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ON THE GAUSS MAP OF A MINIMAL SURFACE IN THE HEISENBERG GROUP

C. B. Figueroa

Abstract

In this paper we study the Gauss map of minimal surfaces in the Heisenberg group,$\mathcal{H}_3$. We obtain a representation formula for minimal surfaces in $\mathcal{H}_3$ by means of the Gauss map.

1 Introduction

It is well-known the classical Weierstrass representation formula describes minimal surfaces in $\mathbb{R}^3$ in terms of their Gauss map. More generally, Kenmotsu [7] show a representation formula for arbitrary surfaces in $\mathbb{R}^3$ with nonvanishing mean curvature, which describes these surfaces in terms of their Gauss map and mean curvature functions.

Motivated by these results, we will show that any minimal surfaces immersed in $\mathcal{H}_3$ satisfies a system of first order partial equations involving its Gauss map, ψ. An interesting feature is that the complete integrability condition for the above equations is a nonlinear second order partial differential equation which is nothing but the tension field of ψ.

On the other hand given a nowhere holomorphic smooth mapping ψ, of a simply connected Riemann surface into the Riemman sphere S^2 satisfying the complete integrability condition, we can construct a minimal immersion of M into $\mathcal{H}_3$. We now review the contents of the paper.

In section 2 we present the basic Riemannian geometry of $\mathcal{H}_3$ equipped with a left-invariant metric and a relationship between the Gauss map and the extrinsic geometry of surfaces in $\mathcal{H}_3$. In the same section we describe, in charts, the tension field of a minimal surface in $\mathcal{H}_3$.

In section 3 we prove that the Gauss map of a minimal immersion in $\mathcal{H}_3$ must satisfy a first order differential equation of Beltrami type.

A representation formula for minimal surfaces in $\mathcal{H}_3$ by means of the Gauss map and the integrability condition is shown in section 4 and 5.

In section 6 we make a little introduction the Hopf differential and show a result which will be important for the next section.

Finally, in the last section we show some examples.

2 Basic Riemannian Geometry of $\mathcal{H}_3$

The Lie algebra, h_3, of $\mathcal{H}_3$ is isomorphic to $I\!R^3$ with the Lie product:

$$\left\{ \begin{array}{rcl} [e_1, e_2] & = & e_3 \\ [e_i, e_3] & = & 0, \ i = 1, 2, 3. \end{array} \right.$$

where $\{e_i\}$ is the canonical basis in $I\!R^3$.

The exponential map, $\exp : h_3 \to \mathcal{H}_3$, is given by:

$$\exp(A) = I + A^2 + A^3$$

and it is a diffeomorphism which induces on h_3, by the Campbell-Hausdorff formula, the group structure on $\mathcal{H}_3$:

$$\mathbf{x}_1 * \mathbf{x}_2 = \mathbf{x}_1 + \mathbf{x}_2 + \tfrac{1}{2}[\mathbf{x}_1, \mathbf{x}_2]. \tag{1}$$

where $\mathbf{x} = xe_1 + ye_2 + ze_3$. Notice that the 1-parameter subgroups are straight lines.

From now on, modulo the identification given by exp, we consider $\mathcal{H}_3$ as $I\!R^3$ with the product given in (1). Using $\{e_i\}$ as the orthonormal frame at the identity , we have an orthonormal basis of left-invariant vector fields:

$$
\begin{aligned}
E_1 &= \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z} \\
E_2 &= \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} \\
E_3 &= \frac{\partial}{\partial z}
\end{aligned}
$$

The left-invariant metric, induced by the Euclidean metric at the identity, is given by

$$ds^2 = dx^2 + dy^2 + (\tfrac{y}{2}dx - \tfrac{x}{2}dy + dz)^2, \tag{2}$$

then the Riemman connection of ds^2, in terms of the basis $\{E_i\}$, is given by:

$$
\begin{aligned}
\nabla_{E_1}E_2 &= \tfrac{1}{2}E_3 &&= -\nabla_{E_2}E_1 \\
\nabla_{E_1}E_3 &= -\tfrac{1}{2}E_2 &&= \nabla_{E_3}E_1 \\
\nabla_{E_2}E_3 &= \tfrac{1}{2}E_1 &&= \nabla_{E_3}E_2 \\
\nabla_{E_i}E_i &= 0.
\end{aligned}
$$

Let M be an oriented 2-dimensional connected Riemannian manifold and $f : M \to \mathcal{H}_3$ an isometric immersion of M into $\mathcal{H}_3$. At a neighborhood of any point of M we shall use an isothermal coordinate (X, U) ,

$$U \xrightarrow{X} M \xrightarrow{f} \mathcal{H}_3$$

and making use of it, the first fundamental form is now written by $ds^2 = \lambda^2\,|dz|^2$, $\lambda > 0$. The coordinate fields, $X_u = f_*\left(\frac{\partial}{\partial u}\right)$ and $X_v = f_*(\frac{\partial}{\partial v})$, are given by:

$$
\begin{aligned}
X_u &= x_uE_1 + y_uE_2 + \alpha E_3 \\
X_v &= x_vE_1 + y_vE_2 + \beta E_3
\end{aligned}
$$

where we set

$$
\begin{aligned}
\alpha &= \tfrac{y}{2}x_u - \tfrac{x}{2}y_u + z_u \\
\beta &= \tfrac{y}{2}x_v - \tfrac{x}{2}y_v + z_v
\end{aligned}
\tag{3}
$$

Hence, it follows that

$$\langle X_u, X_u\rangle = \langle X_v, X_v\rangle = \lambda^2; \;\; \langle X_u, X_v\rangle = 0. \tag{4}$$

A unit normal vector field of the immersion f is given by:

$$\eta = \frac{1}{\lambda^2}\left[(\beta y_u - \alpha y_v)E_1 + (\alpha x_v - \beta x_u)E_2 + (x_u y_v - x_v y_u)E_3\right]$$

where we will denote the coordinates of η, in the basis $\{E_i\}$, by (a, b, c). Is easily verify that

$$\lambda^2(a^2 + b^2) = (\alpha^2 + \beta^2). \tag{5}$$

The tension field of the immersion f is given by

$$\tau(f) = \lambda^{-2}(\nabla_{X_u} X_u + \nabla_{X_v} X_v) = 2\mathbf{H}$$

where $\mathbf{H}$ is the mean curvature vector. If f is minimal, we have :

$$\begin{array}{rcl} \Delta x & = & -(\alpha y_u + \beta y_v) \\ \Delta y & = & \alpha x_u + \beta x_v \\ \frac{y}{2}\Delta x - \frac{x}{2}\Delta y + \Delta z & = & 0. \end{array} \tag{6}$$

Observe that the third equation of the system of (6) is equivalent to:

$$\alpha_u + \beta_v = 0. \tag{7}$$

Finally we recall that in the Euclidean case the differential of the Gauss map is just the second fundamental form for surfaces in $\mathbb{R}^3$. This fact can be generalized for hypersurfaces in any Lie Group. The following theorem, see [8], establishes a relationship between the Gauss map and the extrinsic geometry of S.

Let S be an orientable hypersurfaces of a Lie group G with Lie algebra g and η an unitary normal vector field of S. We define

$$\gamma : S \to S^{n-1} = \{w \in g \colon \|w\| = 1\}$$

with $\gamma(x) = dL_{p^{-1}}(\eta(x))$ for $x \in S$. We call this function, the Gauss map of S.

Theorem 2.1. *Let S be an orientable hypersurfaces of a Lie group. Then*

$$dL_p \circ d\gamma_p(v) = -(A_\eta(v) + \alpha_{\bar{\eta}}(v)), \quad v \in T_pS,$$

where A_η is the Weingarten operator, $\alpha_{\bar{\eta}}(v) = \nabla_v \bar{\eta}$ and $\bar{\eta}$ is a left invariant vector field such that $\eta(p) = \bar{\eta}(p)$.

As a consequence, $dL_p^{-1}(T_pS)$ is a Lie subalgebra of codimension 1 if the Gauss map is constant. In the case of a surface in the Heisenberg group, we have

$$[X_u, X_v] = (x_u y_v - x_v y_u)E_3 = \lambda^2 c E_3$$

So, if the Gauss map is constant, c must be equal to 0 and the surface must be vertical.

3 The Beltrami Equation

In this section we shall prove that the Gauss map of any minimal immersion in $\mathcal{H}_3$ satisfies a Beltrami equation.

With respect to the basis $\{X_u, X_v\}$, the operators A_η and $\alpha_{\bar{\eta}}$ are represented by matrices (h_{ij}) and $(\hat{h}_{ij})$, respectively. If we set $(\gamma_{ij}) = (h_{ij} + \hat{h}_{ij})$, by Theorem (2.1), we have

$$dL_p \circ d\gamma_p = -(\gamma_{ij}).$$

In particular, the coefficients of $\alpha_{\bar{\eta}}$ is given by:

$$\alpha_{\bar{\eta}} = (\hat{h}_{ij}) = \frac{1}{\lambda^2}\begin{pmatrix} -\alpha\beta & \frac{\lambda^2}{2} - \beta^2 \\ \alpha^2 - \frac{\lambda^2}{2} & \alpha\beta \end{pmatrix} \tag{8}$$

From $dL_p \circ d\gamma_p\,(X_u) = -\gamma_{11}X_u - \gamma_{21}X_v$, and using (2.1), we can compute the derivatives of a, b and c with respect to u:

$$\begin{aligned} a_u &= -\gamma_{11}x_u - \gamma_{21}x_v \\ b_u &= -\gamma_{11}y_u - \gamma_{21}y_v \\ c_u &= -\gamma_{11}\alpha - \gamma_{21}\beta \end{aligned} \tag{9}$$

In a similar fashion, we compute the derivatives of a, b and c with respect to v:

$$\begin{aligned} a_v &= -\gamma_{12}x_u - \gamma_{22}x_v \\ b_v &= -\gamma_{12}y_u - \gamma_{22}y_v \\ c_v &= -\gamma_{12}\alpha - \gamma_{22}\beta. \end{aligned} \tag{10}$$

Let S^2 be the unit sphere in $h_3 \simeq T_0\mathcal{H}_3$ and we consider S^2 as the standard Riemann sphere: We cover S^2 by the union of the two open sets U_i, where we set $U_1 = S^2 - \{northpole\}$ and $U_2 - \{southpole\}$ and let ψ_i be the coordinate functions on U_i. Then

$$\begin{aligned} \psi_1(x) &= \tfrac{x_1+ix_2}{1-x_3}, \quad \text{if } x = (x_1, x_2, x_3) \in U_1 \\ \psi_2(x) &= \tfrac{x_1-ix_2}{1+x_3}, \quad \text{if } x = (x_1, x_2, x_3) \in U_2. \end{aligned}$$

We consider, for any surface in $\mathcal{H}_3$, the following sequence of mappings:

$$S \xrightarrow{f} f(S) \xrightarrow{Gauss\ Map} S^2 \xrightarrow{\psi_i} w - plane.$$

with $i = 1, 2$. The composed map, which will be also called the Gauss map of M,

$$\psi_i : S \to \text{Riemann sphere}$$

is considered as a complex mapping of a 1-dimensional complex manifold M to the Riemann sphere. We omit the subscript i in ψ_i and write simply ψ if there is no confusion or if the statement under consideration holds for both ψ_i. We shall now compute the derivatives of the Gauss map ψ.

Proposition 3.1. *Under the above notations, we have*

$$\frac{\partial\psi_1}{\partial\bar{z}} = -(\frac{H + \Theta i}{2})(1 + \psi_1\bar{\psi_1})^2(\frac{\partial x}{\partial\bar{z}} + i\frac{\partial y}{\partial\bar{z}})$$

where $\Theta = \frac{(\hat{h}_{12} - \hat{h}_{21})}{2}$ *and* H *the mean curvature of* S.

Proof: We know that

$$\psi_1(z) = \frac{a(z) + ib(z)}{1 - c(z)}.$$

Since we put $\frac{\partial \psi_1}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial \psi_1}{\partial u} + i\frac{\partial \psi_1}{\partial v})$ we have, using (9) and (4),

$$\frac{\partial \psi_1}{\partial u} = \frac{1}{(1-c)^2}\left\{[(-x_u + y_v) - i(x_v + y_u)]\,\gamma_{11} - [(x_v + y_u) + i(y_v - x_u)]\,\gamma_{21}\right\}.$$

By similar way, using (10), we have that

$$i\frac{\partial \psi_1}{\partial v} = \frac{1}{(1-c)^2}\left\{[(x_v + y_u) + i(y_v - x_u)]\,\gamma_{12} + [(-x_u + y_v) - i(x_v + y_u)]\,\gamma_{22}\right\}.$$

Observe that $(-x_u + y_v) - i(x_v + y_u) = -\,i\,[(x_v + y_u) + i(y_v - x_u)]$. Then, substituting in the above two equation and summing up we obtain that

$$\frac{\partial \psi_1}{\partial \bar{z}} = \frac{1}{2(1-c)^2}\left\{(\gamma_{12} - \gamma_{21}) - i(\gamma_{11} + \gamma_{22})\,[(x_v + y_u) + i(y_v - x_u)]\right\}.$$

Notice that $\gamma_{11} + \gamma_{22} = 2H + (\hat{h}_{11} + \hat{h}_{22}) = 2H$ because $\hat{h}_{11} + \hat{h}_{22}$ is the trace of the matrix $\alpha_{\bar{\eta}}$, which in $\mathcal{H}_3$ is equal to zero, see (8). And $\gamma_{12} - \gamma_{21} = \hat{h}_{12} - \hat{h}_{21}$, because the matrix of the Weingarten operator is symmetric in the basis $\{X_u, X_v\}$. Then,

$$\frac{\partial \psi_1}{\partial \bar{z}} = \frac{(\hat{h}_{12} - \hat{h}_{21}) - 2Hi}{2(1-c)^2}\left\{(x_v + y_u) + i(y_v - x_u)\right\}$$

Now $(x_v + y_u) + i(y_v - x_u) = -2i(\frac{\partial x}{\partial \bar{z}} + i\frac{\partial y}{\partial \bar{z}})$ and using the fact that

$$(1 + \psi_1\bar{\psi}_1)(1 - c) = 2,$$

follow the result.

$\square$

Remark 3.1. *Note that Θ is a function of ψ_1. In fact, by using (8) and (5), we can see that $\Theta = \frac{1}{2}(c^2)$. Then we have*

$$\Theta = \frac{1}{2}(\frac{\psi_1\bar{\psi}_1 - 1}{1 + \psi_1\bar{\psi}_1})^2 \tag{11}$$

Notice that $\Theta = 0$ is equivalent to $\psi_1\bar{\psi}_1 = 1$, so in the minimal case $(H = 0)$ we have, from Proposition 3.1,

$$\begin{aligned} \frac{\partial\psi_1}{\partial\bar{z}} &= 0 \\ \frac{\partial\psi_1}{\partial z} &= 0 \end{aligned}$$

That is, the Gauss map ψ_1, is constant. So the minimal surface is a vertical plane, see [5].

We define here the following functions:

$$\Phi = \tfrac{1}{2}(h_{11} - h_{22}) - ih_{12}; \quad \hat{\Phi} = \tfrac{1}{2}(\hat{h}_{11} - \hat{h}_{22}) - \tfrac{i}{2}(\hat{h}_{12} + \hat{h}_{21})$$

Proposition 3.2. *Under the above notations, we have*

$$\frac{\partial\psi_1}{\partial z} = \frac{(\Phi + \hat{\Phi})}{2}(1 + \psi_1\bar{\psi}_1)^2(\frac{\partial x}{\partial\bar{z}} + i\frac{\partial y}{\partial\bar{z}})$$

Proof: Since $\frac{\partial\psi_1}{\partial z} = \frac{1}{2}(\frac{\partial\psi_1}{\partial u} - i\frac{\partial\psi_1}{\partial v})$, we can prove the Proposition 3.2 in the same way as Proposition 3.1.

□

By the same argument or using the relation $\psi_1\psi_2 = 1$, we can also prove the following

Proposition 3.3. *The complex derivatives of the Gauss map ψ_2 are given*

$$\begin{aligned} \frac{\partial\psi_2}{\partial\bar{z}} &= -(\tfrac{H+\Theta i}{2})(1 + \psi_1\bar{\psi}_1)^2(\tfrac{\partial x}{\partial\bar{z}} - i\tfrac{\partial y}{\partial\bar{z}}) \\ \frac{\partial\psi_2}{\partial z} &= \tfrac{(\Phi+\hat{\Phi})}{2}(1 + \psi_1\bar{\psi}_1)^2(\tfrac{\partial x}{\partial\bar{z}} - i\tfrac{\partial y}{\partial\bar{z}}) \end{aligned} \tag{12}$$

We can calculate the norms of these complex vectors

Corollary 3.1. *Let ψ be the Gauss map of an arbitrary minimal surface in $\mathcal{H}_3$. Then we have*

$$\begin{aligned} \left|\frac{\partial\psi}{\partial\bar{z}}\right| &= \tfrac{\lambda}{2}(1 + \psi\bar{\psi})\,|\Theta i + H| \\ \left|\frac{\partial\psi}{\partial z}\right| &= \tfrac{\lambda}{2}(1 + \psi\bar{\psi})\left|\Phi + \hat{\Phi}\right|. \end{aligned} \tag{13}$$

Proof: For $\psi = \psi_1$. Firstly we prove that

$$4\left|\frac{\partial x}{\partial \bar{z}} + i\frac{\partial y}{\partial \bar{z}}\right|^2 = \lambda^2(1-c)^2.$$

In fact,

$$\frac{\partial x}{\partial \bar{z}} + i\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left[(x_u - y_v) + i(x_v + y_u)\right].$$

Then, using (4), we have

$$4\left|\frac{\partial x}{\partial \bar{z}} + i\frac{\partial y}{\partial \bar{z}}\right|^2 = 2\lambda^2(1-c) - (\alpha^2 + \beta^2).$$

From (5) and using $a^2 + b^2 + c^2 = 1$, follows the result. For $\psi = \psi_2$, we can prove the proposition in the same way.

□

Now we prove that the Gauss map of an arbitrary immersed surface in $\mathcal{H}_3$ must satisfy a first order differential equation which is a natural extension of the Cauchy-Riemman equation.

Theorem 3.1. *The Gauss map ψ of a surface in $\mathcal{H}_3$ satisfies a Beltrami equation:*

$$(\Phi + \hat{\Phi})\frac{\partial \psi}{\partial \bar{z}} = -(H + \Theta i)\frac{\partial \psi}{\partial z} \tag{14}$$

Proof: By Propositions 3.1 and 3.2 we obtain $(\Phi+\hat{\Phi})\frac{\partial \psi_1}{\partial \bar{z}} = -(H+i\Theta)\frac{\partial \psi_1}{\partial z}$ in U_1. On $U_1 \cap U_2$ we have also $(\Phi + \hat{\Phi})\frac{\partial \psi_2}{\partial \bar{z}} = -(H + i\Theta)\frac{\partial \psi_2}{\partial z}$ by virtue of $\psi_1\psi_2 = 1$. By the continuity we have the same formula on U_2.

□

4 The Weierstrass Formula

In this section we shall give a Weierstrass formula for minimal surfaces in $\mathcal{H}_3$. We consider surfaces that their Gauss map nowhere holomorphic, that is $\Theta \neq 0$. Since $\psi_1\psi_2 = 1$, we have

$$\begin{aligned} 1+\psi_1\bar{\psi}_1 &= \bar{\psi}_1(\bar{\psi}_2+\psi_1) \\ 1+\psi_2\bar{\psi}_2 &= \psi_2(\bar{\psi}_2+\psi_1) \\ \psi_1\frac{\partial\psi_2}{\partial\bar{z}}+\psi_2\frac{\partial\psi_1}{\partial\bar{z}} &= 0. \end{aligned}$$

This, together with Propositions (3.1) and (3.2), yields the following equation,

$$(\Theta i)\psi_1(\psi_2)^2(\bar{\psi}_2+\psi_1)^2(\frac{\partial x}{\partial\bar{z}}-i\frac{\partial y}{\partial\bar{z}})+(\Theta i)\psi_2(\bar{\psi}_1)^2(\bar{\psi}_2+\psi_1)^2(\frac{\partial x}{\partial\bar{z}}+i\frac{\partial y}{\partial\bar{z}})=0.$$

Since $(\bar{\psi}_2+\psi_1)^2 \neq 0$ and $\Theta \neq 0$, we have

$$\left[\frac{\partial x}{\partial\bar{z}}-i\frac{\partial y}{\partial\bar{z}}+(\bar{\psi}_1)^2(\frac{\partial x}{\partial\bar{z}}+i\frac{\partial y}{\partial\bar{z}})\right]=0. \tag{15}$$

Proposition 4.1. *Let $f : M \longrightarrow \mathcal{H}_3$ be a minimal immersion of M into $\mathcal{H}_3$ and $\psi : M \longrightarrow S^2$ be the Gauss map of M into S^2 considered as the Riemann sphere. Then we have, on U_1,*

$$\begin{aligned} \frac{\partial x}{\partial\bar{z}} &= \frac{2i(1-\bar{\psi}_1^2)}{(\psi_1\bar{\psi}_1-1)^2}\frac{\partial\psi_1}{\partial\bar{z}} \\ \frac{\partial y}{\partial\bar{z}} &= \frac{2(1+\bar{\psi}_1^2)}{(\psi_1\bar{\psi}_1-1)^2}\frac{\partial\psi_1}{\partial\bar{z}} \\ \frac{\partial\xi}{\partial\bar{z}} &= \frac{-4\bar{\psi}_1}{(\psi_1\bar{\psi}_1-1)^2}\frac{\partial\psi_1}{\partial\bar{z}} \end{aligned} \tag{16}$$

where ξ is such that $\xi_u = -\beta$ and $\xi_v = \alpha$.

On U_2 we have similar equations

$$\begin{aligned} \frac{\partial x}{\partial\bar{z}} &= \frac{2i(1-\bar{\psi}_2^2)}{(\psi_2\bar{\psi}_2-1)^2}\frac{\partial\psi_2}{\partial\bar{z}} \\ \frac{\partial y}{\partial\bar{z}} &= \frac{-2(1+\bar{\psi}_2^2)}{(\psi_2\bar{\psi}_2-1)^2}\frac{\partial\psi_2}{\partial\bar{z}} \\ \frac{\partial\xi}{\partial\bar{z}} &= \frac{4\bar{\psi}_2}{(\psi_2\bar{\psi}_2-1)^2}\frac{\partial\psi_2}{\partial\bar{z}} \end{aligned} \tag{17}$$

Proof: From (15) we have

$$(1+\bar{\psi}_1^2)\frac{\partial x}{\partial\bar{z}} = i(1-\bar{\psi}_1^2)\frac{\partial y}{\partial\bar{z}}. \tag{18}$$

By virtue of Proposition 3.1 and equation (18), we have

$$(1+\bar{\psi}_1^2)\left[\frac{-2}{\Theta(1+\psi_1\bar{\psi}_1)^2}\frac{\partial\psi_1}{\partial\bar{z}}-i\frac{\partial y}{\partial\bar{z}}\right]=i(1-\bar{\psi}_1^2)\frac{\partial y}{\partial\bar{z}}$$

whence we obtain

$$\Theta\frac{\partial y}{\partial\bar{z}}=\frac{i(1+\bar{\psi}_1^2)}{(1+\psi_1\bar{\psi}_1)^2}\frac{\partial\psi_1}{\partial\bar{z}},$$

using (11) follows the second formula of (16). By the similar way we have also the first formula of (16). The last equality of (16) follows from the next formula:

$$\frac{1}{2}(\alpha-i\beta)(\frac{\partial x}{\partial\bar{z}}+i\frac{\partial y}{\partial\bar{z}})=\lambda^2\frac{\psi_1}{(1+\psi_1\bar{\psi}_1)^2}. \tag{19}$$

We shall prove at first this equation. By mean the definition of $\psi_1$we get

$$\begin{aligned}\tfrac{1}{2}(\alpha-i\beta)(\tfrac{\partial x}{\partial z}+i\tfrac{\partial y}{\partial z})-\lambda^2\tfrac{\psi_1}{(1+\psi_1\bar{\psi}_1)^2}=\\ \tfrac{1}{4}\left\{(\alpha-i\beta)(x_u+ix_v+iy_u-y_v)-\lambda^2(1-c)(a+ib)\right\}.\end{aligned} \tag{20}$$

The real part of the above formula is equal to

$$\begin{aligned}\tfrac{1}{4}\left\{\alpha(x_u-y_v)+\beta(x_v+y_u)-\lambda^2(1-c)a\right\}=\\ \tfrac{1}{4}\left\{\alpha x_u-\alpha y_v+\beta x_v+\beta y_u-(1-\tfrac{1}{\lambda^2}(x_uy_v-x_vy_u))(\beta y_u-\alpha y_v)\right\}=\\ \tfrac{1}{4\lambda^2}\left\{\alpha x_u\lambda^2+\beta x_v\lambda^2+(x_uy_v\beta y_u-x_uy_v^2\alpha-x_vy_u^2\beta+x_vy_u\alpha y_v)\right\}=0.\end{aligned}$$

The last equal held using the relations in (4). By the similar way we can see that the imaginary part of (20) is also zero. This prove the formula.

Since $\frac{\partial\psi_1}{\partial\bar{z}}\neq 0$, we have

$$\frac{1}{2}(\alpha+i\beta)\left|\frac{\partial x}{\partial\bar{z}}+i\frac{\partial y}{\partial\bar{z}}\right|^2=\frac{\lambda^2\bar{\psi}_1(\frac{\partial x}{\partial\bar{z}}+i\frac{\partial y}{\partial\bar{z}})}{(1+\psi_1\bar{\psi}_1)^2}.$$

Using (19), we obtain

$$\frac{\Theta}{2}(\alpha+i\beta)=\frac{-2\bar{\psi}_1}{(1+\psi_1\bar{\psi}_1)^2}\frac{\partial\psi_1}{\partial\bar{z}}.$$

But we know, see remark (7), $\frac{\partial\alpha}{\partial u}+\frac{\partial\beta}{\partial v}=0$; then, there exist a differential function ξ such that $\xi_u=-\beta$ and $\xi_v=\alpha$. Hence,

$$\frac{\Theta}{2}(\xi_u+i\xi_v)=\frac{-2i\bar{\psi}_1}{(1+\psi_1\bar{\psi}_1)^2}\frac{\partial\psi_1}{\partial\bar{z}}.$$

(17) can be proved using (16) and the relation $\psi_1\psi_2=1$. This concludes the proof.

□

5 Integrability Condition

We shall show in this section that the Gauss map of a minimal immersion in $\mathcal{H}_3$ satisfies a second order differential equation which help us to find a integrability condition for the system (16) and (17).

Theorem 5.1. *Let $f: M\to\mathcal{H}_3$ be an isometric immersion of M into $\mathcal{H}_3$. Then f is minimal iff the Gauss map ψ satisfy*

$$\frac{\partial^2\psi}{\partial z\partial\bar{z}}-\frac{2\bar{\psi}}{\psi\bar{\psi}-1}\frac{\partial\psi}{\partial\bar{z}}\frac{\partial\psi}{\partial z}=0 \tag{21}$$

Proof: We shall prove (21) for $\psi=\psi_1$. Firstly we derive the system (16) with respect a z. From the first equation of this system we have

$$\frac{\partial^2 x}{\partial z\partial\bar{z}}=\frac{2i(1-\bar{\psi}^2)}{(\psi\bar{\psi}-1)^2}\left[\frac{\partial^2\psi}{\partial z\partial\bar{z}}-\frac{2\bar{\psi}}{(\psi\bar{\psi}-1)}\frac{\partial\psi}{\partial\bar{z}}\frac{\partial\psi}{\partial z}\right]-\frac{4i(\psi-\bar{\psi})}{(\psi\bar{\psi}-1)^3}\frac{\partial\psi}{\partial\bar{z}}\frac{\partial\bar{\psi}}{\partial z}.$$

Notice that the second term of the right side is real and equal to $\frac{\lambda^2 bc}{4}=\frac{-(\alpha y_u+\beta y_v)}{4}$ and by the first equation of (6) it is equal to $\dfrac{\partial^2 x}{\partial z\partial\bar{z}}$. By the similar way, from the second equation of (16), we have

$$\frac{\partial^2 y}{\partial z\partial\bar{z}}=\frac{2i(1+\bar{\psi}^2)}{(\psi\bar{\psi}-1)^2}\left[\frac{\partial^2\psi}{\partial z\partial\bar{z}}-\frac{2\bar{\psi}}{(\psi\bar{\psi}-1)}\frac{\partial\psi}{\partial\bar{z}}\frac{\partial\psi}{\partial z}\right]-\frac{4(\psi+\bar{\psi})}{(\psi\bar{\psi}-1)^3}\frac{\partial\psi}{\partial\bar{z}}\frac{\partial\bar{\psi}}{\partial z}.$$

In this case the second term of the right side is real and equal to $\frac{-\lambda^2 ac}{4}=\frac{\alpha x_u+\beta x_v}{4}$ and it is equal to $\dfrac{\partial^2 y}{\partial z\partial\bar{z}}$. Finally, from the third equation of (16) we have:

$$\frac{\partial^2\xi}{\partial z\partial\bar z} = \frac{\bar\psi}{(\psi\bar\psi-1)^2}\left[\frac{\partial^2\psi}{\partial z\partial\bar z} - \frac{2\bar\psi}{(\psi\bar\psi-1)}\frac{\partial\psi}{\partial\bar z}\frac{\partial\psi}{\partial z}\right] + \frac{4}{(\psi\bar\psi-1)^3}\frac{\partial\psi}{\partial\bar z}\frac{\partial\bar\psi}{\partial z}.$$

The second term of the right side is equal to $\frac{\lambda^2 c}{4}$ and, by using the fact that $\xi_u = -\beta$ and $\xi_v = \alpha$, this is equal to $\dfrac{\partial^2\xi}{\partial z\partial\bar z}$. Then the Gauss map ψ satisfy (21) iff f is a minimal immersion of M into $\mathcal{H}_3$. We also get the same equation for $\psi = \psi_2$ by the same argument

□

Furthermore we can see that equation (21) is just the complete integrability condition for the system (16) and (17). Therefore we have the following

Theorem 5.2. *Let M be a simply connected 2-dimensional smooth Riemannian manifold and $\psi_1 : M \to S^2$ be a nowhere holomorphic smooth mapping which satisfies the differential equation (21). Then ψ_1 is a Gauss map of the following minimal surface of $\mathcal{H}_3$:*

$$\begin{array}{rcl} x &=& Re\int\limits_0^z \frac{-2i(1-\psi_1^2)}{(\psi_1\bar\psi_1-1)^2}\overline{\frac{\partial\psi_1}{\partial\bar z}}dz + c_1 \\ y &=& Re\int\limits_0^z \frac{2(1+\psi_1^2)}{(\psi_1\bar\psi_1-1)^2}\overline{\frac{\partial\psi_1}{\partial\bar z}}dz + c_2 \\ \xi &=& Re\int\limits_0^z \frac{-4\psi_1}{(\psi_1\bar\psi_1-1)^2}\overline{\frac{\partial\psi_1}{\partial\bar z}}dz + c_3 \end{array} \qquad (22)$$

Proof: This follows from Theorems (4.1) and (5.1).

□

So we have found a correspondence from the set of solution of the differential equation (21) to the set of minimal surfaces of $\mathcal{H}_3$. In [1], Akutagawa and Nishikawa showed a similar representation with the same integrability condition, (21) for spacelike CMC surface in $\mathbb{L}^3$. Now we shall study the uniqueness of the correspondence. In [2], B. Daniel also obtained a similar result, that is, he constructs minimal surfaces in $\mathcal{H}_3$ starting with a harmonic map into the hyperbolic disk.

Theorem 5.3. *Let $\psi(z)$ (resp. $\tilde{\psi}(z)$) be a smooth mapping satisfying (21) on a simply connected 2-dimensional manifold M. We define a minimal immersion $X(z)$ (resp. $\tilde{X}(z)$) by the above theorem. Then the two condition are equivalent:*

1. *There exist a holomorphic mapping $w = f(z)$ with $f'(z) \neq 0$ on M and an isometry τ in $\mathcal{H}_3$ such that $\tilde{X}(f(z)) = \tau \circ X(z)$, $z \in M$.*
2. *There exist a holomorphic mapping $w = f(z)$ with $f'(z) \neq 0$ on M such that $\tilde{\psi}(f(z)) = \psi(z)$, $z \in M$.*

Proof: We can repeat the proof of Theorem 5 of [7].

□

6 The Hopf differential

Let us point out some remarks about harmonic maps into the hyperbolic plane $\mathbb{H}^2$. Let M and N two simply connected Riemannian surfaces. If $z = x + iy$ and $w = u + iv$ are local conformal parameters on M and N respectively, and the metric tensor of N is given by

$$\omega = \rho^2 dw d\bar{w}$$

Then $\psi : M \to N$ is harmonic if and only if the tension field $\tau(\psi) = 0$, i.e.

$$\tau(\psi) = \psi_{z\bar{z}} + \frac{2\rho_w}{\rho}\psi_z\psi_{\bar{z}} = 0$$

On the other hand, the complexified first fundamental form of ψ is

$$\psi^*(\omega) = Qdz^2 + \mu dz d\bar{z} + \bar{Q}d\bar{z}^2$$

where,

$$\begin{aligned} Q &= \rho^2\psi_z\bar{\psi}_z &&= \tfrac{\rho^2}{4}(|\psi_u|^2 - |\psi_v|^2 - 2i\langle\psi_u, \psi_v\rangle) \\ \mu &= \rho^2(\psi_z\bar{\psi}_{\bar{z}} + \psi_{\bar{z}}\bar{\psi}_z) &&= \tfrac{\rho^2}{2}(|\psi_u|^2 + |\psi_v|^2) \end{aligned}$$

We calculate

$$Q_{\bar{z}} = \rho^2(\bar{\psi}_z \tau(\psi) + \psi_z \overline{\tau(\psi)})$$

Thus, if ψ is harmonic,

$$Q_{\bar{z}} = 0$$

Since $\psi^*(\omega)$ is a Riemannian metric, we get $\mu^2 - 4|Q|^2 \geq 0$, and equality holds exactly at the singular points of ψ. But

$$\mu^2 - 4|Q|^2 = |\psi_u|^2 |\psi_v|^2 - \langle \psi_u, \psi_v \rangle^2 \geq 0$$

This mean, if p is a singular point, ψ_x is parallel to ψ_y, or $\det(d\psi_p) = 0$.

When N is the Poincaré disk, that is,

$$\mathbb{H}^2 = \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

endowed with the metric,

$$\frac{4dzd\bar{z}}{(1 - |z|^2)^2}$$

we have that any mapping $\psi : M \to \mathbb{H}^2$ is harmonic if

$$\frac{\partial^2 \psi}{\partial z \partial \bar{z}} - \frac{2\bar{\psi}}{\psi\bar{\psi} - 1} \frac{\partial \psi}{\partial \bar{z}} \frac{\partial \psi}{\partial z} = 0.$$

Compare with (21).

Proposition 6.1. *Let $\psi : M \to \mathbb{H}^2$ be a harmonic map , nowhere anti-holomorphic and* $\det(d\psi) = 0$ *on an open subset of M. Then $\psi(M)$ lies in a geodesic of* $\mathbb{H}^2$.

Proof: By the above remark, this imply that Q is holomorphic and $\mu^2 - 4|Q|^2 = 0$. It follows that if $Q = 0$ imply $\mu = 0$ and ψ is constant. If Q is not identically zero, Q only has isolated zeros. Take $Q(z_0) \neq 0$ and follow the proof in [3]

$\square$

7 Examples

In this section we classify the minimal surfaces by the rank of its Gauss map.

1. We have seen that the vertical plane is the unique minimal surface that the Gauss map is constant.

2. Assume that $f : M \to \mathcal{H}_3$ is a minimal inmersion and its Gauss map, ψ_1, is nowhere antiholomorphic and has rank equal 1, $\det(d\psi_1) = 0$. This type of surface we will call minimal surface of rank one. From (6.1), $\psi_1(M)$ is a geodesic in $\mathbb{H}^2$. So, define

$$\psi_1(\zeta) = \frac{e^{\zeta+\bar{\zeta}} - 1}{e^{\zeta+\bar{\zeta}} + 1}$$

which is a geodesic in $\mathbb{H}^2$. The minimal immersion f defined by (16) is

$$\begin{array}{rcl} x(\zeta) & = & i(\zeta - \bar{\zeta}) \\ y(\zeta) & = & \frac{1}{2}(e^{\zeta+\bar{\zeta}} - e^{-(\zeta+\bar{\zeta})}) \\ \xi(\zeta) & = & -\frac{1}{2}(e^{\zeta+\bar{\zeta}} + e^{-(\zeta+\bar{\zeta})}) \end{array}$$

From the last equality and using (3) we obtain

$$z(\zeta) = \frac{i}{2}(\zeta - \bar{\zeta})(e^{\zeta+\bar{\zeta}} - e^{-(\zeta+\bar{\zeta})})$$

This surface is an entire graph over $\mathbb{R}^2$, given by

$$z = xy$$

In general, if T is an isometry of $\mathbb{H}^2$, then $T \circ \psi_1$ is harmonic and is a Gauss map of a minimal surface of rank one, so we classify all this type of surface.In [5], we also obtained a similar result.

3. Let $\psi_1 : \mathbb{D} \to \mathbb{C}$ be $\psi_1(\zeta) = \bar{\zeta}$. Then ψ_1 satisfies (21) and the minimal immersion defined by (22) is written as

$$X(\zeta) = (\frac{2i(\zeta - \bar{\zeta})}{\zeta\bar{\zeta} - 1}, \frac{2(\zeta + \bar{\zeta})}{\zeta\bar{\zeta} - 1}, cte.), \ \zeta \in \mathbb{D}.$$

This is the horizontal plane. Observe that $\det(d\psi_1) < 0$ and the Hopf differential of ψ_1 is $Q = 0$.

For more examples see [2] and [4].

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Pontificia Universidad Católica del Perú
Departamento de Ciências Matemática
Av. Universitária 1801
San Miguel
Lima, Perú
E-mail: cfiguer@puc.edu.pe