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GROMOLL-MEYER ACTIONS AND TRIALITY

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Abstract

We use the triality presentation of the Spin groups $Spin(n)$, $n = 4, \ldots, 8$, to provide extensions of the Gromoll-Meyer and canonical actions on $Sp(2) \cong Spin(5)$.

1 Introduction

The triality presentation of $Spin(8)$ provides an extremely useful algebra to work in low-dimensional Lie groups and their associated bundles and quotient maps; see for example [Ri, CR1, CR2], where among other results Hopf maps are studied under this viewpoint.

In this note we use triality to extend the Gromoll-Meyer non-negative curvature model of the Milnor (2,-1) exotic sphere. Using this technique the action extends to many actions and spaces; to give a detailed description of all actions and quotients involved would make this paper rather unwieldy. Our intention is to remark how the triality presentation allows the actual computation and explicit identification in some of these examples.

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Essentially, we shall see that in many cases the spaces involved have product structures $S^7 \times X$; these product structures are expicitly realized with triality triples. In the "exotic" case, preliminary computation with invariants suggests that they also have product structures $\Sigma^7 \times X$, where Σ^7 is an exotic sphere. The non-cancellation phenomenon implies that $\Sigma^7 \times X$ is diffeomorphic to the standard case $S^7 \times X$ for many cases of manifolds X ([Wa]). Thus these quotients probably do not produce new examples of manifolds of non-negative curvature but instead a testbed for studying non-cancellation phenomena in explicit terms.

2 Preliminaries

A fundamental construction in the geometry of non-negative curvature is the *Gromoll-Meyer action*, ([GM]) which is the following free S^3 action on the group $Sp(2)$ of quaternionic 2×2 matrices A satisfying $A^*A = AA^* = \mathbb{I}$ (the reader should beware that we are expressing the Gromoll-Meyer action as a *right* action instead of as a left action):

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \bigstar q = \begin{pmatrix} \bar{q} & 0 \\ 0 & \bar{q} \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \bar{q}aq & \bar{q}c \\ \bar{q}bq & \bar{q}d \end{pmatrix}.$$

The quotient Σ^7_{GM} is diffeomorphic to the Milnor exotic sphere $\Sigma^7_{2,-1}$ ([Mi]), and O'Neill's theorem [ON] implies that the biinvariant metric of $Sp(2)$ descends to a metric of non-negative curvature on Σ^7_{GM}.

A crucial aspect of the geometry and topology of the Gromoll-Meyer action is the existence of a "canonical" action of S^3 on $Sp(2)$:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \bullet q = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} a & cq \\ b & dq \end{pmatrix}.$$

The quotient of $Sp(2)$ by this action is diffeomorphic to the standard sphere S^7; in fact the projection to the quotient is realized as the projection onto the first column $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$. Let us remark that the metric subduced in S^7 by the bi-invariant metric on $Sp(2)$ is *not* the standard round metric on the sphere ([Du]).

Therefore we have the following "cross diagram":

$$\begin{tikzcd} & S^3 \arrow[d, "\star"'] & \\ S^3 \arrow[r, "\bullet"] & Sp(2) \arrow[r] \arrow[d] & S^7 \\ & \Sigma^7 & \end{tikzcd}$$

The study of the geometry and topology of these two actions, one with a "standard" quotient and the other with a "exotic" quotient, produces far reaching consequences, exploited in [Du, DMR, ADPR, DP].

It has now been realized that this "cross diagram" situation can be studied in more general contexts. In [DPR], we study the geometry and topology of these diagrams associated to arbitrary S^3-principal bundles over S^7, indexed by $k \mod 12$ (and E_1 is the classical Gromoll-Meyer action on $Sp(2)$):

$$\begin{tikzcd} & S^3 \arrow[d, "\star"'] & \\ S^3 \arrow[r, "\bullet"] & E_k \arrow[r] \arrow[d] & S^7\,, \\ & \Sigma^7_k & \end{tikzcd}$$

which provides geometric Gromoll-Meyer type models for *all* 7-dimensional exotic spheres Σ^7_k, (indexed by $k \mod 28$), each one of them in infinitely many ways.

In this note we use the triality presentations of the low-dimensional Spin groups to provide extensions of both the Gromoll-Meyer and canonical actions in another direction. Recall that the group $Sp(2)$ is isomorphic to the universal cover $Spin(5)$ of the special orthogonal group $SO(5)$. We have the chain of inclusions $Spin(4) \subset Spin(5) \subset Spin(6) \subset Spin(7) \subset Spin(8)$; the triality presentation gives “cross digrams” of the form

$$
\begin{tikzcd}
 & S^3 \arrow[d, "\star"'] & \\
S^3 \arrow[r, "\bullet"] & Spin(n) \arrow[r] \arrow[d] & C^\ell , \\
 & M^\ell &
\end{tikzcd}
$$

for $n = 4, 5, 6, 7, 8$, $\ell = \frac{1}{2}n(n-1)-3$ indexes the dimension of the quotients C^ℓ and M^ℓ, and the classical Gromoll-Meyer action corresponds to $n = 5$ above; actually (see section 4) we write the free $Spin(4)$-actions that contain the classical ones as subactions. Since in the biinvariant metric subgroups are totally geodesic, note that the Milnor (2,-1) sphere in its Gromoll-Meyer model is included in a totally geodesic way in the quotients M^ℓ. This, added to the non-cancellation $S^7 \times S^k \cong \Sigma^7 \times S^k$, leads to totally geodesic embeddings of Σ^7 in products $S^7 \times S^k$; however the metric on $S^7 \times S^k$ will have no relationship with the product metric of two round spheres.

A careful study of the topological and geometric invariants of these (in principle exotic) manifolds will be considered in a forthcoming paper.

3 Triality

Here we just provide the basic notation for the triality presentation of $Spin(n)$; for details, the reader can take [Ri] or [CR2] as a starting point. We denote by $\mathbb{H}$, $\mathbb{O}$ the algebra of quaternions and the Cayley algebra of octonions, respectively.

We describe the Cayley algebra of octonions as the algebra structure on pairs of quaternions given by by

$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac - \bar{d}b \\ da + d\bar{c} \end{pmatrix}$$

Consider the the set $\mathcal{S}$ of triples $(A,B,C) \in SO(8)\times SO(8)\times SO(8)$ such that for all $\xi,\eta \in \mathbb{O}$ in the Cayley algebra , $A(\xi\cdot\eta) = B(\xi)\cdot C(\eta)$, where the dot represents Cayley multiplication. Given $A \in SO(8)$, this condition determines B and C modulo a sign, also, this set has a group structure given by componentwise multiplication. With these two comments in mind, it is simple to show that $\mathcal{S}$ is the (universal) double cover $Spin(8) \to SO(8)$, with projection given by $(A,B,C) \mapsto A$. Actually, any element of the triple determines the other two modulo sign, and we also have valid projections $(A,B,C) \mapsto B$, $(A,B,C) \mapsto C$. Denoting the usual orthonormal basis of $\mathbb{O}$ by $\{1 = \mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_7\}$, we have

$$Spin(n) = \{(A,B,C) \in Spin(n+1) \mid A(\mathbf{e}_{8-n}) = \mathbf{e}_{8-n}\}$$

for $n = 1,\ldots 7$, projecting to the canonical $SO(n) \subset SO(8)$. A useful characterization of $Spin(7)$ is given by

$$Spin(7) = \{(A,B,C) \in Spin(8)\ :\ C = \widetilde{B}\}\,,$$

where $\widetilde{B}$ is given by $\widetilde{B}(x) = \overline{B(\bar{x})}$ and the bar denotes Cayley conjugation. Of course in all the others subgroups $Spin(n) \subset Spin(7)$ we can then write $(A,B,\widetilde{B})$ instead of (A,B,C).

The group G_2 of automorphisms of the Cayley algebra can also be seen as the subgroup given by the diagonal in $Spin(8)$: $G_2 = \{(A,A,A) \in Spin(8)\}$.

4 Generalized actions

First let us remark that both of these actions are actually subactions of $S^3 \times S^3 \cong Spin(4)$ actions on $Sp(2)$:

$$\begin{aligned}
\begin{pmatrix} a & c \\ b & d \end{pmatrix} \bigstar (p,q) &= \begin{pmatrix} \bar{p} & 0 \\ 0 & \bar{p} \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \bar{p}aq & \bar{p}c \\ \bar{p}bq & \bar{p}d \end{pmatrix}, \\
\begin{pmatrix} a & c \\ b & d \end{pmatrix} \bullet (p,q) &= \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} ap & cq \\ bp & dq \end{pmatrix},
\end{aligned}$$

with quotients diffeomorphic to S^4 in both cases (see section 5); the Gromoll-Meyer action is given by the diagonal subaction $A\bigstar(q,q)$ and the canonical one by $A \bullet (1,q)$.

Given a quaternion x, we denote by $l_x, r_x : \mathbb{H} \to \mathbb{H}$ the left and right multiplication operators, $l_x(h) = xh$, $r_x(h) = hx$, and the 4×4 identity matrix by 1. Consider now the following monomorphisms $S_1, S_2 : G \to Spin(8)$:

$$\begin{aligned}
S_1(x) &= \left(\begin{pmatrix} 1 & 0 \\ 0 & r_{\bar{x}} \end{pmatrix}, \begin{pmatrix} r_{\bar{x}} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} l_x & 0 \\ 0 & 1 \end{pmatrix} \right), \\
S_2(x) &= \left(\begin{pmatrix} 1 & 0 \\ 0 & l_x \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & l_x \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & l_x \end{pmatrix} \right), \\
G(x) &= \left(\begin{pmatrix} 1 & 0 \\ 0 & l_x \circ r_{\bar{x}} \end{pmatrix}, \begin{pmatrix} r_{\bar{x}} & 0 \\ 0 & l_x \end{pmatrix}, \begin{pmatrix} l_x & 0 \\ 0 & l_x \end{pmatrix} \right).
\end{aligned}$$

Note that

- For all $x \in S^3$, $S_1(x)$ and $S_2(x)$ commute.
- The image of $S_2(x)$ is contained in $G_2 \subset Spin(8)$.
- The inclusions $S_1(x)$ and $S_2(x)$ are conjugate to each other (this will be used in section 5.3).

Consider now the following $S^3 \times S^3$ actions on $Spin(8)$:

$$\begin{aligned}
(A,B,C) \bullet_T (p,q) &= (A,B,C)S_1(p)S_2(q) \\
&= \left(A \begin{pmatrix} 1 & 0 \\ 0 & l_q \circ r_{\bar{p}} \end{pmatrix}, B \begin{pmatrix} r_{\bar{p}} & 0 \\ 0 & l_q \end{pmatrix}, C \begin{pmatrix} l_p & 0 \\ 0 & l_q \end{pmatrix} \right)
\end{aligned}$$

$$(A,B,C)\bigstar_T(p,q) = G(\bar{p})(A,B,C)S_2(q)$$

$$= \left(\begin{pmatrix} 1 & 0 \\ 0 & l_{\bar{p}} \circ r_p \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ 0 & l_q \end{pmatrix}, \begin{pmatrix} r_p & 0 \\ 0 & l_{\bar{p}} \end{pmatrix} B \begin{pmatrix} 1 & 0 \\ 0 & l_q \end{pmatrix}, \begin{pmatrix} l_{\bar{p}} & 0 \\ 0 & l_{\bar{p}} \end{pmatrix} C \begin{pmatrix} 1 & 0 \\ 0 & l_q \end{pmatrix} \right).$$

Note that the matrices multiplying the first component A of (A, B, C) have the 4×4 identity matrix on the upper-left corner. It thus follows that

Proposition 1. *These $Spin(4)$-actions leave the subsets $Spin(n)$, $n = 4, 5, 6, 7$, invariant as sets.*

Now we reach the main results of this section:

Theorem 1. *The $\bullet_T$ and $\bigstar_T$ actions on $Spin(5)$ described above are conjugate to the $\bullet$ and $\bigstar$-actions on $Sp(2)$ given at the beggining of the section.*

Proof: The map $p_1 : Spin(d) \to SO(8)$ is just the projection to the quotient $p : Spin(d) \to SO(d)$ and the inclusion $SO(d) \hookrightarrow SO(8)$; for this proof the other projections p_2, p_3 given by $p_2(A, B, C) = B, p_3(A, B, C) = C$ are much more interesting:

Recall that $Spin(5) = (A, B, \widetilde{B})$ such that $A \in SO(5)$, and that since $Sp(2)$ acts in $\mathbb{H} \times \mathbb{H}$ it is naturally a subgroup of $SO(8)$. We have that if $(A, B, \widetilde{B}) \in Spin(5)$, then $B = p_2(A, B, \tilde{B})$ is a subgroup of $SO(8)$ that is (linearly) conjugate to the "canonical" $Sp(2)$ given by the standard quaternionic structure on $\mathbb{H} \times \mathbb{H}$ (see proposition 7 of [CR2]). Since $C = \widetilde{B}$ which is B conjugated by the Cayley bar involution, the same is true for $C = p_3(A, B, C)$.

Then looking at the C-component of the triality triples gives the desired result. To be more specific, a 2×2 matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ in $Sp(2)$ is determined by its first and second columns $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$, that is, the images $A \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $A \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Both $Spin(4)$-actions preserve the columns; the standard $\bullet$-action acts by

$$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} ap \\ bp \end{pmatrix}, \quad \begin{pmatrix} c \\ d \end{pmatrix} \to \begin{pmatrix} cq \\ dq \end{pmatrix},$$

and the exotic ★-action by

$$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} \bar{p}aq \\ \bar{p}bq \end{pmatrix}, \quad \begin{pmatrix} c \\ d \end{pmatrix} \to \begin{pmatrix} \bar{p}c \\ \bar{p}d \end{pmatrix}.$$

The corresponding columns in the triality presentation are $C(1)$ and $C(e_4)$. Then keeping track of what happens to $C(1)$ and $C(e_4)$ when C is transformed by the actions we see that it is the same as the actions expressed in the canonical $Sp(2)$.

□

Next we show that these extensions of the actions are free:

Theorem 2. *For all $d = 4, 5, 6, 7, 8$, both the • and ★ actions (as $S^3 \times S^3$ actions) are free on $Spin(d)$.*

Proof: For the •-action this is trivial, being the right action of $S^3 \times S^3$ as a subgroup. For the ★-action, this requires the analysis of the action on a component of the triality triple (A, B, C): suppose that $(A, B, C)\bigstar(p, q) = (A, B, C)$. Choosing, for example, the C-component, we get

$$\begin{pmatrix} l_{\bar{p}} & 0 \\ 0 & l_{\bar{p}} \end{pmatrix} C \begin{pmatrix} 1 & 0 \\ 0 & l_q \end{pmatrix} = C$$

Dividing C into 4×4 blocks $C = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$, we have

$$\begin{pmatrix} l_{\bar{p}} & 0 \\ 0 & l_{\bar{p}} \end{pmatrix} C \begin{pmatrix} 1 & 0 \\ 0 & l_q \end{pmatrix} = \begin{pmatrix} l_{\bar{p}} \circ E & l_{\bar{p}} \circ F \circ l_q \\ l_{\bar{p}} \circ G & l_{\bar{p}} \circ H \circ l_q \end{pmatrix}.$$

Since the column $(E, G)^t$ has rank 4 it follows that $l_{\bar{p}}$ is the identity and $p = 1$. Then since the column $(F, H)^t$ has rank 4, $q = 1$ by the same reasoning.

□

Remark. The anonymous referee observed that this construction actually applies to $Spin(n)$ for all $n \geq 4$, not only for $n = 4, 5, 6, 7, 8$. Indeed, construct a block diagonal matrices $A_p, B_q \in SO(n)$ by

$$A_p = \begin{pmatrix} 1 & 0 \\ 0 & l_{\bar{p}} \circ r_p \end{pmatrix} \qquad B_q = \begin{pmatrix} 1 & 0 \\ 0 & l_q \end{pmatrix} ,$$

where the 1 in the upper left positions is the $(n-4) \times (n-4)$ identity matrix. We then get a free action of $SO(3) \times S^3$ in $SO(n)$ which is $(p,q) \cdot X = A_p X B_q$. For $n = 4, 5, 6, 7, 8$, this action is a $\mathbb{Z}_2$ reduction of the actions given above (just look at the A-component). These actions can be lifted to $Spin(n)$; however, for $n > 8$ we lose the triality interpretation.

5 Quotients

In this section we study the quotients of some of these actions. There are several actions in question:

- The $Spin(4)$ action (p,q) $\bullet_T$;
- The restricted actions $(1,q)$, $(p,1)$ and (q,q) of $\bullet_T$;
- The $Spin(4)$ action (p,q) of $\bigstar_T$;
- The restricted actions $(1,q)$, $(p,1)$ and (q,q) of $\bigstar_T$;

Note that in both cases ($\bullet$ and $\bigstar$) the $Spin(4) \cong S^3_p \times S^3_q$ action is given by the commuting actions $(p,1)$ and $(1,q)$, and therefore we have principal bundles

$$S^3_p \cdots Spin(d)/S^3_q \to Spin(d)/Spin(4) \,,$$
$$S^3_q \cdots Spin(d)/S^3_p \to Spin(d)/Spin(4) \,,$$

where we have denoted by S^3_p and S^3_q the typical principal fiber of these actions. Sometimes in the sequel we abuse notation and denote by S^3_p (resp. S^3_q) the actions themselves.

Let us first give a short description of the $Spin(5) \cong Sp(2)$ case; this description is given with respect to the $Sp(2)$ presentation of the actions given at the beginning of section 3.

For the canonical $\bullet$-action this produces Hopf bundles $S^3 \cdots S^7 \to S^4$ in either case: the $(p,1)$-action kills the first column and then the $(1,q)$-action is just the canonical Hopf action on S^7. In the exotic $\bigstar$-action, we find an asymmetry: $S^3_p \cdots Spin(5)/S^3_q \to S^4$ is again the Hopf bundle $S^3 \cdots S^7 \to S^4$ (since the quotient of the $(1,q)$-action is realized by projecting on the second column), whereas in the bundle $S^3_q \cdots Spin(5)/S^3_p \to S^4$ the total space $Spin(5)/S^3_p$ is $Sp(2)$ divided by the diagonal action. It is well-known that this space is diffeomorphic to T_1S^4, the unit tangent bundle of S^4.

5.1 The canonical $\bullet_T$-action

For the $\bullet_T$ action, we have the following table of quotients; $V_{n,k}$ denotes the sets of all orthonormal k-frames in $\mathbb{R}^n$ (in particular, $V_{2,n}$ is the unit tangent bundle of S^{n-1}). The table shows the symmetry between the $(p,1)$-action and the $(1,q)$-action, since they essentially are the same action operating on the first or last columns.

$\bullet_T$	(p,q)	$(1,q)$	$(p,1)$
Spin(4)	point	S^3	S^3
Spin(5)	S^4	S^7	S^7
Spin(6)	$V_{6,2}$	$S^7 \times S^5$	$S^7 \times S^5$
Spin(7)	$V_{7,3}$	$V_{7,2} \times S^7$	$V_{7,2} \times S^7$
Spin(8)	$V_{8,4}$	$V_{7,2} \times S^7 \times S^7$	$V_{7,2} \times S^7 \times S^7$

Let us give a couple of explicit realizations of the quotients; these realizations are easily expressed in terms of the triality presentation.

The most important part is not the quotients in itselves, but the fact that they have non-trivial product structures; as mentioned in the introduction, this reinforces the hope of having product structures and non-cancellation phenomena in an explicit way in this fashion.

Theorem 3. *The quotient $Spin(6)/S^3_q$ is explicitly diffeomorphic to $S^7 \times S^5$, and also to the complex 2-frames on S^7, the homogeneous space $SU(4)/SU(2)$.*

Proof: Let $\Phi : Spin(6) \to S^7 \times S^5$ be given by $\Phi(A,B,C) = (B(1), A(e_2))$. A priori the image falls in $S^7 \times S^7$; however, since we are in $Spin(6)$, $A(1) = 1$ and $A(e_1) = e_1$. Thus $A(e_2)$ lies in the space orthogonal to $1, e_1$ and therefore the image falls in $S^7 \times S^5$. Note that both $B(1)$ and $A(e_2)$ are invariant under the $(1,q)$-action and therefore Φ descends to a map $\phi : Spin(6)/S^3_p \to S^7 \times S^5$. We construct an inverse, noting that the class of $[(A, B, \widetilde{B})]$ under the S^3_q action is determined by the first four columns $B(1), \ldots B(e_3)$ of its B-component. Then, using the basic property $A(\xi \cdot \eta) = B(\xi)\widetilde{B}(\eta)$ (and $\cdot$ denotes the Cayley multiplication), we have

- $B(1) = \alpha$.
- $B(e_2) = A(e_2) \cdot B(1) = \gamma \cdot \alpha$.
- $B(e_3) = A(e_1) \cdot B(e_2) = e_1 \cdot (\gamma \cdot \alpha)$.
- $B(e_1) = A(e_2) \cdot B(e_3) = \gamma \cdot (e_1 \cdot (\gamma \cdot \alpha))$.

Therefore, the map ϕ is smoothly invertible using the formulas above.

Consider now the map $F : Spin(6) \to S^7 \times S^7$, $F(A, B, \widetilde{B}) = (B(1), B(e_2))$. Again these coordinates are invariant under the S^3_q-action and F descends to $f : Spin(6)/S^3_q \to S^7 \times S^7$. However, in $Spin(6)$ we have that $B(e_1) = A(e_1) \cdot B(1) = e_1 \cdot B(e_1)$. Since $B \in SO(8)$, $B(e_2)$ is orthogonal to the span of $B(1), B(e_1)$ which by the previous remark is the same as the span of $B(1), e_1 \cdot B(1)$. Taking left multiplication by e_1 to define the complex structure on $\mathbb{R}^8$ we get that $B(e_2)$ is *complex* orthogonal to $B(1)$ and thus the pair $(B(1), B(e_2))$ lives in the complex unit tangent bundle of S^7, $SU(4)/SU(2)$: The inverse of this map is again given by providing the first four columns of B; given (ϵ, ζ) a complex 2-frame,

- $B(1) = \epsilon$.
- $B(e_1) = e_1 \cdot \epsilon$.
- $B(e_2) = \zeta$.
- $B(e_3) = e_1 \cdot \zeta$.

□

Let us study the $(p, 1)$-action on $Spin(6)$:

Theorem 4. *The quotient $Spin(6)/S^3_p$ is explicitly diffeomorphic to $S^7 \times S^5$.*

Proof: Now the class of $(A, B, C) = (A, B, \widetilde{B})$ is determined by the last four columns of B. Then in a similar vein to the previous theorem we have $\Phi : Spin(6)/S^3_p$ given by

$$\Phi([A, B, \tilde{B}]) = (B(e_4), A(e_2)),$$

with inverse given by

- $B(e_4) = \delta$.
- $B(e_5) = A(e_1) \cdot B(e_4) = e_1 \cdot \delta$.
- $B(e_6) = A(e_2) \cdot B(e_4) = \gamma \cdot \delta$.
- $B(e_7) = A(e_2) \cdot B(e_5) = \gamma \cdot (e_1 \cdot \delta)$.

□

Note that by triality, these maps can be expressed in terms of B, e.g. $\Phi[(A,B,\widetilde{B})] = (B(e_4), -B(e_4)\cdot\overline{B(e_6)})$.

With these examples, the astute reader will have noticed that the quotients of the different $\bullet$-actions are characterized by choosing the adequate columns from (A,B,C). For example, the full $Spin(4)$-action on $Spin(6)$ is the standard homogeneous space $Spin(6)/Spin(4) = V_{6,2}$, the unit tangent bundle of S^5, with identification given by $(A(e_1), A(e_2))$. Let us put together all these actions in the diagrams

$$\begin{tikzcd}
 & & Spin(4) \arrow[d, dotted] & & \\
S^3_q \arrow[dr, dotted] & & Spin(6) \arrow[dl] \arrow[dd] \arrow[dr] & & S^3_p \arrow[dl, dotted] \\
 & S^7\times S^5 \arrow[dr] & & S^7\times S^5 \arrow[dl] & \\
 & Spin(5)/Spin(4) \arrow[r, dotted] & V_{6,2} = Spin(6)/Spin(4) \arrow[r] & S^5 = Spin(6)/Spin(5) &
\end{tikzcd}$$

According to the notation we adopted, i.e., $B(1) = \alpha \in S^7$, $B(e_4) = \delta \in S^7$ and $A(e_2) = \gamma \in S^5_{1,e_1}$, we have

$$\begin{tikzcd}[column sep=small]
 & B \arrow[dl, mapsto] \arrow[dd, mapsto] \arrow[dr, mapsto] & \\
(B(1),A(e_2)) = (\alpha,\gamma) \arrow[dr, mapsto] & & (\delta,\gamma) = (B(e_4),A(e_2)) \arrow[dl, mapsto] \\
 & (A(e_2),A(e_3)) \arrow[d, equal] & \\
 & (\gamma,(\gamma\cdot(e_1\cdot(\gamma\cdot\alpha)))\cdot(\bar{\alpha},\bar{\gamma}) \arrow[d, equal] & \\
 & (\gamma,\delta\cdot(\gamma\cdot(e_1\cdot\delta))) &
\end{tikzcd}$$

The long parentheses are a result of expressing $A(e_3)$ in terms of $(\alpha,\gamma) = (B(1),A(e_2))$, respectively in terms of $(\delta,\gamma) = (B(e_4),A(e_2))$ through Cayley products. Let us remark that the bundle $S^4 \dots V_{6,2} \to S^5$ is *not* trivial in spite of the fact that it has a section; in fact $V_{6,2}$ is not even homotopically equivalent to $S^5 \times S^4$ (see [Ja]).

The corresponding diagrams for $Spin(7)$ is given by

$$\begin{tikzcd}[column sep=small]
 & & Spin(4) \arrow[d, dotted] & & \\
S^3_q \arrow[dr, dotted] & & Spin(7) \arrow[dl] \arrow[dd] \arrow[dr] & & S^3_p \arrow[dl, dotted] \\
 & V_{8,3} \cong S^7 \times V_{7,2} \arrow[dr] & & V_{8,3} \cong S^7 \times V_{7,2} \arrow[dl] & \\
Spin(5)/Spin(4) \arrow[r, dotted] & & V_{7,3} = Spin(7)/Spin(4) \arrow[rr] & & V_{7,2}
\end{tikzcd}$$

Where we have the diffeomorphisms $Spin(7)/S^3_p \cong V_{8,3} \cong S^7 \times V_{7,2} \cong Spin(7)/S^3_q$, and recall that $S^7 \times V_{7,2}$ is diffeomorphic to $V_{8,3}$ via $(\alpha,(J,K)) \mapsto$

$(\alpha, \alpha \cdot J, \alpha \cdot K)$, with inverse $(\alpha, \beta, \epsilon) \mapsto (\alpha, (\bar{\alpha}, \alpha\epsilon))$. The maps are given by

$$\begin{tikzcd}
 & B \arrow[dl, mapsto] \arrow[dr, mapsto] \arrow[dd, mapsto] & \\
(B(1), B(e_1), B(e_2)) = (\alpha, \beta, \epsilon) \arrow[dr, mapsto] & & ((B(e_4), B(e_5), B(e_6)) = (\delta, \zeta, \eta) \arrow[dl, mapsto] \\
 & (A(e_1), A(e_2), A(e_3)) \arrow[d, equal] & \\
 & (-\alpha \cdot \bar{\beta}, -\alpha \cdot \bar{\epsilon}, -\beta) \arrow[d, equal] & \\
 & (-\delta \cdot \bar{\zeta}, -\delta \cdot \bar{\eta}, \zeta \cdot \bar{\eta}) &
\end{tikzcd}$$

and the last horizontal map is $(Ae_1, Ae_2, Ae_3) \mapsto (Ae_1, Ae_2)$.

For the $Spin(8)$ case, recall that $Spin(8)$ is a trivial principal $Spin(7)$ bundle over S^7 and therefore the associated $Spin(7)/S^3_p$, respectively $Spin(7)/S^3_q$ bundles are trivial as well and one has

$$\begin{tikzcd}
 & & Spin(4) \arrow[d, dotted] & & \\
S^3_q \arrow[dr, dotted] & & Spin(8) \arrow[dl] \arrow[dr] \arrow[dd] & & S^3_p \arrow[dl, dotted] \\
 & V_{8,3} \times S^7 \cong V_{7,2} \times S^7 \times S^7 \arrow[dr] & & V_{8,3} \times \cong V_{7,2} \times S^7 \times S^7 \arrow[dl] & \\
 & Spin(5)/Spin(4) \arrow[r, dotted] & V_{8,4} = Spin(8)/Spin(4) \arrow[r] & V_{8,3} = Spin(8)/Spin(5) &
\end{tikzcd}$$

Given the diffeomorphism

$$Spin(8) \ni (A, B, C) \mapsto \Big(L_{\overline{A(1)}} \cdot A, L_{\overline{A(1)}} \cdot R_{\overline{A(1)}} \cdot B, L_{A(1)} \cdot C); A(1)\Big) \in Spin(7) \times S^7 \,,$$

whose inverse is

$$Spin(7) \times S^7 \ni ((\Lambda, M, \tilde{M}), \alpha) \mapsto (L_\alpha \cdot \Lambda, L_\alpha \cdot R_\alpha \cdot M, L_{\bar{\alpha}} \cdot \tilde{M}) \in Spin(8)\,,$$

then $Spin(8)/Spin(4) \cong Spin(7)/Spin(4) \times S^7$ as $[(A,B,C)]_{Spin(4)} \mapsto ((\overline{A}(1) \cdot A(e_1), \overline{A}(1) \cdot A(e_2), \overline{A}(1) \cdot A(e_3)), A(1) \in V_{7,3} \times S^7$ and since $V_{7,3}$ fibers over $V_{7,2}$ with fiber S^4 and $V_{7,2} \times S^7$ is diffeomorphic to $V_{8,3}$ as we saw, we get back

$$\begin{tikzcd}
Spin(5)/Spin(4) \arrow[r, dotted] \arrow[d, equal] & Spin(8)/Spin(4) \arrow[r] \arrow[d, equal] & Spin(8)/Spin(5) \arrow[d, equal] \\
S^4 \arrow[r, dotted] & V_{7,3} \times S^7 \arrow[r] & V_{8,3}
\end{tikzcd}$$

5.2 Associated Bundles

Let us remark that the (p,q) $Spin(4)$-actions and the restricted diagonal S^3 actions are related by the passage from a principal bundle to associated bundles; indeed, if we let $S^3_p \times S^3_q$ act on S^3 via the canonical projection to $SO(4)$, i.e., $(p,q) \cdot s = ps\bar{q}$, then the quotient of the diagonal restrictions $\Delta = (q,q)$ for both the $\bullet$ and the $\bigstar$ actions is given by the associated

$$Spin(n)/\Delta \cong Spin(n) \times_{S^3_p \times S^3_q} S^3\,;$$

and we have a (non-principal) bundle

$$S^3 \cdots Spin(n)/\Delta \to Spin(n)/S^3_p \times S^3_q\,.$$

In the classical case, this produces the unit tangent bundle of the 4-sphere $S^3 \cdots V_{5,2} \to S^4$ (in the $\bullet$-action), and the classical Milnor fibration $S^3 \cdots \Sigma^7_{2,-1} \to S^4$ in the $\bigstar$-action ([GM]). In the next section we will study this diagonal action for the first of the extension, namely the generalized Gromoll-Meyer action on $Spin(6)$.

5.3 The exotic $\bigstar_T$ actions

In the "canonical" $\bullet$ actions, the maps $[(A,B,C)] \mapsto A(e_1)$ (or $A(e_2)$ or $A(e_3)$) defines a bundle with total space the respective quotient and base a sphere; in most cases, the product structure of the quotient arises from the triviality of these bundles.

What will happen in the exotic $\bigstar$-actions is that the corresponding bundles will *not* be trivial; however, in spite of the non-triviality of the bundle, the total space can be diffeomorphic to a product. But in these cases the "triviality" of the total space has to be detected by more powerful invariants. The most interesting action in here is the diagonal $\bigstar$-actions (q,q) on $Spin(n)$, since they all contain the Gromoll-Meyer sphere embedded in a totally geodesic fashion. In this section we will study just the first case, namely $M^{12} = Spin(6)/S^3_{GM}$, the quotient of the diagonal Gromoll-Meyer extension on $Spin(6)$. As we saw in 5.2, the manifold M^{12} is a non-principal S^3-bundle over a 9-dimensional manifold Q^9.

On the other hand, since the two separate actions S^3_p and S^3_q commute we also have a principal $S^3_p ... Spin(6)/S^3_q \to Q^9$ and saw that $Spin(6)/S^3_q$ is diffeomorphic to $S^7 \times S^5$. (recall that the inclusions $S_1(q)$ and $S_2(q)$ are conjugate so that the $(q,1)$-quotient is the same for the $\bigstar$ and $\bullet$-actions). Using the formula for this diffeomorphism and the explicit S^3_p action we get:

Theorem 5. *For $p \in S^3$, $(\alpha,\gamma) \in S^7 \times S^5$ the free action with quotient Q is $p * (\alpha,\gamma) = (\begin{pmatrix} ap \\ \bar{p}b \end{pmatrix}, (\xi, \bar{p}\eta p))$ where $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{H} \oplus \mathbb{H}$ and $(\xi,\eta) \in Im\mathbb{H} \oplus Im\mathbb{H}$ are both unit vectors.*

The above, in turn says that Q is the total space of the associated 5-Sphere bundle to the $\mathbb{R}^6$ vector bundle over S^4, obtained from the principal Hopf bundle $S^3 ... S^7 \to S^4$, the representation $S^3 \to SO(3)$ (universal covering) that induces canonically an $\mathbb{R}^3-$vector bundle and the Whitney sum with a trivial 3-dimensional bundle. I.e., $S^5 ... Q \to S^4$ is the unit sphere bundle of $\mathbb{R}^6 ... (S^7 \times_{S^3} \mathbb{R}^3) \oplus \epsilon_3 \to S^4$. In particular, Q is not trivial over S^4, since the Hopf bundle represents the generator of the stable group $\pi_4 BSO \cong \mathbb{Z}$.

Note that there are more bundles related to this action:

$$\begin{array}{ccccc} Spin(4)/S^3_q & \dashrightarrow & Spin(6)/S^3_q & \longrightarrow & Spin(6)/Spin(4) \\ \| & & \| & & \| \\ S^3 & \dashrightarrow & S^7 \times S^5 & \longrightarrow & V_{6,2} \end{array}$$

a non principal bundle, and

$$\begin{array}{ccccc} S^3_q & \cdots\rightarrow & Spin(6)/S^3_q & \rightarrow & Spin(6)/Spin(4) \\ \| & & \| & & \| \\ S^3 & \cdots\cdots\rightarrow & S^7 \times S^5 & \longrightarrow & Q \end{array}$$

a principal bundle.

The base spaces of these bundles also are total spaces: we know that $V_{6,2} = T_1 S^5$ is the total space of the non trivial $S^4 ... V_{6,2} \to S^5$ and Q is the total space of a non trivial bundle $S^5 ... Q \to S^4$.

The diffeomorphism types of the spaces involved may still be the same, while they are not bundle isomorphic. We intend to complete this study in a forthcoming paper and we close now with a few comments that seem to indicate that this phenomenon is indeed present, and the exoticity of the Gromoll-Meyer construction is lost when we pass to the bigger quotients: the manifold $M^{12} \cong Spin(6)/S^3_{GM}$ is diffeomorphic to $\Sigma^7_k \times S^5$ for any homotopy 7-sphere. In particular, M^{12} is diffeomorphic to $S^7 \times S^5$.

First note that the basic homotopy invariants of M^{12} coincide with the respective ones of $S^7 \times S^5$; the exact homotopy sequence shows that the homotopy groups coincide, and a Gysin sequence argument shows that the homology groups are the same. In fact lifting constructions as in [Du] could be used to provide explict homeomorphisms (*not* diffeomorphisms!) between these spaces and the standard ones. However, differential topological invariants show diffeomorphisms in a non-explicit way:

Fix an arbitrary Σ^7_k that may even be S^7. We know that it is parallelizable since $\pi_7 BSO(7) = 0$. Let $Z^{12} = \Sigma^7_k \times S^5$; using the fact that TS^5 has a section we conclude that Z^{12} is also parallelizable. Since Z^{12} has zero sixth homology it follows that its signature is zero.

Now the corresponding facts for M^{12} are also valid. From $S^3 \cdots M \to Q$ and $S^5 \cdots Q \to S^4$ we conclude that $S^3 \times S^5 ... M \to S^4$ is a (linear) bundle and the analogous argument as above can be used to conclude that M is also parallelizable with zero signature. Then, standard methods of Differential Topology[1] imply

[1]Due to the work of many people, in particular, W. Browder, C.T.C. Wall, S. Smale, S. Novikov, D. Sullivan and others.

that they are diffeomorphic.

The same kind of argument shows that $\Sigma^7_k \times S^3$, for any k has just one differentiable structure: that of the cartesian product $S^7 \times S^3$; this has already been observed in the non-cancellation context in [Ri2]. Here, however, one has more structure that could be used to realize this non-cancellation in by an explicit geometric diffeomorphism.

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