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SPACELIKE SURFACES IN $\mathbb{L}^4$ WITH PRESCRIBED GAUSS MAP AND NONZERO MEAN CURVATURE

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Dedicated to Francesco Mercuri & Renato Tribuzy on their sixtieth birthdays

Abstract

In this work we study the generalized Gauss map of spacelike surfaces in the Lorentz-Minkowski space $\mathbb{L}^n$, with emphasis to the case $n = 4$. We present necessary and sufficient conditions for a complex map to be a Gauss map of a spacelike surface in $\mathbb{L}^4$ and a representation formula of Kenmotsu type, and this gives a method to construct spacelike surfaces with prescribed nonvanishing scalar mean curvature and Gauss map. We also present the extension to L^4 of the complete integrability conditions for existence of surface in $\mathbb{R}^3$ and $\mathbb{L}^3$ of Kenmotsu and Akutagawa-Nishikawa.

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1 Introduction

Let $\{e_i : 1 \leq i \leq n\}$ be the canonical basis of the real vector space $\mathbb{R}^n$, $n \geq 2$, and define a symmetric bilinear form $\langle , \rangle$ of index 1 in $\mathbb{R}^n$ by $\langle u, v\rangle = u^1v^1 + \cdots + u^{n-1}v^{n-1} - u^nv^n$, where $u = \sum u^ie_i$ and $v = \sum v^ie_i$ are vectors of $\mathbb{L}^n$. The form $\langle , \rangle$, is the *Lorentz scalar product* and the n–dimensional *Lorentz–Minkowski* space $\mathbb{L}^n$ is the space $\mathbb{R}^n$ endowed with this form. A spacelike surface in $\mathbb{L}^n$, $n \geq 3$, is the image $S = X(M^2)$ of a connected, oriented abstract surface M^2 through a locally conformal immersion $X : M^2 \to \mathbb{L}^n$ such that the induced form $ds^2 = X^*\langle , \rangle$ is a Riemannian metric in M^2.

The classical Gauss map for surfaces in $\mathbb{R}^3$, was introduced by Gauss in his fundamental work on the theory of surfaces and he used it to define what today is known as *Gauss curvature.* For surfaces of higher codimension in Euclidian space, we have a natural generalization of the Gauss map, where the image space is the Grassmannian $G_{2,n}$ of the oriented 2-planes in $\mathbb{R}^n$, which may be identified with the complex quadric Q_{n-2} of the complex projective space $\mathbb{CP}^{n-1}$. D. Hoffman and R. Osserman in [6] studied in detail the properties of the *generalized Gauss map* of a surface S immersed in $\mathbb{R}^n$.

For connected, oriented spacelike surfaces S the n–dimensional Lorentz-Minkowski space, it is also natural to think in the generalized Gauss map. This concept was introduced by F. J. M. Estudillo and A. Romero [4] as follows: let $G^+_{2,n}$ be the set of all oriented spacelike 2–planes in $\mathbb{L}^n$. It is known that $G^+_{2,n}$ is an open subset of the classical Grassmannian of 2-planes in $\mathbb{L}^n$. Also, we may identify $G^+_{2,n}$ with the complex quadric $Q^{n-2}_1 := \{[z] \in \mathbb{CP}^{n-1}_1 : (z^1)^2 + \cdots + (z^{n-1})^2 - (z^n)^2 = 0\}$, where $\mathbb{CP}^{n-1}_1 := \{z \in \mathbb{C}^n \setminus \{0\} : \ll z, z \gg > 0\}/\mathbb{C}^*$ and $\ll z, w \gg = z^1\overline{w^1} + \ldots - z^n\overline{w^n}$ is the indefinite hermitian form in $\mathbb{C}^n$, see [3, 12]. Therefore, if we associate to each point $p \in M^2$, the tangent plane T_pM in $G^+_{2,n}$, we can define a map $G : M^2 \to Q^{n-2}_1$ which is called the *generalized Gauss map* of the spacelike surface $S = X(M^2)$ in $\mathbb{L}^n$.

Motivated by these results, in this work we firstly study what conditions a generalized Gauss map $G : M^2 \to Q_1^{n-2}$ of a spacelike surface in $\mathbb{L}^n$ must satisfy by the virtue of being such a map. This is done in section 2, where the main result is Theorem 2.2. The similar problem in the Euclidian space $\mathbb{R}^n$ was studied by D. Hoffman and R. Osserman in [7, 8]. In section 3 we study the generalized Gauss map of spacelike surfaces $S = X(M^2)$ in $\mathbb{L}^4$ and prove that if it is given locally by a function Φ, that is, $X_z = \mu\Phi$, with $\Phi = \big(1 + \mathbf{ab}, i(1 - \mathbf{ab}), \mathbf{a} - \mathbf{b}, \mathbf{a} + \mathbf{b}\big)$[2, 5], then $\overline{F_1}F_2 = F_1\overline{F_2}$ and $\mathfrak{Im}\{T_1 + T_2\} = 0$ see Theorem 3.3. This result is very important in the study of a special class of the spacelike surfaces in $\mathbb{L}^4$, the surfaces with degenerate Gauss map [15].

In section 4 we relate our work with the work of K. Kenmotsu, and K. Akutagawa and S. Nishikawa. In [9], Kenmotsu proved that the Gauss map and the mean curvature of an arbitrary surface in $\mathbb{R}^3$ satisfy a second order differential equation, and later he extended this result for surfaces in $\mathbb{R}^4$, see [10]. In [1] Akutagawa and Nishikawa, showed that a same type of equation is satisfied for arbitrary spacelike surfaces in $\mathbb{L}^3$. In Theorem 4.1, we show that the generalized Gauss map and the mean curvature vector of a spacelike surface in $\mathbb{L}^4$ satisfy a second order differential equation which generalizes the above mentioned equations. In Proposition 4.3, we introduce a complex representation formula of Kenmotsu's type for simply connected spacelike surfaces conformally immersed in $\mathbb{L}^4$ with nonzero mean curvature $h = \sqrt{|\langle H, H\rangle|}$; we emphasize that this is equivalent to say that the *mean curvature vector H is nonzero nor lightlike.*

Finally in section 5 we prove our main result Theorem 5.3, which gives sufficient conditions for a complex map $G : M^2 \to Q_1^2$ to be the generalized Gauss map of a conformal spacelike immersion X of M^2 onto S in $\mathbb{L}^4$ with nonzero mean curvature h.

We observe that besides the Weierstrass type representation for spacelike surfaces in $\mathbb{L}^4$ given in [2, 5], there exists another Weierstrass type representation that is discussed in [11].

2 The generalized Gauss map of spacelike surfaces in $\mathbb{L}^n$

The generalized Gauss map G of a spacelike surface S in $\mathbb{L}^n$, given by a conformal immersion $X : M^2 \to \mathbb{L}^n$, is locally defined by

$$G(z) = [X_{\overline{z}}], \tag{1}$$

where $z = u + iv$ is a conformal parameter for M^2. The local conformality of S is expressed by $\lambda^2 = \langle X_u, X_u \rangle = \langle X_v, X_v \rangle$, $\langle X_u, X_v \rangle = 0$. Thus

$$ds^2 = \lambda^2 |dz|^2, \quad \lambda^2 = 2 \ll X_z, X_z \gg . \tag{2}$$

Moreover,

$$\Delta_M X = 2H, \quad X_{z\overline{z}} = \frac{\lambda^2}{2} H, \tag{3}$$

where $\Delta_M X$ is the Laplacia-Beltrami operator on M^2 and H is the mean curvature vector field.

Now given a map of M^2 in the quadric Q_1^{n-2} of $\mathbb{CP}_1^{n-1}$, we can represent it locally in the form $[\Phi(z)]$, where $\Phi(z) = (\phi^1(z), \dots, \phi^n(z))$ satisfies

$$(\phi^1)^2 + (\phi^2)^2 + \dots + (\phi^{n-1})^2 - (\phi^n)^2 = 0. \tag{4}$$

This map describes the Gauss map of a spacelike surface S in $\mathbb{L}^n$, given by $X : M^2 \to \mathbb{L}^n$, if

$$X_z = \mu \Phi, \tag{5}$$

for some map $\mu : M^2 \to \mathbb{C}$. Note that S is regular where μ is nonvanishing. Therefore, from (2), (3) and (5) it follows that

$$\lambda^2 = 2 \ll X_z, X_z \gg = 2|\mu|^2 \ll \Phi, \Phi \gg, \quad \frac{\lambda^2}{2} H = (\mu\Phi)_{\overline{z}} = \mu_{\overline{z}} \Phi + \mu \Phi_{\overline{z}}.$$

Thus,

$$\overline{\mu} \ll \Phi, \Phi \gg H = \Phi (\log \mu)_{\overline{z}} + \Phi_{\overline{z}}, \tag{6}$$

whenever $\mu \neq 0$.

Lemma 2.1. *Let W be a vector of $\mathbb{C}_1^n$ in the form $W = A + iB$ where A and B are spacelike vectors of $\mathbb{L}^n$ satisfying $\langle A, A\rangle = \langle B, B\rangle$ and $\langle A, B\rangle = 0$. Let Π^2 be the spacelike 2–plane spanned by A and B. For any pair of vectors C and D of $\mathbb{L}^n$, let $Z = C + iD \in \mathbb{C}_1^n$ and define $Z^\top := C^\top + iD^\top$, where $C^\top$ and $D^\top$ are the projections of C and D on Π^2. Then,*

$$Z^\top = \frac{\ll Z, W \gg}{\ll W, W \gg} W + \frac{\ll Z, \overline{W} \gg}{\ll W, W \gg} \overline{W}, \tag{7}$$

where $\mathbb{C}_1^n := (\mathbb{C}^n, \ll, \gg)$, with $\ll z, w \gg := \sum_{k=1}^{n-1} z^k \overline{w^k} - z^n \overline{w^n}$, $z, w \in \mathbb{C}^n$.

Proof: Since

$$C^\top = \frac{\langle C, A\rangle}{\langle A, A\rangle} A + \frac{\langle C, B\rangle}{\langle B, B\rangle} B, \;\; D^\top = \frac{\langle D, A\rangle}{\langle A, A\rangle} A + \frac{\langle D, B\rangle}{\langle B, B\rangle} B$$

are the projections of C, D on Π^2 and

$$A = \frac{W + \overline{W}}{2}, \;\; B = \frac{W - \overline{W}}{2i}, \;\; C = \frac{Z + \overline{Z}}{2}, \;\; D = \frac{Z - \overline{Z}}{2i},$$

the result follows.

□

We can apply Lemma 2.1 for $W = \Phi$, in such a way that Π^2 is the tangent plane of a surface S, given by X such that $X_z = \mu\Phi$ and $Z = \Phi_{\overline{z}}$. Since $[\Phi] \in Q_1^{n-2}$, then $\ll \Phi_{\overline{z}}, \overline{\Phi} \gg = 0$. Hence,

$$(\Phi_{\overline{z}})^\top = \eta\Phi, \quad \eta := \frac{\ll \Phi_{\overline{z}}, \Phi \gg}{\ll \Phi, \Phi \gg}. \tag{8}$$

Denoting by $\mathbb{V}$ the component of $\Phi_{\overline{z}}$ orthogonal to the plane $\Pi^2 = T_pS$, that is, $\mathbb{V} := \Phi_{\overline{z}} - (\Phi_{\overline{z}})^\top$, we have

$$\mathbb{V} = \Phi_{\overline{z}} - \eta\Phi. \tag{9}$$

The mean curvature vector H is orthogonal to the tangent plane of S, thus from $\overline{\mu} \ll \Phi, \Phi \gg H = (\Phi_{\overline{z}} - \eta\Phi) + ((\log\mu)_{\overline{z}} + \eta)\Phi$, it follows that

$$(\log\mu)_{\overline{z}} + \eta = 0, \tag{10}$$

and by (9) we have

$$\mathbb{V} = \overline{\mu} \ll \Phi, \Phi \gg H. \tag{11}$$

The proofs of Theorem 2.2 and of Lemma 2.3 below are similar to the proofs given by Hoffman and Osserman to Theorem 2.3 and to Lemma 2.4 in the case $\mathbb{R}^n$ [7]. For this reason these proofs will be omitted here.

Theorem 2.2. *Let S be a spacelike surface in $\mathbb{L}^n$ locally given by a conformal map $X : \Omega \to \mathbb{L}^n$. Let Φ be the Gauss map of S, that is, $X_z = \mu\Phi$. Then, for every $z \in \Omega$, $\mathbb{V}$ is of the form*

$$\mathbb{V}(z) = e^{i\alpha(z)} R(z), \tag{12}$$

where $R(z)$ is a vector of $\mathbb{L}^n$. Furthermore, on the set $\{z \in \Omega : \mathbb{V}(z) \neq 0\}$, the function $\alpha : \Omega \to \mathbb{R}$ is uniquely defined modulo 2π, and satisfies

$$\alpha_{z\overline{z}} = \Im m(\eta_z). \tag{13}$$

Note that $\mathbb{V}$ and η given in (12) and (13), are expressed explicitly by (8) and (9) in terms of the local representation Φ of the Gauss map G. By Lemma 2.3 below, $\mathbb{V}$ and η are independent of the particular representation of G.

Lemma 2.3. *Given a map $\Phi : \Omega \to \mathbb{C}^n \setminus \{0\}$, set $\widehat{\Phi} = f\Phi$ where f is a smooth nonvanishing complex function. Let $\eta, \mathbb{V}$ be defined in terms of Φ as in* (8) *and* (9) *and the correspondents $\widehat{\eta}, \widehat{\mathbb{V}}$ in terms of $\widehat{\Phi}$. Then*

$$\widehat{\mathbb{V}} = f\mathbb{V}$$

and in the set where $\widehat{\mathbb{V}}$ and $\mathbb{V}$ are nonzero, the functions $\alpha, \widehat{\alpha}$ defined by (12) *satisfy*

$$\widehat{\alpha}_{\overline{z}z} - \Im m(\widehat{\eta}_z) = \alpha_{\overline{z}z} - \Im m(\eta_z).$$

3 The generalized Gauss map for spacelike surfaces in $\mathbb{L}^4$

In this section we shall obtain explicit conditions that the generalized Gauss map of a spacelike surface S in $\mathbb{L}^4$ must satisfy. Let S be a spacelike surface, immersed in $\mathbb{L}^4$ by $X : M^2 \to \mathbb{L}^4$ with generalized Gauss map $G : M^2 \to Q_1^2$, given locally by $G = [X_z]$ where $(U, z = u + iv)$ are local isothermal coordinates of M^2. It follows from (4) that we may express G by a pair of complex functions $\mathbf{a}(z) := \dfrac{-\phi^3 + \phi^4}{\phi^1 - i\phi^2}$, $\mathbf{b}(z) := \dfrac{\phi^3 + \phi^4}{\phi^1 - i\phi^2}\cdot$ Since $\lambda^2 = 4|\mu|^2|1-\mathbf{a}\overline{\mathbf{b}}|^2$, then $\mathbf{a}\overline{\mathbf{b}} \neq 1$. Hence, we can write $G(z) = [\Phi(z)]$ where

$$\Phi(z) = \Big(1 + \mathbf{a}(z)\mathbf{b}(z), i(1 - \mathbf{a}(z)\mathbf{b}(z)), \mathbf{a}(z) - \mathbf{b}(z), \mathbf{a}(z) + \mathbf{b}(z)\Big). \quad (14)$$

We now introduce certain auxiliary functions derived from the functions $\mathbf{a}(z)$ and $\mathbf{b}(z)$ describing the Gauss map, as follows:

$$\begin{aligned} F_1 = F_1(\mathbf{a},\mathbf{b}) &:= \frac{\mathbf{a}_{\overline{z}}}{1-\mathbf{a}\overline{\mathbf{b}}}, \ F_2 = F_2(\mathbf{a},\mathbf{b}) := \frac{\mathbf{b}_{\overline{z}}}{1-\overline{\mathbf{a}}\mathbf{b}}, \\ \widehat{F}_1 = \widehat{F}_1(\mathbf{a},\mathbf{b}) &:= \frac{\mathbf{a}_{z}}{1-\mathbf{a}\overline{\mathbf{b}}}, \ \widehat{F}_2 = \widehat{F}_2(\mathbf{a},\mathbf{b}) := \frac{\mathbf{b}_{z}}{1-\overline{\mathbf{a}}\mathbf{b}}, \end{aligned} \quad (15)$$

$$S_1(\mathbf{a},\mathbf{b}) := \frac{\mathbf{a}_{z\overline{z}}}{\mathbf{a}_{\overline{z}}} + 2\overline{\mathbf{b}}\cdot\frac{\mathbf{a}_z}{1-\mathbf{a}\overline{\mathbf{b}}}\ ,\ S_2(\mathbf{a},\mathbf{b}) := \frac{\mathbf{b}_{z\overline{z}}}{\mathbf{b}_{\overline{z}}} + 2\overline{\mathbf{a}}\cdot\frac{\mathbf{b}_z}{1-\overline{\mathbf{a}}\mathbf{b}}, \quad (16)$$

$$T_1(\mathbf{a},\mathbf{b}) := \left(\frac{\mathbf{a}_{z\overline{z}}}{\mathbf{a}_{\overline{z}}} + 2\overline{\mathbf{b}}\widehat{F}_1\right)_{\overline{z}},\ \ T_2(\mathbf{a},\mathbf{b}) := \left(\frac{\mathbf{b}_{z\overline{z}}}{\mathbf{b}_{\overline{z}}} + 2\overline{\mathbf{a}}\widehat{F}_2\right)_{\overline{z}}, \quad (17)$$

where T_1 and T_2 are defined on the set $\{z : \mathbf{a}_{\overline{z}} \neq 0, \mathbf{b}_{\overline{z}} \neq 0\}$.

Lemma 3.1. *Let* $R(w) = \dfrac{\alpha w + \beta}{\overline{\beta} w + \overline{\alpha}}$, $|\alpha|^2 - |\beta|^2 = 1$, *be a Möbius transformation and* Y *any of the auxiliary functions* $\overline{F}_1 F_2$, $F_1\overline{F}_2$, $\overline{\widehat{F}_1}\widehat{F}_2$, $\widehat{F}_1\overline{\widehat{F}_2}$, S_k *and* $T_k, k = 1, 2$. *Then* Y *is invariant by* $(R(\mathbf{a}), R(\mathbf{b}))$, *that is ,* $Y(R(\mathbf{a}), R(\mathbf{b})) = Y(\mathbf{a}, \mathbf{b})$.

Proof: We will indicate the proof only for the function S_1 and the proof for the other functions is analogous. By a straightforward calculation, we obtain that

$$R(\mathbf{a})_z = \frac{\mathbf{a}_z}{(\overline{\beta}\mathbf{a}+\overline{\alpha})^2}, \;\; R(\mathbf{a})_{\overline{z}} = \frac{\mathbf{a}_{\overline{z}}}{(\overline{\beta}\mathbf{a}+\overline{\alpha})^2}, \;\; R(\mathbf{a})_{z\overline{z}} = \frac{\mathbf{a}_{z\overline{z}}(\overline{\beta}\mathbf{a}+\overline{\alpha}) - 2\mathbf{a}_z\mathbf{a}_{\overline{z}}\overline{\beta}}{(\overline{\beta}\mathbf{a}+\overline{\alpha})^3}.$$

Since the same is true for $R(\mathbf{b})$, we obtain that

$$1 - R(\mathbf{a})\overline{R(\mathbf{b})} = \frac{1-\mathbf{a}\overline{\mathbf{b}}}{(\overline{\beta}\mathbf{a}+\overline{\alpha})(\beta\overline{\mathbf{b}}+\alpha)}.$$

Thus,

$$S_1(R(\mathbf{a}), R(\mathbf{b})) = \frac{R(\mathbf{a})_{z\overline{z}}}{R(\mathbf{a})_{\overline{z}}} + \frac{2\overline{R(\mathbf{b})}R(\mathbf{a})_z}{1-R(\mathbf{a})\overline{R(\mathbf{b})}},$$

$$\begin{aligned} S_1(R(\mathbf{a}), R(\mathbf{b})) &= \frac{\mathbf{a}_{z\overline{z}}}{\mathbf{a}_{\overline{z}}} - \frac{2\mathbf{a}_z\overline{\beta}}{(\overline{\beta}\mathbf{a}+\overline{\alpha})} + \frac{2(\overline{\alpha}\overline{\mathbf{b}}+\overline{\beta})}{(\overline{\beta}\mathbf{a}+\overline{\alpha})}\frac{\mathbf{a}_z}{1-\mathbf{a}\overline{\mathbf{b}}} \\ &= \frac{\mathbf{a}_{z\overline{z}}}{\mathbf{a}_{\overline{z}}} + \frac{2\overline{\mathbf{b}}\mathbf{a}_z}{1-\mathbf{a}\overline{\mathbf{b}}} = S_1(\mathbf{a}, \mathbf{b}). \end{aligned}$$

□

Clearly the functions F_k and T_k above are smooth whenever $w_k(w_1 = \mathbf{a}(z), w_2 = \mathbf{b}(z))$ are finite. Now if $\mathbf{a}(z) = \infty$ or $\mathbf{b}(z) = \infty$, we may apply a Möbius transformation as above in a way that the functions $\mathbf{a}(z)$ and $\mathbf{b}(z)$ stay both finite in a neighborhood of a point . This corresponds to a Lorentz transformation in the surface S of $\mathbb{L}^4$. In fact, consider the group $SU(1,1) = U(1,1) \cap SL(2,\mathbb{C})$, that is,

$$SU(1,1) = \left\{ \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix} : |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

Therefore, for $A = \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix}$ in $SU(1,1)$, we have that the matrix

$$\Lambda_A = \begin{bmatrix} \Re e(\alpha^2+\beta^2) & -\Im m(\alpha^2-\beta^2) & 0 & -2\Re e(\alpha\beta) \\ \Im m(\alpha^2+\beta^2) & \Re e(\alpha^2-\beta^2) & 0 & -2\Im m(\alpha\beta) \\ 0 & 0 & 1 & 0 \\ -2\Re e(\alpha\overline{\beta}) & -2\Im m(\alpha\overline{\beta}) & 0 & |\alpha|^2+|\beta|^2 \end{bmatrix}$$

is in $O^{++}(3,1)$, the group of Lorentz transformations of $\mathbb{L}^4$ which preserve space and time orientation.

We conclude that to each change of conformal coordinates in the spacelike surface S of $\mathbb{L}^4$, through a Möbius transformation $R(w)$ in such a way $\mathbf{a}(z)$ and $\mathbf{b}(z)$ are finite, corresponds a Lorentz transformation $\Lambda_A \in O^{++}(3,1)$ of $\mathbb{L}^4$ (for more details see [14]).

Lemma 3.2. *Let F_k and $\widehat{F}_k$, $k = 1, 2$, defined in (15). Then*

$$\mathfrak{Im}\Big\{\big(\overline{\mathbf{b}}F_1\big)_z + \big(\overline{\mathbf{a}}F_2\big)_z\Big\} = \mathfrak{Im}\Big\{\big(\overline{\mathbf{b}}\widehat{F}_1\big)_{\overline{z}} + \big(\overline{\mathbf{a}}\widehat{F}_2\big)_{\overline{z}}\Big\}.$$

Proof: By a direct calculation, we obtain that

$$\big(\overline{\mathbf{b}}F_1\big)_z - \big(\overline{\mathbf{b}}\widehat{F}_1\big)_{\overline{z}} = \frac{\mathbf{a}_{\overline{z}}\,\overline{\mathbf{b}_{\overline{z}}} - \mathbf{a}_z\,\overline{\mathbf{b}_z}}{(1-\mathbf{a}\overline{\mathbf{b}})^2},$$

since $\big(\overline{\mathbf{b}}\big)_z = \overline{\mathbf{b}_{\overline{z}}}$ and $\big(\overline{\mathbf{b}}\big)_{\overline{z}} = \overline{\mathbf{b}_z}$. Analogously,

$$\big(\overline{\mathbf{a}}F_2\big)_z - \big(\overline{\mathbf{a}}\widehat{F}_2\big)_{\overline{z}} = \frac{\overline{\mathbf{a}_{\overline{z}}}\,\mathbf{b}_{\overline{z}} - \overline{\mathbf{a}_z}\,\mathbf{b}_z}{(1-\overline{\mathbf{a}}\mathbf{b})^2}.$$

Therefore,

$$\big(\overline{\mathbf{a}}F_2\big)_z - \overline{\big(\overline{\mathbf{b}}F_1\big)_z} = \big(\overline{\mathbf{a}}\widehat{F}_2\big)_{\overline{z}} - \overline{\big(\overline{\mathbf{b}}\widehat{F}_1\big)_{\overline{z}}}. \tag{18}$$

On the other hand,

$$\begin{aligned} 2i\mathfrak{Im}\Big\{\big(\overline{\mathbf{b}}F_1\big)_z + \big(\overline{\mathbf{a}}F_2\big)_z\Big\} &= \Big(\big(\overline{\mathbf{a}}F_2\big)_z - \overline{\big(\overline{\mathbf{b}}F_1\big)_z}\Big) - \Big(\overline{\big(\overline{\mathbf{a}}F_2\big)_z} - \big(\overline{\mathbf{b}}F_1\big)_z\Big) \\ &\overset{(18)}{=} \Big(\big(\overline{\mathbf{a}}\widehat{F}_2\big)_{\overline{z}} - \overline{\big(\overline{\mathbf{b}}\widehat{F}_1\big)_{\overline{z}}}\Big) - \overline{\Big(\big(\overline{\mathbf{a}}\widehat{F}_2\big)_{\overline{z}} - \overline{\big(\overline{\mathbf{b}}\widehat{F}_1\big)_{\overline{z}}}\Big)} \\ &= \Big(\big(\overline{\mathbf{b}}\widehat{F}_1\big)_{\overline{z}} + \big(\overline{\mathbf{a}}\widehat{F}_2\big)_{\overline{z}}\Big) - \overline{\Big(\big(\overline{\mathbf{b}}\widehat{F}_1\big)_{\overline{z}} + \big(\overline{\mathbf{a}}\widehat{F}_2\big)_{\overline{z}}\Big)} \\ &= 2i\mathfrak{Im}\Big\{\big(\overline{\mathbf{b}}\widehat{F}_1\big)_{\overline{z}} + \big(\overline{\mathbf{a}}\widehat{F}_2\big)_{\overline{z}}\Big\}. \end{aligned}$$

This proves the lemma.

□

Now we present the necessary conditions for a map in Q_1^2 to be a generalized Gauss map of a spacelike surface in $\mathbb{L}^4$.

Theorem 3.3. *Let S be an oriented spacelike surface given by the immersion $X : M^2 \to \mathbb{L}^4$, with generalized Gauss map G locally given by* (14) *via the pair of functions $\mathbf{a}(z)$ and $\mathbf{b}(z)$, where z is a local conformal parameter on M^2. Then,*

$$\overline{F_1} F_2 = F_1 \overline{F_2} \ , \tag{19}$$

$$\mathfrak{Im}\{T_1 + T_2\} = 0 \quad whenever \quad \mathbf{a}_{\overline{z}} \neq 0, \ \mathbf{b}_{\overline{z}} \neq 0. \tag{20}$$

Proof: We apply Theorem 2.2 to show that (12) and (13) imply respectively (19) and (20). The first step is to express the functions $\eta(z)$ and $\mathbb{V}(z)$, defined in (8) and (9), in terms of the functions $\mathbf{a}(z), \mathbf{b}(z)$ and $F_k(z)$. From (14) we have

$$\Phi_{\overline{z}}(z) = \mathbf{a}_{\overline{z}}\big(\mathbf{b}, -i\mathbf{b}, 1, 1\big) + \mathbf{b}_{\overline{z}}\big(\mathbf{a}, -i\mathbf{a}, -1, 1\big), \tag{21}$$

$$\ll \Phi, \Phi \gg = 2|1 - \mathbf{a}\overline{\mathbf{b}}|^2 > 0, \tag{22}$$

because $\mathbf{a}\overline{\mathbf{b}} \neq 1$. It follows that

$$\ll \Phi_{\overline{z}}, \Phi \gg = 2\overline{\mathbf{b}}\mathbf{a}_{\overline{z}}(\overline{\mathbf{a}}\mathbf{b} - 1) + 2\overline{\mathbf{a}}\mathbf{b}_{\overline{z}}(\overline{\mathbf{b}} - 1).$$

Hence,

$$\eta(z) = -\frac{\overline{\mathbf{b}}\mathbf{a}_{\overline{z}}}{1 - \mathbf{a}\overline{\mathbf{b}}} - \frac{\overline{\mathbf{a}}\mathbf{b}_{\overline{z}}}{1 - \overline{\mathbf{a}}\mathbf{b}} \quad \Leftrightarrow \quad \eta(z) = -(\overline{\mathbf{b}}F_1 + \overline{\mathbf{a}}F_2). \tag{23}$$

Now, substituting (14), (21) and (23) in $\mathbb{V}(z) = \Phi_{\overline{z}} - \eta\Phi$, yields

$$\begin{aligned} \mathbb{V}(z) &= \frac{\mathbf{a}_{\overline{z}}}{1 - \mathbf{a}\overline{\mathbf{b}}}\big(2\mathfrak{Re}(\mathbf{b}), 2\mathfrak{Im}(\mathbf{b}), 1 - |\mathbf{b}|^2, 1 + |\mathbf{b}|^2\big) + \\ &+ \frac{\mathbf{b}_{\overline{z}}}{1 - \overline{\mathbf{a}}\mathbf{b}}\big(2\mathfrak{Re}(\mathbf{a}), 2\mathfrak{Im}(\mathbf{a}), |\mathbf{a}|^2 - 1, |\mathbf{a}|^2 + 1\big) \\ &= F_1\mathcal{B} + F_2\mathcal{A}, \end{aligned} \tag{24}$$

where

$$\mathcal{A} := \big(2\mathfrak{Re}(\mathbf{a}), 2\mathfrak{Im}(\mathbf{a}), |\mathbf{a}|^2 - 1, |\mathbf{a}|^2 + 1\big)\,, \tag{25}$$

$$\mathcal{B} := \big(2\mathfrak{Re}(\mathbf{b}), 2\mathfrak{Im}(\mathbf{b}), 1 - |\mathbf{b}|^2, 1 + |\mathbf{b}|^2\big)\,. \tag{26}$$

Note that $\mathcal{A}$ and $\mathcal{B}$ are nonzero, lightlike and future directed vectors of $\mathbb{L}^4$, such that $\ll \mathcal{A}, \mathcal{B} \gg = -2|1-\mathbf{a}\overline{\mathbf{b}}|^2 < 0$. So, they are linearly independent vectors of the light cone of $\mathbb{L}^4$. From (24), we conclude that $\mathbb{V}(z) = 0$ if and only if $F_1(z) = F_2(z) = 0$, which is equivalent to $\mathbf{a}_{\overline{z}}(z) = \mathbf{b}_{\overline{z}}(z) = 0$. In particular, wherever $\mathbb{V}(z) = 0$, the condition (19) is trivially satisfied.

Letting $\mathbb{V} = (\mathbb{V}^1, \mathbb{V}^2, \mathbb{V}^3, \mathbb{V}^4)$, $\mathcal{A} = (\mathcal{A}^1, \mathcal{A}^2, \mathcal{A}^3, \mathcal{A}^4)$ and $\mathcal{B} = (\mathcal{B}^1, \mathcal{B}^2, \mathcal{B}^3, \mathcal{B}^4)$, the condition (12) of Theorem 2.2, implies that $\mathbb{V}^t\overline{\mathbb{V}} = R^tR$ wherever $\mathbb{V}(z) \neq 0$. Therefore, the entries $\mathbb{V}^j\overline{\mathbb{V}^k}$, $1 \leq j, k \leq 4$ of the matrix $\mathbb{V}^t\overline{\mathbb{V}}$ must be real. On the other hand, from (24) we have

$$\begin{aligned}
\mathbb{V}^j\overline{\mathbb{V}^k} &= \left(F_1\mathcal{B}^j + F_2\mathcal{A}^j\right)\overline{\left(F_1\mathcal{B}^k + F_2\mathcal{A}^k\right)} = \\
&= \left(|F_1|^2\mathcal{B}^j\mathcal{B}^k + |F_2|^2\mathcal{A}^j\mathcal{A}^k\right) + \left(\overline{F_1}F_2\mathcal{A}^j\mathcal{B}^k + F_1\overline{F_2}\mathcal{A}^k\mathcal{B}^j\right).
\end{aligned}$$

Thus, $\mathbb{V}^j\overline{\mathbb{V}^k} \in \mathbb{R}$ if and only if $\left(\overline{F_1}F_2\mathcal{A}^j\mathcal{B}^k + F_1\overline{F_2}\mathcal{A}^k\mathcal{B}^j\right) \in \mathbb{R}$ because $\left(|F_1|^2\mathcal{B}^j\mathcal{B}^k + |F_2|^2\mathcal{A}^j\mathcal{A}^k\right)$ is a real number. Hence,

$$\mathbb{V}^j\overline{\mathbb{V}^k} \in \mathbb{R} \quad \Leftrightarrow \quad \left(\overline{F_1}F_2 - F_1\overline{F_2}\right)\left(\mathcal{A}^j\mathcal{B}^k - \mathcal{A}^k\mathcal{B}^j\right) = 0.$$

This implies that

$$\overline{F_1}F_2 - F_1\overline{F_2} = 0 \text{ or } \mathcal{A}^j\mathcal{B}^k - \mathcal{A}^k\mathcal{B}^j = 0, \ 1 \leq j < k \leq 4. \tag{27}$$

We will show that $(\mathcal{A}^j\mathcal{B}^k - \mathcal{A}^k\mathcal{B}^j)$ is nonzero for all $z_0 \in U$, where $(U, z = u + iv)$ are local isothermal coordinates of S. In fact, since S is a spacelike surface, the metric $ds^2 = 4|\mu|^2|1-\mathbf{a}\overline{\mathbf{b}}|^2|dz|^2$ induced by $X : M^2 \to \mathbb{L}^4$ is Riemannian, hence $\mathbf{a}\overline{\mathbf{b}} \neq 1$ on U. Let z_0 be any point in U such that $\mathbb{V}(z_0) \neq 0$ and

$$\mathcal{A}^j(z_0)\mathcal{B}^k(z_0) - \mathcal{A}^k(z_0)\mathcal{B}^j(z_0) = 0, \quad 1 \leq j < k \leq 4.$$

These six equations imply that

1. $\overline{\mathbf{a}}\mathbf{b} = \mathbf{a}\overline{\mathbf{b}}$, 2. $\Re e(\mathbf{a}+\mathbf{b})(1-\mathbf{a}\overline{\mathbf{b}}) = 0$,
3. $\Re e(\mathbf{a}-\mathbf{b})(1-\mathbf{a}\overline{\mathbf{b}}) = 0$, 4. $\Im m(\mathbf{a}+\mathbf{b})(1-\mathbf{a}\overline{\mathbf{b}}) = 0$,
5. $\Im m(\mathbf{a}-\mathbf{b})(1-\mathbf{a}\overline{\mathbf{b}}) = 0$, 6. $|\mathbf{a}|^2|\mathbf{b}|^2 = 1$.

By the equations 1 and 6 above we have that

$$|\mathbf{a}|^2|\mathbf{b}|^2 = 1 \Leftrightarrow (\mathbf{a}\overline{\mathbf{b}})(\overline{\mathbf{a}}\mathbf{b}) = 1 \Rightarrow (\mathbf{a}\overline{\mathbf{b}})^2 = 1 \Rightarrow \mathbf{a}(z_0)\overline{\mathbf{b}}(z_0) = \pm 1. \tag{28}$$

Now using equations $2, 3, 4$ and 5 we may easily conclude that $\mathbf{a}(z_0) = \mathbf{b}(z_0) = 0$, which contradicts (28). Hence, $\mathbb{V}^t\overline{\mathbb{V}}$ is a real hermitian matrix if and only if $\overline{F_1}F_2 = F_1\overline{F_2}$, which proves (19).

To prove (20), we need first to establish a relation between the argument of the coordinates of $\mathbb{V}$ and the argument of the functions F_1 and F_2. From (19) and (24) we have that

$$\begin{aligned}\overline{F}_1\mathbb{V}(z) &= \overline{F}_1(F_1\mathcal{B} + F_2\mathcal{A}) = F_1\overline{F}_1\mathcal{B} + \overline{F}_1F_2\mathcal{A} \\ &= F_1\overline{F}_1\mathcal{B} + F_1\overline{F}_2\mathcal{A} = F_1(\overline{F}_1\mathcal{B} + \overline{F}_2\mathcal{A}) = F_1\overline{\mathbb{V}}(z).\end{aligned}$$

In a similar way, we prove that $\overline{F}_2\mathbb{V}(z) = F_2\overline{\mathbb{V}}(z)$. Since we are assuming that $\mathbf{a}_{\overline{z}} \neq 0$ and $\mathbf{b}_{\overline{z}} \neq 0$, we have $F_1 \neq 0$ and $F_2 \neq 0$. So $\mathbb{V}(z) \neq 0$ and there exists some nonzero component $\mathbb{V}^j$ of $\mathbb{V}$ which satisfies

$$\overline{F}_1\mathbb{V}^j = F_1\overline{\mathbb{V}^j}, \quad \overline{F}_2\mathbb{V}^j = F_2\overline{\mathbb{V}^j}. \tag{29}$$

Then (29) implies that $\arg(\overline{F}_1\mathbb{V}^j) = 0 \pmod{\pi}$ and $\arg(\overline{F}_2\mathbb{V}^j) = 0 \pmod{\pi}$, which are equivalent to $-\arg(F_k) + \arg(\mathbb{V}^j) = 0 \pmod{\pi}, k = 1, 2$. Now from (12) we have $\mathbb{V}(z) = e^{i\alpha(z)}R(z)$, then $\arg(\mathbb{V}^j) = \alpha(z) \pmod{2\pi}$. Therefore,

$$\alpha(z) = \arg(F_k) \pmod{\pi}, \; k = 1, 2.$$

Hence

$$\alpha(z) = \frac{1}{2}\big(\arg(F_1) + \arg(F_2)\big) \Leftrightarrow \alpha(z) = \frac{1}{2}\big(\arg(F_1F_2)\big) \pmod{\frac{\pi}{2}}. \tag{30}$$

Now (15) yields

$$\arg(F_1F_2) = \arg\left(\frac{\mathbf{a}_{\overline{z}}}{1 - \mathbf{a}\overline{\mathbf{b}}} \cdot \frac{\mathbf{b}_{\overline{z}}}{1 - \overline{\mathbf{a}}\mathbf{b}}\right) = \Im m\left\{\log\left(\frac{\mathbf{a}_{\overline{z}}\mathbf{b}_{\overline{z}}}{|1 - \mathbf{a}\overline{\mathbf{b}}|^2}\right)\right\}.$$

Therefore,

$$\alpha(z) = \frac{1}{2}\Im m\Big(\log \mathbf{a}_{\overline{z}} + \log \mathbf{b}_{\overline{z}}\Big) + 2n\pi, n \in \mathbb{Z}.$$

Thus,

$$\alpha_{z\overline{z}} = \frac{1}{2}\Im m\left\{\left(\frac{\mathbf{a}_{\overline{z}z}}{\mathbf{a}_{\overline{z}}}\right)_{\overline{z}} + \left(\frac{\mathbf{b}_{\overline{z}z}}{\mathbf{b}_{\overline{z}}}\right)_{\overline{z}}\right\}. \tag{31}$$

From (31) we have that

$$\begin{aligned}
\alpha_{z\overline{z}} &= \frac{1}{2}\Im m\left\{\left(\frac{\mathbf{a}_{\overline{z}z}}{\mathbf{a}_{\overline{z}}} + 2\overline{\mathbf{b}}\widehat{F}_1\right)_{\overline{z}} + \left(\frac{\mathbf{b}_{\overline{z}z}}{\mathbf{b}_{\overline{z}}} + 2\overline{\mathbf{a}}\widehat{F}_2\right)_{\overline{z}} - 2\Big(\big(\overline{\mathbf{b}}\widehat{F}_1\big)_{\overline{z}} + \big(\overline{\mathbf{a}}\widehat{F}_2\big)_{\overline{z}}\Big)\right\}, \\
\alpha_{z\overline{z}} &= \frac{1}{2}\Im m\Big\{T_1(\mathbf{a},\mathbf{b}) + T_2(\mathbf{a},\mathbf{b})\Big\} - \Im m\Big(\big(\overline{\mathbf{b}}\widehat{F}_1\big)_{\overline{z}} + \big(\overline{\mathbf{a}}\widehat{F}_2\big)_{\overline{z}}\Big). \qquad (32)
\end{aligned}$$

On the other hand, we have seen from (13) that $\alpha_{z\overline{z}} = \Im m(\eta_z)$. Hence from (23) we have that

$$\alpha_{z\overline{z}} = -\Im m\Big(\big(\overline{\mathbf{b}}F_1\big)_z + \big(\overline{\mathbf{a}}F_2\big)_z\Big). \tag{33}$$

Therefore, comparing (32) with (33) and using the Lemma 3.2, we prove (20). Finally, observe that (19) and (20) are independent of the choice of the conformal coordinates of M^2. This concludes the proof of the theorem.

□

In the following corollary we will be using the notation of Akutagawa and Nishigawa for spacelike surfaces in $\mathbb{L}^3$ [1].

Corollary 3.4. *Let S be a spacelike surface in $\mathbb{L}^3$ defined by the immersion $X : M^2 \to \mathbb{L}^3$, and let $w = f(z)$ be the local representation of the Gauss map, that is $N = \frac{1}{1-|f|^2}(2\Re e(f), 2\Im m(f), 1+|f|^2)$ and $|f(z)| \neq 1$, in local isothermal coordinates $(U, z = u+iv)$ of M^2. Then at every point of M^2, one of the following two conditions must hold:*

$$(i)\ f_{\overline{z}} = 0 \quad or \quad (ii)\ f_{\overline{z}} \neq 0 \quad and \quad \Im m\left\{\left(\frac{f_{z\overline{z}}}{f_{\overline{z}}} + \frac{2\overline{f}f_z}{1-|f|^2}\right)_{\overline{z}}\right\} = 0.$$

Remark 3.5. We can give an example of a map $[\Phi] : U \to Q_1^2$ which cannot represent the Gauss map of any spacelike surface in $\mathbb{L}^4$. In fact, for instance setting $\mathbf{a}(z) = z + \overline{z}$ and $\mathbf{b}(z) = -i(z - \overline{z})$ in (14), we have $\mathbf{a}_{\overline{z}} = 1$ and $\mathbf{b}_{\overline{z}} = i$. Hence,

$$\overline{F}_1 F_2 = \frac{i}{(1 - 4\Re e(z)\Im m(z))^2}, \quad F_1\overline{F}_2 = \frac{-i}{(1 - 4\Re e(z)\Im m(z))^2}.$$

This tells us that the condition (19) does not hold.

4 The Kenmotsu's type formula for spacelike surfaces in $\mathbb{L}^4$

In this section we prove that the mean curvature vector field H and the components $\mathbf{a}(z)$, $\mathbf{b}(z)$ of the Gauss map of a spacelike surface in $\mathbb{L}^4$ must satisfy a second order partial differential equation. We also write the integration factor μ, defined in (5), explicitly in terms of $\mathbf{a}(z)$, $\mathbf{b}(z)$ and H. This allows us to give a representation formula for spacelike surfaces in $\mathbb{L}^4$ in terms of the Gauss map and of mean curvature vector H.

In [1] it was proved that the Gauss map f and the mean curvature h of a spacelike surface in $\mathbb{L}^3$ need to satisfy the equation

$$h\left(f_{z\overline{z}} + \frac{2\overline{f} f_z f_{\overline{z}}}{1-|f|^2}\right) = h_z f_{\overline{z}} \quad \Leftrightarrow \quad S_1(f,f) = (\log h)_z \tag{34}$$

wherever $f_{\overline{z}} \neq 0$. Since h is real, we have that $(\log h)_{z\overline{z}}$ is a real number. Hence,

$$\mathfrak{Im}\{(S_1(f,f))_{\overline{z}}\} = 0 \quad \Leftrightarrow \quad \mathfrak{Im}\{T_1(f,f)\} = 0.$$

Our goal now is to generalize the equation (34) for spacelike surfaces in $\mathbb{L}^4$, in such a way that the necessary condition $\mathfrak{Im}\{T_1 + T_2\} = 0$, given in (20), becomes a consequence of this generalized equation. More precisely, we have the following.

Theorem 4.1. *Let S be a spacelike surface immersed in $\mathbb{L}^4$ by $X : M^2 \to \mathbb{L}^4$, with generalized Gauss map G given locally by* (14) *via the pair of functions $\mathbf{a}(z)$ and $\mathbf{b}(z)$, where z is a local conformal parameter on M^2. Then the mean curvature vector H and the functions $\mathbf{a}(z)$ and $\mathbf{b}(z)$ satisfy the second order PDE*

$$\langle H,H\rangle\left(\mathbf{b}_{\overline{z}}\left(\mathbf{a}_{z\overline{z}}+\frac{2\overline{\mathbf{b}}\mathbf{a}_{\overline{z}}\mathbf{a}_z}{1-\mathbf{a}\overline{\mathbf{b}}}\right)+\mathbf{a}_{\overline{z}}\left(\mathbf{b}_{z\overline{z}}+\frac{2\overline{\mathbf{a}}\mathbf{b}_{\overline{z}}\mathbf{b}_z}{1-\overline{\mathbf{a}}\mathbf{b}}\right)\right)=\mathbf{a}_{\overline{z}}\mathbf{b}_{\overline{z}}\langle H,H\rangle_z. \tag{35}$$

Proof: The Gauss map of S is locally defined by $G(z)=[X_z]$ and given by (14), that is, $X_z=\mu\Phi(z)$ for some function $\mu:M^2\to\mathbb{C}$. From (8) and (9) we obtain

$$\overline{\mu}\ll\Phi,\Phi\gg H=\Phi_{\overline{z}}-\eta\Phi,\quad \eta:=\frac{\ll\Phi_{\overline{z}},\Phi\gg}{\ll\Phi,\Phi\gg}.$$

We can easily verify that

$$\ll\overline{\mu}\ll\Phi,\Phi\gg H,\overline{(\overline{\mu}\ll\Phi,\Phi\gg H)}\gg=$$

$$=\ll\Phi_{\overline{z}},\overline{\Phi_{\overline{z}}}\gg-2\eta\ll\Phi_{\overline{z}},\overline{\Phi}\gg+\eta^2\ll\Phi,\overline{\Phi}\gg.$$

Since $[\Phi]\in Q_1^2$, that is $\ll\Phi,\overline{\Phi}\gg=0$, we have that $\ll\Phi_{\overline{z}},\overline{\Phi}\gg=0$. Thus, the above equation is reduced to

$$\ll\Phi,\Phi\gg^2\langle H,H\rangle\overline{\mu}^2=\ll\Phi_{\overline{z}}.\overline{\Phi_{\overline{z}}}\gg. \tag{36}$$

Let us now explicitly calculate $\ll\Phi_{\overline{z}},\overline{\Phi_{\overline{z}}}\gg$ in function of $\mathbf{a}(z)$ and $\mathbf{b}(z)$. From (21) we have that

$$\ll\Phi_{\overline{z}},\overline{\Phi_{\overline{z}}}\gg=-4\mathbf{a}_{\overline{z}}\mathbf{b}_{\overline{z}}. \tag{37}$$

Therefore,

$$\ll\Phi,\Phi\gg^2\langle H,H\rangle\overline{\mu}^2=-4\mathbf{a}_{\overline{z}}\mathbf{b}_{\overline{z}}. \tag{38}$$

Now differentiating $\overline{\mu}\ll\Phi,\Phi\gg H=\Phi_{\overline{z}}-\eta\Phi$ with respect to z, we obtain

$$\ll\Phi,\Phi\gg_z(\overline{\mu}H)+\ll\Phi,\Phi\gg(\overline{\mu}H)_z=\Phi_{\overline{z}z}-\eta_z\Phi-\eta\Phi_z.$$

On the other hand,

$$\ll\Phi,\Phi\gg_z(\overline{\mu}H)+\ll\Phi,\Phi\gg(\overline{\mu}H)_z=$$

$$= (\ll \Phi_z, \Phi \gg + \ll \Phi, \Phi_{\overline{z}} \gg)\overline{\mu}H + \ll \Phi, \Phi \gg ((\overline{\mu})_z H + \overline{\mu}H_z).$$

Thus,

$$\begin{aligned} \ll \Phi, \Phi \gg \overline{\mu}H_z &= \Phi_{\overline{z}z} - \eta_z\Phi - \eta\Phi_z - (\ll \Phi_z, \Phi \gg + \overline{\ll \Phi_{\overline{z}}, \Phi \gg})\overline{\mu}H - \\ &- (\ll \Phi, \Phi \gg H)\overline{\mu_{\overline{z}}}. \end{aligned}$$

Moreover, note that $\overline{\mu}H = \dfrac{\Phi_{\overline{z}} - \eta\Phi}{\ll \Phi, \Phi \gg}$, $\ll \Phi, \Phi \gg H = \dfrac{\Phi_{\overline{z}} - \eta\Phi}{\overline{\mu}}$. Therefore,

$$\begin{aligned} \ll \Phi, \Phi \gg \overline{\mu}H_z &= \Phi_{\overline{z}z} - \eta\Phi_z - \\ &- \left(\frac{\ll \Phi_z, \Phi \gg}{\ll \Phi, \Phi \gg} + \frac{\overline{\ll \Phi_{\overline{z}}, \Phi \gg}}{\ll \Phi, \Phi \gg} + \overline{\left(\frac{\mu_{\overline{z}}}{\mu}\right)} \right) \Phi_{\overline{z}} + \\ &+ \left(\left(\frac{\ll \Phi_z, \Phi \gg}{\ll \Phi, \Phi \gg} + \frac{\overline{\ll \Phi_{\overline{z}}, \Phi \gg}}{\ll \Phi, \Phi \gg} + \overline{\left(\frac{\mu_{\overline{z}}}{\mu}\right)} \right) \eta - \eta_z \right) \Phi. \end{aligned}$$

From (10) we know that $\dfrac{\mu_{\overline{z}}}{\mu} = -\eta$. Also, $\eta = \dfrac{\ll \Phi_{\overline{z}}, \Phi \gg}{\ll \Phi, \Phi \gg}$, so that

$$\ll \Phi, \Phi \gg \overline{\mu}H_z = \Phi_{\overline{z}z} - \eta\Phi_z - \frac{\ll \Phi_z, \Phi \gg}{\ll \Phi, \Phi \gg}\Phi_{\overline{z}} + \left(\frac{\ll \Phi_z, \Phi \gg}{\ll \Phi, \Phi \gg}\eta - \eta_z \right)\Phi. \tag{39}$$

Calculating the symmetric product, $\ll \cdot, \bar{\cdot} \gg$ between (39) and $\overline{\mu} \ll \Phi, \Phi \gg H = \Phi_{\overline{z}} - \eta\Phi$, we have that

$$\begin{aligned} \ll \Phi, \Phi \gg^2 \langle H, H_z \rangle \overline{\mu}^2 &= \ll \Phi_{\overline{z}}, \overline{\Phi_{\overline{z}z}} \gg - \eta(\ll \Phi_{\overline{z}}, \overline{\Phi_z} \gg + \ll \Phi, \overline{\Phi_{\overline{z}z}} \gg) - \\ &- \frac{\ll \Phi_z, \Phi \gg}{\ll \Phi, \Phi \gg} \ll \Phi_{\overline{z}}, \overline{\Phi_{\overline{z}}} \gg . \end{aligned}$$

Now using that $0 = \ll \Phi, \overline{\Phi_z} \gg_{\overline{z}} = \ll \Phi_{\overline{z}}, \overline{\Phi_z} \gg + \ll \Phi, \overline{\Phi_{z\overline{z}}} \gg$ in the above equation, we get

$$\ll \Phi, \Phi \gg^2 \langle H, H_z \rangle \overline{\mu}^2 = \frac{1}{2} \ll \Phi_{\overline{z}}, \overline{\Phi_{\overline{z}}} \gg_z - \frac{\ll \Phi_z, \Phi \gg}{\ll \Phi, \Phi \gg} \ll \Phi_{\overline{z}}.\overline{\Phi_{\overline{z}}} \gg . \tag{40}$$

From (21) and $\ll \Phi, \Phi \gg = 2(1 - \mathbf{a}\overline{\mathbf{b}})(1 - \overline{\mathbf{a}}\mathbf{b})$, we can verify that

$$\frac{\ll \Phi_z, \Phi \gg}{\ll \Phi, \Phi \gg} = -\left(\overline{\mathbf{b}}\widehat{F}_1(\mathbf{a}, \mathbf{b}) + \overline{\mathbf{a}}\widehat{F}_2(\mathbf{a}, \mathbf{b}) \right), \tag{41}$$

where $\widehat{F}_i$ is given in (15). Finally, substituting (37) and (41) into (40), we obtain

$$\begin{aligned} \ll \Phi, \Phi \gg^2 \overline{\mu}^2 \langle H, H_z \rangle &= -2\left(\mathbf{a}_{\overline{z}\overline{z}z}\mathbf{b}_{\overline{z}} + \mathbf{a}_{\overline{z}}\mathbf{b}_{\overline{z}\overline{z}z}\right) + 2\left(\overline{\mathbf{b}}\widehat{F}_1 + \overline{\mathbf{a}}\widehat{F}_2\right)\left(-2\mathbf{a}_{\overline{z}}\mathbf{b}_{\overline{z}}\right) = \\ &= -2\left(\mathbf{b}_{\overline{z}}\left(\mathbf{a}_{z\overline{z}} + \frac{2\overline{\mathbf{b}}\mathbf{a}_{\overline{z}}\mathbf{a}_z}{1-\mathbf{a}\overline{\mathbf{b}}}\right) + \mathbf{a}_{\overline{z}}\left(\mathbf{b}_{z\overline{z}} + \frac{2\overline{\mathbf{a}}\mathbf{b}_{\overline{z}}\mathbf{b}_z}{1-\overline{\mathbf{a}}\mathbf{b}}\right)\right). \end{aligned}$$

Taking the product of the above equation with $\langle H, H\rangle$ and using (38), we get

$$\left(\mathbf{b}_{\overline{z}}\left(\mathbf{a}_{z\overline{z}} + \frac{2\overline{\mathbf{b}}\mathbf{a}_{\overline{z}}\mathbf{a}_z}{1-\mathbf{a}\overline{\mathbf{b}}}\right) + \mathbf{a}_{\overline{z}}\left(\mathbf{b}_{z\overline{z}} + \frac{2\overline{\mathbf{a}}\mathbf{b}_{\overline{z}}\mathbf{b}_z}{1-\overline{\mathbf{a}}\mathbf{b}}\right)\right)\langle H, H\rangle = 2\mathbf{a}_{\overline{z}}\mathbf{b}_{\overline{z}}\langle H, H_z\rangle,$$

which proves the theorem.

□

Let S be a spacelike surface in $\mathbb{L}^3$. We may identify $\mathbb{L}^3$ with the subset of $\mathbb{L}^4$ given by $\{(x^1, x^2, x^3, x^4) \in \mathbb{L}^4 : x^3 = 0\}$. So S can be considered as a spacelike surface in $\mathbb{L}^4$ and, in this case, we denote it by $\check{S}$ and we may assume that $\mathbf{b}(z) = \mathbf{a}(z)$ in (14). In analogous way, any surface S in $\mathbb{R}^3$ can be viewed as a surface $\hat{S}$ in $\mathbb{R}^3 \equiv \{(x^1, x^2, x^3, x^4) \in \mathbb{L}^4 : x^4 = 0\}$ and in this case we may assume that $\mathbf{b}(z) = -\mathbf{a}(z)$ in (14). Now we will use formula (35) to obtain, in a new way, the integrability conditions for a spacelike surface in $\mathbb{L}^3$ and a surface in $\mathbb{R}^3$ with prescribed nonvanishing mean curvature h and Gauss map $\mathbf{a}$.

The vectors $\mathcal{A}$ and $\mathcal{B}$ given in (25) and (26), are future directed lightlike vectors such that $\langle \mathcal{A}, \mathcal{B}\rangle = -2|1-\mathbf{a}\overline{\mathbf{b}}|^2 < 0$, therefore are linearly independent wherever $\mathbf{a}(z)$ and $\mathbf{b}(z)$ describe the Gauss map Φ of a spacelike surface S in $\mathbb{L}^4$. It is easy to see that $\langle \Phi, \mathcal{A}\rangle = 0$ and $\langle \Phi, \mathcal{B}\rangle = 0$. Thus, $\{\mathcal{A}, \mathcal{B}\}$ is a basis for $(T_pS)^\perp$. Moreover, $H \in (T_pS)^\perp$ and then we can write H in the form

$$H = \frac{\langle H, \mathcal{B}\rangle \mathcal{A} + \langle H, \mathcal{A}\rangle \mathcal{B}}{\langle \mathcal{A}, \mathcal{B}\rangle}.$$

Hence

$$\langle H, H\rangle = \frac{2\langle H, \mathcal{A}\rangle\langle H, \mathcal{B}\rangle}{\langle \mathcal{A}, \mathcal{B}\rangle}. \tag{42}$$

1^{st} **case**: If $\mathbf{b}(z) = \mathbf{a}(z)$, we know that $\Phi(z) = (1 + \mathbf{a}^2, i(1 - \mathbf{a}^2), 0, 2\mathbf{a})$ locally represents the Gauss map of a spacelike surface $\check{S}$ in $\mathbb{L}^3 \subset \mathbb{L}^4$. In this case, the mean curvature vector $H = \check{H}$ is timelike and parallel to the classical Gauss map $\check{N} : M^2 \to H_0^2(-1)$ of $\check{S}$, where $H_0^2(-1) = \{x \in \mathbb{L}^3 : \langle x, x\rangle = -1\}$. From (42) we have that

$$\langle \check{H}, \check{H}\rangle = -\frac{\langle \check{H}, \check{\mathcal{A}}\rangle\langle \check{H}, \check{\mathcal{B}}\rangle}{(1 - |\mathbf{a}|^2)^2},$$

with $\check{\mathcal{A}} = \check{\mathcal{B}} = (2\Re e(\mathbf{a}), 2\Im m(\mathbf{a}), 0, |\mathbf{a}|^2 + 1)$. It is well-known that

$$\check{N}(z) = \frac{1}{1 - |\mathbf{a}|^2}\left(2\Re e(\mathbf{a}), 2\Im m(\mathbf{a}), 0, 1 + |\mathbf{a}|^2\right). \tag{43}$$

Hence from (43), we have that

$$\check{\mathcal{A}} = (1 - |\mathbf{a}|^2)\check{N}, \quad \check{\mathcal{B}} = (1 - |\mathbf{a}|^2)\check{N}.$$

Therefore,

$$\langle \check{H}, \check{H}\rangle = -\langle \check{H}, \check{N}\rangle^2 = -h^2,$$

where h is the mean curvature of $\check{S}$. Substituting $\mathbf{b}(z) = \mathbf{a}(z)$ in (35), we have

$$-2h^2\mathbf{a}_{\overline{z}}\left(\mathbf{a}_{z\overline{z}} + \frac{2\overline{\mathbf{a}}\mathbf{a}_z\mathbf{a}_{\overline{z}}}{1 - |\mathbf{a}|^2}\right) = -2hh_z(\mathbf{a}_{\overline{z}})^2 \quad \Leftrightarrow$$

$$ha_{\overline{z}}\left(h\left(\mathbf{a}_{z\overline{z}} + \frac{2\overline{\mathbf{a}}\mathbf{a}_z\mathbf{a}_{\overline{z}}}{1 - |\mathbf{a}|^2}\right) - h_z\mathbf{a}_{\overline{z}}\right) = 0.$$

Since $h = 0$ if and only if $\mathbf{a}_{\overline{z}} = 0$, the above equation implies that

$$h\left(\mathbf{a}_{z\overline{z}} + \frac{2\overline{\mathbf{a}}\mathbf{a}_z\mathbf{a}_{\overline{z}}}{1 - |\mathbf{a}|^2}\right) = h_z\mathbf{a}_{\overline{z}}.$$

This equation is a complete integrability condition for the existence of a spacelike surface with prescribed nonvanishing mean curvature and Gauss map in $\mathbb{L}^3$ and was firstly obtained by Akutagawa and Nishikawa [1].

2^{nd} **case**: If $\mathbf{b}(z) = -\mathbf{a}(z)$, we know that $\Phi(z) = (1 - \mathbf{a}^2, i(1 + \mathbf{a}^2), 2\mathbf{a}, 0)$ locally represents the Gauss map of spacelike surfaces $\hat{S}$ in $\mathbb{R}^3 \subset \mathbb{L}^4$. In this case, the mean curvature vector $H = \hat{H}$ is spacelike and parallel to the classical Gauss map $\hat{N} : M^2 \to \mathbb{S}^2(1)$ of $\hat{S}$. It is well-known that

$$\hat{N}(z) = \frac{1}{1+|\mathbf{a}|^2}\left(2\Re e(\mathbf{a}), 2\Im m(\mathbf{a}), |\mathbf{a}|^2 - 1, 0\right). \tag{44}$$

Now from (42) we have that

$$\langle \hat{H}, \hat{H}\rangle = -\frac{\langle \hat{H}, \hat{\mathcal{A}}\rangle\langle \hat{H}, \hat{\mathcal{B}}\rangle}{(1+|\mathbf{a}|^2)^2},$$

with $\hat{\mathcal{A}} = -\hat{\mathcal{B}} = (2\Re e(\mathbf{a}), 2\Im m(\mathbf{a}), |\mathbf{a}|^2 - 1, 0)$. Hence from (44) it follows that

$$\hat{\mathcal{A}} = (1+|\mathbf{a}|^2)\hat{N}, \quad \hat{\mathcal{B}} = -(1+|\mathbf{a}|^2)\hat{N}.$$

Therefore,

$$\langle \hat{H}, \hat{H}\rangle = \langle \hat{H}, \hat{N}\rangle^2 = h^2,$$

where h is the mean curvature of $\hat{S}$. Substituting $\mathbf{b}(z) = -\mathbf{a}(z)$ in (35), we get

$$-2h^2\mathbf{a}_{\overline{z}}\left(\mathbf{a}_{z\overline{z}} - \frac{2\overline{\mathbf{a}}\mathbf{a}_z\mathbf{a}_{\overline{z}}}{1+|\mathbf{a}|^2}\right) = -2hh_z(\mathbf{a}_{\overline{z}})^2 \quad \Leftrightarrow$$

$$ha_{\overline{z}}\left(h\left(\mathbf{a}_{z\overline{z}} - \frac{2\overline{\mathbf{a}}\mathbf{a}_z\mathbf{a}_{\overline{z}}}{1+|\mathbf{a}|^2}\right) - h_z\mathbf{a}_{\overline{z}}\right) = 0.$$

Since $h = 0$ if and only if $\mathbf{a}_{\overline{z}} = 0$, the above equation implies that

$$h\left(\mathbf{a}_{z\overline{z}} - \frac{2\overline{\mathbf{a}}\mathbf{a}_z\mathbf{a}_{\overline{z}}}{1+|\mathbf{a}|^2}\right) = h_z\mathbf{a}_{\overline{z}}.$$

This is a complete integrability condition for the existence of a surface with prescribed nonvanishing mean curvature and Gauss map in $\mathbb{R}^3$ and was firstly obtained by Kenmotsu in [9].

As an immediate consequence of Theorem 4.1 we have the following

Corollary 4.2. *Under the hypothesis of Theorem* 4.1*, if the mean curvature $h = \langle H, H\rangle$ is nonvanishing for every point of M^2, then*

$$S_1(\mathbf{a}, \mathbf{b}) + S_2(\mathbf{a}, \mathbf{b}) = (\log\langle H, H\rangle)_z, \tag{45}$$

where z is a local conformal parameter on M^2.

Observe that since $\langle H, H\rangle$ is a real function, then $(\log\langle H, H\rangle)_{z\overline{z}} \in \mathbb{R}$. Hence, (45) implies that

$$\mathfrak{Im}\{(S_1(\mathbf{a},\mathbf{b}))_{\overline{z}} + (S_2(\mathbf{a},\mathbf{b}))_{\overline{z}}\} = 0 \quad \Leftrightarrow \quad \mathfrak{Im}\{T_1(\mathbf{a},\mathbf{b}) + T_2(\mathbf{a},\mathbf{b})\} = 0.$$

Therefore, the equation (45) implies the equation (20) of Theorem 3.3.

We can give another interesting consequence of (the proof) Theorem 4.1. From equation (38) it follows that

$$\overline{\mu}^2 = \frac{-4\mathbf{a}_{\overline{z}}\mathbf{b}_{\overline{z}}}{\langle H, H\rangle \ll \Phi, \Phi \gg^2}$$

wherever $\langle H, H\rangle \neq 0$. In other words, the integration factor μ can be given explicitly by

$$\overline{\mu}^2 = \frac{-\mathbf{a}_{\overline{z}} \cdot \mathbf{b}_{\overline{z}}}{\langle H, H\rangle |1 - \mathbf{a}\overline{\mathbf{b}}|^4}. \tag{46}$$

Proposition 4.3. *Let S be a spacelike surface in $\mathbb{L}^4$ whose Gauss map is given locally by* (14)*. If the mean curvature vector H is nonvanishing nor lightlike, then the surface S can be obtained explicitly by*

$$X(z) = 2\mathfrak{Re}\int_{z_0}^{z} \mu\,(1 + \mathbf{ab}, i(1 - \mathbf{ab}), \mathbf{a} - \mathbf{b}, \mathbf{a} + \mathbf{b})\, dw + X(z_0), \tag{47}$$

where μ is given by (46)*.*

5 Sufficient conditions for a map to be a Gauss map

We will show now that the conditions (19) and(45) are also sufficient for the existence for spacelike surface with prescribed Gauss map and nonzero mean curvature. Therefore, it will also holds the converse of Proposition 4.3. That is, given an arbitrary map of a simply connected Riemann surface into the quadric Q_1^2, locally represented by $\Phi(z)$, as in (14), via the pair of functions $(\mathbf{a}(z), \mathbf{b}(z))$, such that they satisfy the conditions $\overline{F}_1 F_2 = F_1 \overline{F}_2$, $\langle \mathbb{V}, \mathbb{V}\rangle \neq 0$ and $S_1 + S_2 = (\log h)_z$, where $h : M^2 \to \mathbb{R} \setminus \{0\}$

is a smooth function, then (47) defines a spacelike surface immersed in $\mathbb{L}^4$ with locally Gauss map given by Φ, $\langle H, H\rangle = h$ and induced metric $ds^2 = 4|\mu|^2|1-\mathbf{a}\overline{\mathbf{b}}|^2|dz|^2$ given, using (46), by

$$ds^2 = \left(\frac{4|\mathbf{a}_{\overline{z}}||\mathbf{b}_{\overline{z}}|}{|\langle H, H\rangle||1-\mathbf{a}\overline{\mathbf{b}}|^2}\right)|dz|^2.$$

To prove these results we need two lemmas.

Lemma 5.1. *Let U be a simply connected open set of $\mathbb{C}$ and let $\Phi : U \to \mathbb{C}^4$ be a C^2 map. Then there exists a spacelike surface S given by $X : U \to \mathbb{L}^4$ such that $X_z = \Phi$ if, and only if $\Im m(\Phi_{\overline{z}}) = 0$.*

Proof: We know that $X_{z\overline{z}} = \frac{\lambda^2}{2}H \in \mathbb{R}^4$. Hence, if there exists $X : U \to \mathbb{L}^4$ such that $X_z = \Phi$ it follows that $\Im m(\Phi_{\overline{z}}) = 0$.

Conversely, the condition $\Im m(\Phi_{\overline{z}}) = 0$ implies that the system

$$\begin{cases} X_u = 2\Re e\Phi \\ X_v = -2\Im m\Phi \end{cases}$$

is integrable and the solution is given by $X(z) = 2\int^z \Phi dw$.

□

Lemma 5.2. *If the system $X_z = \mu\Phi$, with*

$$\overline{\mu}^2 = \frac{-\mathbf{a}_{\overline{z}}\cdot\mathbf{b}_{\overline{z}}}{h|1-\mathbf{a}\overline{\mathbf{b}}|^4},\quad \Phi(z) = (1+\mathbf{ab}, i(1-\mathbf{ab}), \mathbf{a}-\mathbf{b}, \mathbf{a}+\mathbf{b}),\quad h : \mathbb{C} \to \mathbb{R}\setminus\{0\}$$

satisfies $S_1(\mathbf{a},\mathbf{b}) + S_2(\mathbf{a},\mathbf{b}) = (\log h)_{\overline{z}}$, then

$$X_{z\overline{z}} = \mu\mathbb{V}, \tag{48}$$

where $\mathbb{V} = F_1\mathcal{B} + F_2\mathcal{A}$.

Proof: Putting $\mathbf{P} = h(1-\mathbf{a}\overline{\mathbf{b}})^2(1-\overline{\mathbf{a}}\mathbf{b})^2$, we can write $\mu^2 = -\dfrac{\overline{\mathbf{a}_{\overline{z}}}\cdot\overline{\mathbf{b}_{\overline{z}}}}{\mathbf{P}}$. Thus,

$$-2\mu_{\overline{z}}\mu = \frac{\left(\overline{\mathbf{a}_{\overline{z}}}\,\overline{\mathbf{b}_{\overline{z}}}\,\right)_{\overline{z}}\mathbf{P} - \left(\overline{\mathbf{a}_{\overline{z}}}\,\overline{\mathbf{b}_{\overline{z}}}\,\right)\mathbf{P}_{\overline{z}}}{\mathbf{P}^2},$$

and since $(\overline{\mathbf{a}_{\overline{z}}})_{\overline{z}} = \overline{\mathbf{a}_{\overline{z}z}}$ and $h_{\overline{z}} = \overline{h_z}$ we have that

$$-2\mu_{\overline{z}}\mu = \frac{\left(\overline{\mathbf{a}_{\overline{z}z}}\ \overline{\mathbf{b}_{\overline{z}}} + \overline{\mathbf{a}_{\overline{z}}}\ \overline{\mathbf{b}_{\overline{z}z}}\ \right)\mathbf{P} - \overline{\mathbf{a}_{\overline{z}}}\ \overline{\mathbf{b}_{\overline{z}}}\ \mathbf{P}_{\overline{z}}}{\mathbf{P}^2}, \tag{49}$$

$$\begin{aligned}\mathbf{P}_{\overline{z}} &= h_{\overline{z}}|1-\mathbf{a}\overline{\mathbf{b}}|^4 + h\Big(2|1-\mathbf{a}\overline{\mathbf{b}}|^2(1-\overline{\mathbf{a}}\mathbf{b})(-\mathbf{a}_{\overline{z}}\overline{\mathbf{b}} - \mathbf{a}\overline{\mathbf{b}_z}) + \\ &+ 2|1-\mathbf{a}\overline{\mathbf{b}}|^2(1-\mathbf{a}\overline{\mathbf{b}})(-\overline{\mathbf{a}_z}\mathbf{b} - \overline{\mathbf{a}}\mathbf{b}_{\overline{z}})\Big).\end{aligned} \tag{50}$$

Substituting (50) in (49) we have that

$$\begin{aligned}-2\mu_{\overline{z}}\mu\mathbf{P}^2 &= |1-\mathbf{a}\overline{\mathbf{b}}|^4\left(h\left(\left(\overline{\mathbf{b}_{\overline{z}}}\ \overline{\mathbf{a}_{\overline{z}z}} + \frac{2\mathbf{b}\ \overline{\mathbf{a}_{\overline{z}}}\ \overline{\mathbf{a}_z}\ \overline{\mathbf{b}_{\overline{z}}}(1-\mathbf{a}\overline{\mathbf{b}})}{|1-\mathbf{a}\overline{\mathbf{b}}|^2}\right)+\right.\right. \\ &+ \left.\left.\left(\overline{\mathbf{a}_{\overline{z}}}\ \overline{\mathbf{b}_{\overline{z}z}} + \frac{2\mathbf{a}\ \overline{\mathbf{b}_{\overline{z}}}\ \overline{\mathbf{b}_z}\ \overline{\mathbf{a}_{\overline{z}}}\ (1-\overline{\mathbf{a}}\mathbf{b})}{|1-\mathbf{a}\overline{\mathbf{b}}|^2}\right)\right) - \overline{\mathbf{a}_{\overline{z}}}\ \overline{\mathbf{b}_{\overline{z}}}\ \overline{h_z}\right) + \\ &+ \overline{\mathbf{a}_{\overline{z}}}\ \overline{\mathbf{b}_{\overline{z}}}\Big(2h\overline{\mathbf{b}}\mathbf{a}_{\overline{z}}(1-\overline{\mathbf{a}}\mathbf{b})|1-\mathbf{a}\overline{\mathbf{b}}|^2\Big) + \\ &+ \overline{\mathbf{a}_{\overline{z}}}\ \overline{\mathbf{b}_{\overline{z}}}\Big(2h\overline{\mathbf{a}}\mathbf{b}_{\overline{z}}(1-\mathbf{a}\overline{\mathbf{b}})|1-\mathbf{a}\overline{\mathbf{b}}|^2\Big),\end{aligned} \tag{51}$$

$$\begin{aligned}-2\mu_{\overline{z}}\mu\mathbf{P}^2 = |1-\mathbf{a}\overline{\mathbf{b}}|^4&\left(h\left(\overline{\mathbf{b}_{\overline{z}}}\left(\overline{\mathbf{a}_{\overline{z}z}} + \frac{2\mathbf{b}\ \overline{\mathbf{a}_{\overline{z}}}\ \overline{\mathbf{a}_z}}{(1-\overline{\mathbf{a}}\mathbf{b})}\right) + \right.\right. \\ &+ \left.\left.\overline{\mathbf{a}_{\overline{z}}}\left(\overline{\mathbf{b}_{\overline{z}z}} + \frac{2\mathbf{a}\ \overline{\mathbf{b}_{\overline{z}}}\ \overline{\mathbf{b}_z}}{(1-\mathbf{a}\overline{\mathbf{b}})}\right)\right) - \overline{\mathbf{a}_{\overline{z}}}\ \overline{\mathbf{b}_{\overline{z}}}\ \overline{h_z}\right) + \\ +\overline{\mathbf{a}_{\overline{z}}}\ \overline{\mathbf{b}_{\overline{z}}}\left(2h\overline{\mathbf{b}}\mathbf{a}_{\overline{z}}(1-\overline{\mathbf{a}}\mathbf{b})|1-\mathbf{a}\overline{\mathbf{b}}|^2\right) &+ \overline{\mathbf{a}_{\overline{z}}}\ \overline{\mathbf{b}_{\overline{z}}}\left(2h\overline{\mathbf{a}}\mathbf{b}_{\overline{z}}(1-\mathbf{a}\overline{\mathbf{b}})|1-\mathbf{a}\overline{\mathbf{b}}|^2\right).\end{aligned} \tag{52}$$

By hypothesis $S_1(\mathbf{a},\mathbf{b}) + S_2(\mathbf{a},\mathbf{b}) = (\log h)_{\overline{z}}$ hence,

$$\mu\mu_{\overline{z}} = -\frac{1}{h|1-\mathbf{a}\overline{\mathbf{b}}|^6}\left(\overline{\mathbf{b}}\ \overline{\mathbf{b}_{\overline{z}}}(1-\overline{\mathbf{a}}\mathbf{b})|\mathbf{a}_{\overline{z}}|^2 + \overline{\mathbf{a}}\ \overline{\mathbf{b}_{\overline{z}}}(1-\mathbf{a}\overline{\mathbf{b}})|\mathbf{b}_{\overline{z}}|^2\right). \tag{53}$$

On the other hand,

$$\mu X_{z\overline{z}} = \mu\mu_{\overline{z}}\Phi + \mu^2\Phi_{\overline{z}}, \tag{54}$$

$$\Phi_{\overline{z}} = \mathbf{a}_{\overline{z}}(\mathbf{b}, -i\mathbf{b}, 1, 1) + \mathbf{b}_{\overline{z}}(\mathbf{a}, -i\mathbf{a}, -1, 1). \tag{55}$$

Substituting (53) and (55) in (54), yields

$$\mu X^1_{z\overline{z}} = -\frac{\overline{\mathbf{a}_{\overline{z}}}\,\overline{\mathbf{b}_{\overline{z}}}}{h|1-\mathbf{a}\overline{\mathbf{b}}|^4}\left(\frac{\mathbf{a}_{\overline{z}}}{(1-\mathbf{a}\overline{\mathbf{b}})}\cdot 2\Re e(\mathbf{b}) + \frac{\mathbf{b}_{\overline{z}}}{(1-\overline{\mathbf{a}}\mathbf{b})}\cdot 2\Re e(\mathbf{a})\right),$$

$$\mu X^2_{z\overline{z}} = -\frac{\overline{\mathbf{a}_{\overline{z}}}\,\overline{\mathbf{b}_{\overline{z}}}}{h|1-\mathbf{a}\overline{\mathbf{b}}|^4}\left(\frac{\mathbf{a}_{\overline{z}}}{(1-\mathbf{a}\overline{\mathbf{b}})}\cdot 2\Im m(\mathbf{b}) + \frac{\mathbf{b}_{\overline{z}}}{(1-\overline{\mathbf{a}}\mathbf{b})}\cdot 2\Im m(\mathbf{a})\right),$$

$$\mu X^3_{z\overline{z}} = -\frac{\overline{\mathbf{a}_{\overline{z}}}\,\overline{\mathbf{b}_{\overline{z}}}}{h|1-\mathbf{a}\overline{\mathbf{b}}|^4}\left(\frac{\mathbf{a}_{\overline{z}}}{(1-\mathbf{a}\overline{\mathbf{b}})}\cdot (1-|\mathbf{b}|^2) + \frac{\mathbf{b}_{\overline{z}}}{(1-\overline{\mathbf{a}}\mathbf{b})}\cdot (|\mathbf{a}|^2-1)\right),$$

$$\mu X^4_{z\overline{z}} = -\frac{\overline{\mathbf{a}_{\overline{z}}}\,\overline{\mathbf{b}_{\overline{z}}}}{h|1-\mathbf{a}\overline{\mathbf{b}}|^4}\left(\frac{\mathbf{a}_{\overline{z}}}{(1-\mathbf{a}\overline{\mathbf{b}})}\cdot (1+|\mathbf{b}|^2) + \frac{\mathbf{b}_{\overline{z}}}{(1-\overline{\mathbf{a}}\mathbf{b})}\cdot (|\mathbf{a}|^2+1)\right).$$

Thus, in agreement with the definitions of $F_1, F_2, \mathcal{A}, \mathcal{B}$ and $\mathbb{V}$ we have that

$$\mu X_{z\overline{z}} = \mu^2\left(\frac{\mathbf{a}_{\overline{z}}}{(1-\mathbf{a}\overline{\mathbf{b}})}\cdot \mathcal{B} + \frac{\mathbf{b}_{\overline{z}}}{(1-\overline{\mathbf{a}}\mathbf{b})}\cdot \mathcal{A}\right),$$

that is,

$$X_{z\overline{z}} = \mu\mathbb{V}.$$

□

Theorem 5.3. *Let M^2 be a simply connected Riemann surface and $G : M^2 \to Q^2_1$ be a smooth map given locally by $\Phi(z) = \Phi(\mathbf{a}(z), \mathbf{b}(z))$ as in* (14) *and let $h : M^2 \to \mathbb{R}\setminus\{0\}$ be a smooth function. Then there exists a spacelike surface S given by a conformal immersion $X : M^2 \to \mathbb{L}^4$ with generalized Gauss map G and mean curvature vector H satisfying $\langle H, H\rangle = h$, if and only if Φ and h satisfy:*
1) $\overline{F_1}F_2 = F_1\overline{F_2}, \quad \mathbf{a}_{\overline{z}}\mathbf{b}_{\overline{z}} \neq 0;$
2) $S_1(\mathbf{a},\mathbf{b}) + S_2(\mathbf{a},\mathbf{b}) = (\log h)_{\overline{z}}.$
Moreover, S is given explicitly by

$$X(z) = 2\Re e\int_{z_0}^{z} \mu\,(1+\mathbf{ab}, i(1-\mathbf{ab}), \mathbf{a}-\mathbf{b}, \mathbf{a}+\mathbf{b})\,dw + X(z_o), \qquad (56)$$

with $\overline{\mu}^2 = -\dfrac{\mathbf{a}_{\overline{z}}\cdot\mathbf{b}_{\overline{z}}}{h|1-\mathbf{a}\overline{\mathbf{b}}|^4}.$

Proof: The necessity of conditions 1) and 2) is an immediate consequence of Theorem 3.3 and Corollary 4.2.

Conversely, suppose that $\Phi = \Phi(\mathbf{a}, \mathbf{b})$ and h satisfy 1) and 2). To prove that Φ locally represent the generalized Gauss map of some spacelike surface S given by the immersion $X : M^2 \to \mathbb{L}^4$ we need to show that $X_z = \mu\Phi$ for some function $\mu : M^2 \to \mathbb{C}^*$. Therefore, we need to prove that the system

$$X_z = \mu(1 + \mathbf{ab}, i(1 - \mathbf{ab}), \mathbf{a} - \mathbf{b}, \mathbf{a} + \mathbf{b}), \quad \mu^2 = -\frac{\overline{\mathbf{a}_{\overline{z}}} \cdot \overline{\mathbf{b}_{\overline{z}}}}{h|1 - \mathbf{a}\overline{\mathbf{b}}|^4} \tag{57}$$

is integrable.

Now by Lemma 5.2, it follows that

$$X_{z\overline{z}} = \mu\mathbb{V}$$

On the other hand, the equation $\overline{F_1}F_2 = F_1\overline{F_2}$ implies that

$$\mathbb{V}^j\overline{\mathbb{V}^k} \in \mathbb{R}, \quad 1 \leq j, k \leq 4. \tag{58}$$

Set $\mathbb{V}^j = \rho_j e^{i\theta_j}$, $\mathbb{V}^k = \rho_k e^{i\theta_k}$, hence $\mathbb{V}^j\overline{\mathbb{V}^k} = \rho_j\rho_k e^{i(\theta_j - \theta_k)}$ is a real number if, and only if, $\theta_j \equiv \theta_k (\mathrm{mod}\pi)$. Putting $\theta(z) = \theta_{j_0}$, for some j_0 between 1 and 4, we have that $\mathbb{V} = e^{i\theta}R$, with $R = (R^1, R^2, R^3, R^4) \in \mathbb{L}^4$, so

$$arg(\mathbb{V}^j) = \theta(\mathrm{mod}\pi), \quad \forall\ j. \tag{59}$$

Since $\overline{F_k}\mathbb{V}^j = F_k\overline{\mathbb{V}^j}$ for all $j = 1, 2, 3, 4$ and $k = 1, 2$ we also have that

$$arg(\mathbb{V}^j) = arg(F_k)(\mathrm{mod}\pi). \tag{60}$$

From (59) and (60) it follows that $\theta = arg(F_k)(\mathrm{mod}\pi)$, and this implies that

$$2\theta = arg(F_1F_2)(\mathrm{mod}\pi). \tag{61}$$

On the other hand, from expression of μ given in (57) we get that

$$2arg(\overline{\mu}) = arg(F_1F_2)(\mathrm{mod}\pi). \tag{62}$$

Suppose that μ is written in the form $\mu = \rho e^{-i\alpha}$, that is $arg(\overline{\mu}) = \alpha$. Then from (61) and (62) we conclude that $\theta \equiv \alpha(\mathrm{mod}\pi)$. Then

$$\mathbb{V} = e^{i\alpha} R. \tag{63}$$

Thus, $X_{z\overline{z}} = \mu\mathbb{V} \;\Leftrightarrow\; X_{z\overline{z}} = \rho R$. Therefore, by Lemma 5.1 the system (57) is integrable and has the solution:

$$\begin{aligned} X(z) &= 2\Re e \int_{z_0}^{z} \mu(1+\mathbf{ab}, i(1-\mathbf{ab}), \mathbf{a}-\mathbf{b}, \mathbf{a}+\mathbf{b}) dw + X(z_0), \\ \mu^2 &= -\frac{\overline{\mathbf{a}_{\overline{z}}} \cdot \overline{\mathbf{b}_{\overline{z}}}}{h|1-\mathbf{a}\overline{\mathbf{b}}|^4}. \end{aligned}$$

Let H be the mean curvature vector of the spacelike surface S given by $X(z)$ above. It remains to show that $\langle H, H\rangle = h$. From Corollary 4.2, it follows that

$$S_1(\mathbf{a},\mathbf{b}) + S_2(\mathbf{a},\mathbf{b}) = (\log\langle H, H\rangle)_{\overline{z}}.$$

Since $\langle H, H\rangle$ and h are nonzero real numbers we have that

$$(\log\langle H, H\rangle)_{\overline{z}} = (\log h)_{\overline{z}} \;\Leftrightarrow\; \left(\log \frac{\langle H, H\rangle}{h}\right)_{\overline{z}} = 0 \;\Rightarrow\; \langle H, H\rangle = ch,$$

where c is a nonzero constant. By a homothety of $\mathbb{L}^4$, we obtain $\langle H, H\rangle = h$. This completes the proof of the Theorem.

□

Example 5.4. Consider $M^2 = \mathbb{C}$,

$$\mathbf{a}(z) = \sqrt{k} e^{\frac{z+\overline{z}}{2t}}, \quad \mathbf{b}(z) = \frac{k}{\sqrt{k}} e^{-\frac{z+\overline{z}}{2t}}, \quad h = \frac{1-t^2}{4t^4},$$

where $z = u + iv \in \mathbb{C}$ and $k = \dfrac{1-t}{1+t}$, with $t \in (-1,1) \setminus \{0\}$. Since $\mathbf{a}_{\overline{z}} = \dfrac{\mathbf{a}(z)}{2t}$, $\mathbf{b}_{\overline{z}} = -\dfrac{\mathbf{b}(z)}{2t}$ and $|1-\mathbf{ab}|^4 = (1-k)^4$, from (57) we have that

$$\overline{\mu}^2 = \frac{k}{4t^2 h} \left(\frac{1+t}{2t}\right)^4 = \frac{k(1+t)^4}{2^4 t^2 (1-t^2)} \;\Rightarrow\; \mu = \frac{1+t}{4t}.$$

Moreover,

$$\Phi(z) = \left(\frac{2}{1+t},\ \frac{2t}{1+t} i,\ \mathbf{a}(z) - \frac{k}{\mathbf{a}(z)},\ \mathbf{a}(z) + \frac{k}{\mathbf{a}(z)} \right).$$

It follows that

$$\Phi(z) = \frac{2t}{1+t} \left(\frac{1}{t},\ i,\ \frac{\sqrt{1-t^2}}{t} \sinh \frac{z+\overline{z}}{2t},\ \frac{\sqrt{1-t^2}}{t} \cosh \frac{z+\overline{z}}{2t} \right).$$

Thus,

$$X_z = \frac{1}{2} \left(\frac{1}{t},\ i,\ \frac{\sqrt{1-t^2}}{t} \sinh \frac{z+\overline{z}}{2t},\ \frac{\sqrt{1-t^2}}{t} \cosh \frac{z+\overline{z}}{2t} \right).$$

Therefore,

$$X_t(u,v) = \left(\frac{u}{t},\ -v,\ \sqrt{1-t^2} \cosh \frac{u}{t},\ \sqrt{1-t^2} \sinh \frac{u}{t} \right).$$

Letting $e_1 := X_u$ and $e_2 := X_v$, from $H = \frac{1}{2} \left(\overline{\nabla}_{e_1} e_1 \right)^{\perp} = \frac{1}{2}(X_{uu})^{\perp} = \frac{1}{2} X_{uu}$, we conclude that $H = \frac{\sqrt{1-t^2}}{2t^2}(0, 0, \cosh \frac{u}{t}, \sinh \frac{u}{t})$. Hence,

$$\langle H, H \rangle = \frac{1-t^2}{4t^4}.$$

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