{\sup V + \alpha}\right)^{\frac{1}{2}} \end{aligned}$$

3 Gradient estimate for $\dfrac{u_2}{u_1}$

Let u_2 be the second eigenfunction of $-\Delta + V$ on Ω. Let $u = \dfrac{u_2}{u_1}$.

Then

$$\Delta u = -(\lambda_2 - \lambda_1)u + 2\nabla\varphi\,\nabla u \tag{3.1}$$

Let $c > \sup_{\Omega_2} \dfrac{u_2}{u_1}$ and $\psi = -\ln(c-u)$.
Then

$$\Delta\psi = (\lambda_2 - \lambda_1)(1 - ce^{\psi}) + 2\nabla\varphi\nabla\psi + |\nabla\psi|^2. \tag{3.2}$$

Let

$$F = \rho^2(V+\alpha)^{-1}\left[|\nabla\psi|^2 + (\lambda_2 - \lambda_1)(1 - ce^{\psi})\right] \tag{3.3}$$

when $\alpha > 0$ is a constant to be chosen.
Then

$$\begin{aligned} \nabla F &= 2F\nabla\log\rho - F\nabla\log(V+\alpha) \\ &\quad + \rho^2(V+\alpha)^{-1}\left[2\nabla\psi\,\nabla\nabla\psi - c(\lambda_2-\lambda_1)e^{\psi}\nabla\psi\right] \end{aligned} \tag{3.4}$$

and

$$
\begin{aligned}
\Delta F &= \nabla F\left(2\nabla\log\rho-\nabla\log(V+\alpha)\right)+F\left(2\Delta\log\rho-\Delta\log(V+\alpha)\right)\\
&\quad+4\rho(V+\alpha)^{-1}\nabla\rho\cdot\nabla\psi\cdot\nabla\nabla\psi-2\rho^2(V+\alpha)^{-2}\ \nabla V\cdot\nabla\nabla\psi\cdot\nabla\psi\\
&\quad+2\rho^2(V+\alpha)^{-1}\ |\nabla\nabla\psi|^2+2\rho^2(V+\alpha)^{-1}\ \nabla\psi\cdot\nabla(\Delta\psi)\\
&\quad-2c\rho(V+\alpha)(\lambda_2-\lambda_1)e^{\psi}\ \nabla\rho\cdot\nabla\psi+c\rho^2(V+\alpha)^{-2}(\lambda_2-\lambda_1)e^{\psi}\ \nabla V\cdot\nabla\psi\\
&\quad-c\rho^2(V+\alpha)^{-1}(\lambda_2-\lambda_1)e^{\psi}\ |\nabla\psi|^2\\
&\quad-c\rho^2(V+\alpha)^{-1}(\lambda_2-\lambda_1)e^{\psi}\left[|\nabla\psi|^2+2\nabla\varphi\cdot\nabla\psi+(\lambda_2-\lambda_1)(1-ce^{\psi})\right]\\
&= \nabla F(2\nabla\log\rho-\nabla\log(V+\alpha))+F(2\Delta\log\rho-\Delta\log(V+\alpha))\\
&\quad+2\nabla F\cdot\nabla\log\rho-4F|\nabla\log\rho|^2+2F\nabla\log(V+\alpha)\cdot\nabla\log\rho\\
&\quad+2c\rho^2(V+\alpha)^{-1}(\lambda_2-\lambda_1)e^{\psi}\ \nabla\psi\cdot\nabla\log\rho-(V+\alpha)^{-1}\ \nabla F\cdot\nabla V\\
&\quad+2F\nabla\log\rho\cdot\nabla\log(V+\alpha)-F|\nabla\log(V+\alpha)|^2\\
&\quad-c\rho^2(V+\alpha)^{-2}(\lambda_2-\lambda_1)e^{\psi}\ \nabla\psi\cdot\nabla V+2\rho^2(V+\alpha)^{-1}|\nabla\nabla\psi|^2\\
&\quad+2\rho^2(V+\alpha)^{-1}(2\nabla\psi\cdot\nabla\nabla\psi\cdot\nabla\psi+2\nabla\psi\cdot\nabla\nabla\psi\cdot\nabla\psi\\
&\qquad\qquad+2\nabla\psi\cdot\nabla\psi\cdot\nabla\nabla\psi-c(\lambda_2-\lambda_1)e^{\psi}|\nabla\psi|^2)\\
&\quad-2c\rho(V+\alpha)(\lambda_2-\lambda_1)e^{\psi}\nabla\psi\cdot\nabla\psi+c\rho^2(V+\alpha)^{-2}(\lambda_2-\lambda_1)e^{\psi}\ \nabla V\cdot\nabla\psi\\
&\quad-c\rho^2(V+\alpha)^{-1}(\lambda_2-\lambda_1)e^{\psi}|\nabla\psi|^2\\
&\quad-c\rho^2(V+\alpha)^{-1}(\lambda_2-\lambda_1)e^{\psi}\left[|\nabla\psi|^2+2\nabla\varphi\cdot\nabla\psi+(\lambda_2-\lambda_1)(1-ce^{\psi})\right]
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
= \quad & \nabla F(4\nabla \log\rho - 2\nabla\log(V+\alpha)) + F(2\Delta\log\rho - \Delta\log(V+\alpha) - 4|\nabla\log\rho|^2 \\
& \qquad +4\nabla\log(V+\alpha)\cdot\nabla\log\rho - |\nabla\log(V+\alpha)|^2) \\
& +2\rho^2(V+\alpha)^{-1}\ |\nabla\nabla\psi|^2 + 2\nabla F\cdot\nabla\psi - 2F\nabla\log\rho\cdot\nabla\psi \\
& -2F\nabla\log(V+\alpha)\cdot\nabla\psi + 4\rho^2(V+\alpha)^{-1}\ \nabla\psi\cdot\nabla\nabla\psi\cdot\nabla\psi + 2\nabla F\cdot\nabla\varphi \\
& -2F\nabla\log\rho\cdot\nabla\varphi + 2F\nabla\log(V+\alpha)\nabla\varphi \\
& +2c\rho^2(V+\alpha)^{-1}(\lambda_2-\lambda_1)e^{\psi}\ \nabla\psi\cdot\nabla\varphi \\
& -c\rho^2(V+\alpha)^{-1}(\lambda_2-\lambda_1)e^{\psi}|\nabla\psi|^2 \\
& -c\rho^2(V+\alpha)^{-1}(\lambda_2-\lambda_1)e^{\psi}\left[|\nabla\psi|^2 + 2\nabla\varphi\cdot\nabla\psi + (\lambda_2-\lambda_1)(1-ce^{\psi})\right] \\
= \quad & \nabla F(4\nabla\log\rho - 2\nabla\log(V+\alpha) + 2\nabla\psi + 2\nabla\psi) \\
& +F(2\Delta\log\rho - \Delta\log(V+\alpha) - 4|\nabla\log\rho|^2 \\
& \qquad +4\nabla\log(V+\alpha)\cdot\nabla\log\rho - |\nabla\log(V+\alpha)|^2 - 2\nabla\log\rho\cdot\nabla\psi \\
& \qquad -2\nabla\log(V+\alpha)\cdot\nabla\psi - 2\nabla\log\rho\cdot\nabla\varphi + 2\nabla\log(V+\alpha)\cdot\nabla\varphi) \\
& +2\rho^2(V+\alpha)^{-1}\ |\nabla\nabla\psi|^2 + 4\rho^2(V+\alpha)^{-1}\ \nabla\psi\cdot\nabla\nabla\psi\cdot\nabla\psi \\
& -2c\rho^2(V+\alpha)^{-1}(\lambda_2-\lambda_1)e^{\psi}\ |\nabla\psi|^2 - c(\lambda_2-\lambda_1)^2\ \rho^2\ (V+\alpha)^{-1}\ e^{\psi}(1-ce^{\psi})
\end{aligned}
$$

Now

$$
|\nabla\nabla\psi|^2 \geq \frac{1}{n}(\Delta\psi)^2 \tag{3.6}
$$

and

$$
\Delta\psi - 2\nabla\psi\cdot\nabla\varphi = \rho^{-2}\ (V+\alpha)F \tag{3.7}
$$

Hence

$$
\begin{aligned}
\rho^2(V+\alpha)^{-1}\ |\nabla\nabla\psi|^2 \geq \rho^2(V+\alpha)^{-1} & \\
\left[(V+\alpha)^2\ \rho^{-4}\ F^2 + 4\rho^{-2}\ (V+\alpha)F\nabla\psi\cdot\nabla\varphi + 4(\nabla\psi\cdot\nabla\varphi)^2\right] &
\end{aligned} \tag{3.8}
$$

When F achieves its maximum, $\nabla F = 0$ and $\Delta F \leq 0$. Therefore,

$$\begin{aligned}0 \geq\ & F(2\Delta\log\rho - \Delta\log(V+\alpha) - 4|\nabla\log\rho|^2 + 4\nabla\log(V+\alpha)\cdot\nabla\log\rho \\ & \quad -|\nabla\log(V+\alpha)|^2 - 2\nabla\log\rho\cdot\nabla\psi - 2\nabla\log(V+\alpha)\cdot\nabla\psi \\ & \quad -2\nabla\log\rho\cdot\nabla\varphi + 2\nabla\log(V+\alpha)\cdot\nabla\varphi) \\ & + \frac{2}{n}\rho^{-2}\ (V+\alpha)F^2 + \frac{8}{n}F\ \nabla\psi\cdot\nabla\varphi + \frac{8}{n}\rho^2(V+\alpha)^{-1}\ (\nabla\psi\cdot\nabla\varphi)^2 \qquad (3.9) \\ & + 4\rho^2(V+\alpha)^{-1}\ \nabla\psi\cdot\nabla\nabla\varphi\cdot\nabla\psi \\ & - 2c\rho^2(V+\alpha)^{-1}\ (\lambda_2-\lambda_1)e^{\psi}\left[|\nabla\psi|^2 + (\lambda_2-\lambda_1)(1-ce^{\psi})\right] \\ & + c(\lambda_2-\lambda_1)^2\ \rho^2\ (V+\alpha)^{-1}\ e^{\psi}(1-ce^{\psi})\end{aligned}$$

Note that

$$\rho^2(V+\alpha)^{-1}\ \nabla\psi\cdot\nabla\nabla\varphi\cdot\nabla\psi \geq \inf\left(\rho^2(V+\alpha)^{-1}\ g(x)\right)F \qquad (3.10)$$

where g is defined in Theorem 1.1.
Hence

$$\begin{aligned}0 \geq\ & \frac{2}{n}F^2 + 4\inf\left(\rho^2\ (V+\alpha)^{-1}\ g(x)\right)F \\ & -F\left(\rho^2\,(V+\alpha)^{-1}\,|\nabla\psi|^2\right) \\ & \quad \left[\tfrac{8}{n}\left(\rho^2\ (V+\alpha)^{-1}\ |\nabla\varphi|^2\right)^{\frac{1}{2}} + 2\left(\rho^2\ (V+\alpha)^{-1}\ |\nabla\log\rho|^2\right)\right. \\ & \qquad \left. +2\left(\rho^2\ (V+\alpha)^{-1}\ |\nabla\log(V+\alpha)|^2\right)^{\frac{1}{2}}\right] \qquad (3.11) \\ & +\rho^2\,(V+\alpha)^{-1}\,F \\ & \quad \left[2\Delta\log\rho - 7|\nabla\log\rho|^2 - 2|\nabla\varphi|^2 - \Delta\log(V+\alpha) - |\nabla\log(V+\alpha)|^2\right] \\ & -2c\rho^2(V+\alpha)^{-1}\ (\lambda_2-\lambda_1)e^{\psi}\ F \\ & +c(\lambda_2-\lambda_1)^2\,\rho^{\psi}\ (V+\alpha)^{-2}\ e^{\psi}\ (1-ce^{\psi})\end{aligned}$$

By definition of F, either

$$\rho^2\ (V+\alpha)^{-1}\ |\nabla\psi|^2 \leq 2F \qquad (3.12)$$

or

$$F \leq (\lambda_2-\lambda_1)(V+\alpha)^{-1}\ (1-ce^{\psi}) \qquad (3.13)$$

Let us assume (3.12) first. In that case, either

$$F^{\frac{1}{2}} \leq \frac{3n}{2}\left[\frac{8}{n}\left(\rho^2\ (V+\alpha)^{-1}\ |\nabla\varphi|^2\right)^{\frac{1}{2}} + 2\left(\rho^2\ (V+\alpha)^{-1}\ |\nabla\log\rho|^2\right)^{\frac{1}{2}}\right. \\ \left. +2\left(\rho^2\ (V+\alpha)^{-1}\ |\nabla\log(V+\alpha)|^2\right)^{\frac{1}{2}}\right] \tag{3.14}$$

or

$$F + 6n\inf\left(\rho^2\ (V+\alpha)^{-1}\ g(x)\right) \\ \leq \frac{3n}{2}\rho^2\ (V+\alpha)^{-1}\Big[2\Delta\log\rho + 7|\nabla\log\rho|^2 + \Delta\log(V+\alpha) \\ +|\nabla\log(V+\alpha)|^2 + 2|\nabla\varphi|^2 + 2c(\lambda_2-\lambda_1)e^{\psi}\Big] \tag{3.15}$$

or

$$F^2 \leq \frac{3n}{2}c(\lambda_2-\lambda_1)^2\ \rho^2\ (V+\alpha)^{-2}\ e^{\psi}\ |(1-ce^{\psi})| \tag{3.16}$$

From this, we conclude:

Theorem 3.1. *Let Ω be a domain and ρ be smooth function with compact support in Ω. Let u_i be smooth function satisfying the equation $(-\Delta+V)u_i = \lambda_i\, u_i$ so that $u_1 > 0$ and $\varphi = -\log u_1$ satisfies the conclusion of Theorem 1.1. Let c be a constant so that $c > \sup u$ where $u = \dfrac{u_2}{u_1}$. Let $\psi = -\log(c-u)$. Then one of the following inequalities hold:*

(1)

$$\rho^2(V+\alpha)^{-1}\left[|\nabla\psi|^2 + (\lambda_2-\lambda_1)(1-ce^{\psi})\right] \\ \leq (\lambda_2-\lambda_1)\sup(V+\alpha)^{-1}(1-ce^{\psi}) \tag{3.17}$$

(2)

$$\rho^2\ (V+\alpha)^{-1}\left[|\nabla\psi|^2 + (\lambda_2-\lambda_1)(1-ce^{\psi})\right] \\ \leq 144\left(\frac{\sup V-\lambda_1}{\sup V+\alpha}\right) + 20n^2\ \sup(V+\alpha)^{-1} \\ \rho^2\ \left(|\nabla\log\rho|^2 + |\Delta\log\rho|\right) \\ +4n\ \sup\left((V+\alpha)^{-2}\ (\Delta V)_+ + (V+\alpha)^{-3}\ |\nabla V|^2\right) \\ +6n\ \sup(V+\alpha)^{-\frac{1}{2}}\ \left(\frac{\sup V-\lambda_1}{\sup V+\alpha}\right)^{\frac{1}{2}} \tag{3.18}$$

(3)

$$\begin{aligned}
&\rho^2\left[(V+\alpha)^{-1}\,|\nabla\psi|^2+(\lambda_2-\lambda_1)(1-ce^{\psi})\right]\\
&+6n\inf\left(\rho^2\,(V+\alpha)^{-1}\,g(x)\right)\\
\leq\;&20n^2\,\sup(V+\alpha)^{-1}\,\left(|\nabla\rho|^2+\rho|\Delta\rho|\right)\\
&+10n\,\sup\rho^2\left[(V+\alpha)^{-2}\,(\Delta V)_+ +(V+\alpha)^{-3}\,|\nabla V|^2\right] \qquad (3.19)\\
&+10\left(\frac{\sup V-\lambda_1}{\sup V+\alpha}\right)+6n\,\sup(V+\alpha)^{-\frac{1}{2}}\left(\frac{\sup V-\lambda_1}{\sup V+\alpha}\right)^{\frac{1}{2}}\\
&+3nc(\lambda_2-\lambda_1)\,\sup\rho^2\,(V+\alpha)^{-1}\,e^{\psi}
\end{aligned}$$

(4)

$$\begin{aligned}
&\rho^2\,(V+\alpha)^{-1}\,\left(|\nabla\psi|^2+(\lambda_2-\lambda_1)(1-ce^{\psi})\right)\\
\leq\;&\sqrt{3nc}\,(\lambda_2-\lambda_1)\,\sup\rho^2\,(V+\alpha)^{-1}\,\left[e^{\psi}\,(1-ce^{\psi})\right]^{\frac{1}{2}}
\end{aligned} \qquad (3.20)$$

4 Estimate of the gap $\lambda_2-\lambda_1$ in terms of the potential

Let $u=\dfrac{u_2}{u_1}$ be defined as in S3. We assume that it is bounded on Ω and is zero somewhere in Ω.

Assume that for some $\delta>0$, $u(x_0)=\sup u$ and $u(x_1)=\delta\sup u$. Then for each smooth function ρ with compact support and constant $\alpha\geq 0$, we can define

$$L(\rho,\alpha,\delta)=\inf_x\int_0^1\left(\rho^{-1}\sqrt{V+\alpha}\right)(x(t))\,|\dot{x}|dt \qquad (4.1)$$

where x is any path in Ω with $x(0)=x_0$ and $x(1)=x_1$.

Now we can define

$$\begin{aligned}
L_\delta\;=\;&\inf_{\alpha,\rho}L(\rho,\alpha,\delta)\Big\{20n^2\sup(V+\alpha)^{-1}\,\rho^2\,\left(|\Delta\rho|+|\nabla\rho|^2\right)\\
&+10n\sup\rho^2\,(V+\alpha)^{-3}\,\left((\Delta V)_+ +|\nabla V|^2\right)+10\frac{\sup V-\lambda_1}{\sup V+\alpha}\\
&+6n\sup(V+\alpha)^{-\frac{1}{2}}\,\left(\frac{\sup V-\lambda_1}{\sup V+\alpha}\right)^{\frac{1}{2}}-6n\inf\left(\rho^2\,(V+\alpha)^{-1}\,g(x)\right)\Big\}
\end{aligned}$$

where $\alpha > 0$ is a constant and ρ is any smooth function with compact support in Ω.

Based on Theorem 3.1, we conclude that

Theorem 4.1.

$$\left|\log \frac{1}{\delta}\right| \leq L_\delta + (\lambda_2 - \lambda_1)\left(\frac{1}{\delta} + \inf_{\rho,\alpha}\left\{L(\rho,\alpha,\delta)\sup \frac{\rho^2\,(V+\alpha)^{-1}}{\delta}\right\}\right)$$

In particular if for some δ, $\log \dfrac{1-\delta}{\delta} - L_\delta > 0$, there is a lower estimate of $\lambda_2 - \lambda_1$, in terms of L_δ and $\inf_{\rho,\alpha}\left\{L(\rho,\alpha,\delta)\sup\left(\dfrac{\rho^2\,(V+\alpha)^{-1}}{\delta}\right)\right\}$.

5 Oscillation of the function $\dfrac{u_2}{u_1}$

Note that in S4, we do not need to assume u_i satisfies any boundary coordinates on Ω.

If we assume $u_i = 0$ on $\partial\Omega$, $u_1 > 0$ and

$$\int_\Omega u_i^2 = 1 \tag{5.1}$$

$$\int_\Omega u_1\, u_2 = 0 \tag{5.2}$$

we find

$$\int_\Omega u^2\, u_1^2 = 1 \tag{5.3}$$

$$\int_\Omega u\, u_1^2 = 0 \tag{5.4}$$

The eigenfunction equations give

$$\int_\Omega |\nabla u_i|^2 + \int_\Omega V u_i^2 = \lambda_i \tag{5.5}$$

Note that assuming (5.1), (5.2) can also be written as

$$\int_\Omega (u_1 + u_2)^2 = 2 \tag{5.6}$$

or

$$\int_{\Omega}(u_1 - u_2)^2 = 2 \tag{5.7}$$

Let

$$\Omega_t = \{x \in \Omega |\ d(x_1\, \partial\Omega) \geq t\} \tag{5.8}$$

We are interested in the behavior of $\int_{\Omega_t} u_i^2$ and $\int_{\Omega_t} u_1\, u_2$. Hence we compute

$$\begin{aligned} \frac{d^2}{dt^2}\int_{\Omega_t} u_i^2 &= -\frac{d}{dt}\int_{\delta\Omega_t} u_i^2 \\ &= \int_{\partial\Omega_t} u_i^2\, H_t + 2\int_{\partial\Omega_t} u_i \frac{\partial u_i}{\partial \nu} \\ &= \int_{\partial\Omega_t} u_i^2 H_t + \int_{\Omega_t} \Delta(u_i^2) \end{aligned} \tag{5.9}$$

where H_t is the mean curvature of $\partial\Omega_t$, measured by the outward normal. But

$$\begin{aligned} \int_{\Omega_t} \Delta(u_i)^2 &= 2\int_{\Omega_t} |\nabla u_i|^2 + 2\int_{\Omega_t} u_i \Delta u_i \\ &= 2\int_{\Omega_t} (|\nabla u_i|^2 + V u_i^2) - 2\lambda_i \int_{\Omega_t} u_i^2 \\ &= 2\inf(V - \lambda_i) \end{aligned} \tag{5.10}$$

Since $u_i = 0$ on $\partial\Omega$,

$$\frac{d}{dt}\int_{\Omega_t} u_i^2 = 0 \tag{5.11}$$

where $t = 0$.

We conclude that

$$\begin{aligned} \int_{\Omega_t} u_i^2 &= \int_0^t \frac{d}{dt}\left(\int_{\Omega_t} u_i^2\right) + \int_{\Omega} u_i^2 \\ &= \int_0^t \left(\int_0^s \frac{d^2}{ds^2}\left(\int_{\Omega_s} u_i^2\right)\right) + \int_{\Omega} u_i^2 \\ &\geq \inf(V - \lambda_i)t^2 + 1 \end{aligned} \tag{5.12}$$

Similarly,

$$\begin{aligned}
&\frac{d^2}{dt^2}\int_{\Omega_t}(u_1+u_2)^2\\
\geq\ & 2\int_{\Omega_t}(\nabla u_1+\nabla u_2)^2+2\int_{\Omega_t}(u_1+u_2)V(u_1+u_2)\\
&-2\int_{\Omega_t}(u_1+u_2)(\lambda_1\,u_1+\lambda_2\,u_2)\\
\geq\ & 2\int_{\Omega_t}\left(V-\frac{\lambda_1+\lambda_2}{2}\right)(u_1+u_2)^2+(\lambda_2-\lambda_1)\int_{\Omega_t}u_1^2+(\lambda_1-\lambda_2)\int_{\Omega_t}u_2^2\\
\geq\ & 4\inf\left(V-\frac{(\lambda_1+\lambda_2)}{2}\right)+(\lambda_1-\lambda_2)\\
\geq\ & 4\inf(V-\lambda_2)
\end{aligned}\tag{5.13}$$

$$\int_{\Omega_t}(u_1+u_2)^2 \quad\geq\quad 2+2\left(\inf_{\Omega}V-\lambda_2\right)t^2 \tag{5.14}$$

$$\int_{\Omega_t}(u_1-u_2)^2 \quad\geq\quad 2+2\left(\inf_{\Omega}V-\lambda_2\right)t^2 \tag{5.15}$$

From (5.14), we obtain

$$\int_{\Omega_t}u_1\,u_2\geq\left[\inf_{\Omega}(V-\lambda_2)\right]t^2 \tag{5.16}$$

From (5.15), we obtain

$$\int_{\Omega_t}u_1\,u_2\leq-\left[\inf_{\Omega}(V-\lambda_2)\right]t^2 \tag{5.17}$$

Let $u=\dfrac{u_2}{u_1}$. Then from (5.12),

$$\begin{aligned}
\int_{\Omega_t}u^2\,u_1^2 \quad&=\quad \int_{\Omega_t}u_2^2\\
&\geq\quad 1+\inf_{\Omega}(V-\lambda_2)t^2
\end{aligned}\tag{5.18}$$

Hence

$$\sup_{\Omega_t}u^2\geq 1+\inf_{\Omega}(V-\lambda_2)t^2 \tag{5.19}$$

On the other hand,

$$\begin{aligned} \inf_{\Omega_t}|u| &\le \left|\int_{\Omega_t} u\, u_1^2\right| \\ &\le t^2\left[-\inf_{\Omega}(V-\lambda_2)\right] \end{aligned} \tag{5.20}$$

Combining (5.19) and (5.20), we conclude

$$\frac{\inf_{\Omega_t}|u|}{\sup_{\Omega_t}|u|} \le \frac{t^2\left[-\inf_{\Omega}(V-\lambda_2)\right]}{1+\inf_{\Omega}(V-\lambda_1)t^2} \tag{5.21}$$

In the next section, we shall apply the estimates in S4.

6 Distance function and the estimate of the gap

For simplicity, we shall assume that Ω to be convex in this section. We also assume that

$$|\nabla V| \le c_\alpha (V+\alpha)^{\frac{3}{2}} \tag{6.1}$$

$$|(\Delta V)_+| \le c_\alpha (V+\alpha)^3 \tag{6.2}$$

We introduce

$$d_\alpha(x_0, x_1) = \inf \int_0^1 \sqrt{V+\alpha}(x(t))\ |\dot{x}| dt \tag{6.3}$$

where the infinitum is taken over all paths $x : [0,1] \to \Omega$ joining x_0 to x_1.
If

$$u(x_0) = \sup u \tag{6.4}$$

and

$$u(x_1) = \delta \sup u \tag{6.5}$$

We define $L_\Omega(\delta)$ to be $d(x_0, x_1)$. It is of course dominated by

$$L(\Omega) = \sup_{\tilde{x_0}, \tilde{x_1} \in \Omega} d(\tilde{x_0}, \tilde{x_1}) \tag{6.6}$$

Using the terminology of S4 and S5, we set ρ to be function of t so that $\rho = 0$ on $\partial\Omega$ and $\rho = 1$ when $t \ge \left(\inf_{\partial\Omega} V\right)^{-\frac{1}{2}}$.

We can assume

$$\rho^2\left(|\nabla \log \rho|^2 + |\Delta \log \rho|\right) \le 3\left(\inf_{\partial\Omega} V\right) \tag{6.7}$$

Note that

$$\begin{aligned} \frac{1}{V+\alpha} - \frac{1}{\inf_{\partial\Omega} V+\alpha} &\le \frac{|\nabla V|}{(V+\alpha)^2} t \\ &\le \frac{c_\alpha t}{(V+\alpha)^{\frac{1}{2}}} \\ &\le \frac{c_\alpha t}{(\inf_{\partial\Omega} V+\alpha)^{\frac{1}{2}}} + \frac{1}{2} c^2 t^2 \end{aligned} \tag{6.8}$$

Hence,

$$\sup(V+\alpha)^{-1}\rho^2(\Delta\rho + |\nabla\rho|^2) \le \frac{3(\inf V)}{(\inf_{\partial\Omega} V+\alpha)} + \frac{c_\alpha(\inf_{\partial\Omega} V)^{\frac{1}{2}}}{(\inf_{\partial\Omega} V+\alpha)^{\frac{1}{2}}} + \frac{3c_\alpha^2}{2} \tag{6.9}$$

From (5.21), we obtain

$$\frac{\inf_{\Omega_t}|u|}{\sup_{\Omega_t}|u|} \le \frac{-\inf_\Omega(V-\lambda_2)}{\inf_{\partial\Omega} V + \inf_\Omega(V-\lambda_1)} \tag{6.10}$$

Now assume

$$|\inf_\Omega(V-\lambda_2)| \le \varepsilon \inf_{\partial\Omega} V \tag{6.11}$$

Then

$$\frac{\inf_{\Omega_t}|u|}{\sup_{\Omega_t}|u|} \le \frac{\varepsilon}{1-\varepsilon} \tag{6.12}$$

According to Theorem 4.1, we have proved

Theorem 6.1. *Assume (6.1), (6.2) and (6.11) and* $t = (\inf_{\partial\Omega} V)^{-\frac{1}{2}}$

$$\left|\log\left(\frac{\varepsilon}{1-\varepsilon}\right)\right| \le \tilde{c}_\alpha L(\Omega_t) + \frac{2(\lambda_2-\lambda_1)}{\varepsilon}\left[1 + L(\Omega_t)\sup_\Omega(V+\alpha)^{-1}\right] \tag{6.13}$$

Here $\tilde{c}_\alpha$ *depends on* c_α *and* $-\inf((V+\alpha)^{-1} g(x))$.

Note that when V grows fast

$$\left|\log\left(\frac{\varepsilon}{1-\varepsilon}\right)\right| > \tilde{c}_\alpha L(\Omega_t)$$

and we have a lower bound for $\lambda_2 - \lambda_1$ from (6.13).

If we know the location of the points to achieve $\inf_{\Omega_t}(u) = u(x_0)$ and $\sup_{\Omega_t}|u| = u(x_1)$, then we can replace $L(\Omega_t)$ by $d_\alpha(x_0, x_1)$. This can be applied when we have the double well potential.

References

[1] Brascamp, H.; Lieb, E., *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Func. Anal., 22 (1976), 366–389.

[2] Li, P., *A lower bound for the first eigenvalue of the Laplacian on a compact manifold*, Indiana Univ. Math. J. 28 (1979), no. 6, 1013–1019.

[3] Li, P.; Yau, S.-T., *Estimates of eigenvalues of a compact Riemannian manifold*, AMS. Proc. Symp. Pure Math., 36 (1980), 205–239.

[4] Korevaar, N. J., *Convexity properties of solutions to elliptic P.D.E.'s*, Variational methods for free surface interfaces (Menlo Park, Calif., 1985), 115–121, Springer, New York, 1987.

[5] Ling, J., *A lower bound for the gap between the first two eigenvalues of Schrödinger operators on the convex domain in S^n or R^n*, Michigan Mathematical Journal, 40 (1993), No. 2, 259–270.

[6] Perelman, G., *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math/0211159.

[7] Singer, I. M.; Wang, B.; Yau, S.-T.; Yau, S.S.-T., *An estimate of the gap of the first two eigenvalues*, Ann. Scuola Norm. Sup. Pisa. Cl. Sci., 12 (1985), 319–333.

[8] Yau, S.-T., *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math., 28 (1975), 201–228.

[9] Yau, S.-T., *An estimate of the gap of the first two eigenvalues in the Schrödinger operator*, Lectures on Partial differential equations: procedings in honor of Louis Nirenberg's 75th Birthday, 223-235, International Press, (2003).

[10] Zhong, J. Q.; Yang, H. C., *On the estimate of the first eigenvalues of a compact Riemannian manifold*, Sci. Sinica Ser. A, (1984), No. 12, 1265–1273.

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