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MAXIMUM PRINCIPLES AT INFINITY ON RIEMANNIAN MANIFOLDS: AN OVERVIEW

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To Renato on his 60th birthday

Introduction

Maximum principles at infinity in the spirit of H. Omori and S.T. Yau are related to a number of properties of the underlying Riemannian manifold, ranging from the realm of stochastic analysis to that of geometry and PDEs. We will survey some of these interplays, with a special emphasis on results recently obtained by the authors, and we shall move a first step in some quite new directions. We will also present crucial applications of the maximum principles both to analytic and to geometric problems. Along the way, we will take the opportunity to introduce some unanswered questions that we feel are interesting for a deeper understanding of the subject.

1 Stochastic completeness and the weak maximum principle

We recall that stochastic completeness is the property for a stochastic process to have infinite (intrinsic) life-time. In other words, the total probability of the particle being found in the state space is constantly equal to 1. From now on,

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until otherwise specified, we shall concentrate only on the Brownian particle. A classical analytic condition to express stochastic completeness is given in the following

Definition 1 *A Riemannian manifold $(M, \langle\,,\rangle)$ is called stochastically complete if, for some (and hence any) $(x,t) \in M \times (0,+\infty)$*

$$\int_M p(x,y,t)\,dy = 1, \tag{1}$$

where $p(x,y,t)$ is the (minimal) heat kernel of the Laplace-Beltrami operator Δ.

We note that in this definition the metric $\langle\,,\rangle$ is not assumed to be geodesically complete. Indeed, following J. Dodziuk, [6], one can construct a minimal heat kernel on an arbitrary Riemannian manifold as the supremum of the Dirichlet heat kernels on an exhausting sequence of relatively compact domains with smooth boundary. The analytic condition expressed in (1) is equivalent to a number of other properties for instance reported in [9] or [22]. In what follows we will be interested in the next recently proved characterization, [24].

Theorem 2 *A Riemannian manifold $(M, \langle\,,\rangle)$ is stochastically complete if and only if the following holds. For every $u \in C^2(M)$ with $u^* = \sup_M u < +\infty$, and for every $\gamma < u^*$,*

$$\inf_{\Omega_\gamma} \Delta u \leq 0, \tag{2}$$

where $\Omega_\gamma = \{x \in M : u(x) > \gamma\}$.

We shall refer to the above property as to the validity of the weak maximum principle (at infinity).

It is immediate to realize that this "function theoretic" characterization of stochastic completeness often enables us to analyze consequences of this latter in a simple way. This is the case, for instance, of Khas'minskii's test; for a proof which uses the maximum principle approach see [22]. See also Theorem 20 below.

Theorem 3 *Let $(M, \langle\, , \rangle)$ be a Riemannian manifold and assume that M supports a C^2 function $\gamma(x)$ such that*

$$\gamma(x) \to +\infty, \text{ as } x \to \infty \tag{3}$$

and

$$\Delta\gamma \leq \lambda\gamma \tag{4}$$

for some $\lambda > 0$, outside a compact set. Then $(M, \langle\, , \rangle)$ is stochastically complete.

Note that, since the manifold is not assumed to be geodesically complete, condition (3) means that $\gamma(x)$ gets positive and arbitrarily large outside compact sets.

It is also worth to observe that condition (4) is equivalent to

$$\Delta\gamma \leq f(\gamma) \tag{5}$$

outside a compact set, with

$$\text{i) } f \in C^1(\mathbb{R}), \qquad \text{ii) } f(t) > 0 \text{ for } t >> 1, \qquad \text{iii) } \frac{1}{f(t)} \in L^1(+\infty)$$

and

$$\text{iv) } f'(t) \geq (1-A)\prod_{j=1}^{N} \log^{(j)} t,$$

for some $N \geq 1$, $A > 0$, $t >> 1$ and where $\log^{(j)}$ is the j-th iterated logarithm.

If we restate the weak maximum principle in the form:

For every $u \in C^2(M)$ with $u^* = \sup_M u < +\infty$, there exists a sequence $\{x_n\} \subset M$ such that, for every n,

$$\text{i) } u(x_n) > u^* - \frac{1}{n}, \qquad \text{ii) } \Delta u(x_n) < \frac{1}{n}, \tag{6}$$

its relations with the Omori-Yau maximum principle becomes apparent. Indeed, this latter is obtained by adding in (6) the further condition

$$|\nabla u(x_n)| < \frac{1}{n}. \tag{7}$$

In [22] it has been pointed out that (6) together with (7) are a consequence of the following "function theoretic" setting:

Theorem 4 *Let $(M, \langle\,,\rangle)$ be a Riemannian manifold and assume that there exists a C^2 function γ and a compact set $K \subset M$ such that*

$$\gamma(x) \to +\infty, \text{ as } x \to \infty \tag{3}$$

$$|\nabla \gamma| \leq A\sqrt{\gamma}, \text{ on } M \backslash K, \tag{8}$$

for some $A > 0$,

$$\Delta \gamma \leq B\sqrt{\gamma G(\sqrt{\gamma})}, \text{ on } M \backslash K, \tag{9}$$

for some $B > 0$, where G is a smooth function on $[0, +\infty)$ satisfying

$$\begin{array}{ll} \text{i) } G(0) > 0, & \text{ii) } G'(t) \geq 0 \text{ on } [0, +\infty) \\ \text{iii) } \dfrac{1}{\sqrt{G(t)}} \notin L^1(+\infty) & \text{iv) } \limsup\limits_{t \to +\infty} \dfrac{tG(\sqrt{t})}{G(t)} < +\infty. \end{array} \tag{10}$$

Then (6) and (7) are satisfied.

Remark 5 (a) The choice of $G(t) = t^2$ yields (4) in Khas'minskii test

(b) Although in the statement of Theorem 4 $(M, \langle\, , \rangle)$ is not requiered to be geodesically complete, this follows by (3) and (8). See Remark 1.12 in [22].

(c) A significant choice of the function G is, for instance,

$$G(t) = t^2 \prod_{j=1}^{N} \left(\log^{(j)} t\right)^2, \; t >> 1$$

and some $N \geq 1$.

This "function theoretic" approach to the Omori-Yau maximum principle enables us to apply it in different situations directly related to the geometric setting at hand. The next are two typical examples respectively of an intrinsic and extrinsic nature.

Example 6 Let G be a smooth function, which is even at the origin, i.e., $G^{(2k+1)}(0) = 0$ for $k = 0, 1, ...,$ and satisfies the conditions listed in (10). Assume that

$$Ricci\,(\nabla r, \nabla r) \geq -G\,(r\,(x))\,,$$

where $r(x) = dist_{(M,\langle\, , \rangle)}(x, o)$ for some origin $o \in M$. Assume that $(M, \langle\, , \rangle)$ is geodesically complete and set $\gamma(x) = r(x)^2$. Then (3), (8) and (9) are satisfied.

Note that the fact that G is even at the origin is used to construct an appropriate model manifold in the sense of R. Greene and H.H. Wu, [12], and to estimate Δr from above.

Example 7 Let $f : (M, \langle\, , \rangle) \to (N, (\, ,))$ be an isometric immersion. Let $p \in N$ and assume that $f(M) \cap \mathrm{cut}\,(p) = \emptyset$. Let G be a function with the properties listed in Example 6 and set $\rho(y) = dist_{(N,(,))}(y, p)$. Suppose that the radial sectional curvature of N, say ${}^{N}K_{rad}$, satisfies

$$ {}^{N}K_{rad} \geq -G\left(\rho\left(x\right)\right). $$

Let $H\left(x\right)$ be the mean curvature vector of the immersion f. If

$$ \left|H\left(x\right)\right| \leq B\sqrt{G\left(\rho \circ f\left(x\right)\right)} $$

for some constant $B > 0$ in the complement of a compact set of M, then $\gamma\left(x\right) = \rho^2 \circ f\left(x\right)$ satisfies (8) and (9). If we further assume that the immersion is proper, then (3) is satisfied too.

Note that an example of D. Stroock, [31] page 133, shows that there exists a proper, isometric, minimal immersion into Euclidean space with a bad behavior of the curvature. Indeed, in this example, one can prove that along a special sequence $\{x_n\} \subset M$ one has

$$ {}^{M}K_{rad}\left(x_n\right) \leq -Ce^{r(x_n)}, $$

for some constant $C > 0$ and n sufficiently large. This shows that Examples 6 and 7 are somewhat independent.

The following observation shows that in Example 6 the requirements on the lower bound of the radial Ricci curvature cannot be relaxed. To fix ideas, let M_g be a surface model in the sense of Greene-Wu. Define on $(0, +\infty) \times S^1$ the metric

$$ \langle\,,\rangle = dr^2 + g\left(r\right)^2 d\theta^2 \tag{11} $$

and choose the function $g\left(r\right)$ to be smooth on $[0, +\infty)$, positive on $(0, +\infty)$ and such that

$$ g\left(r\right) = \begin{cases} r & \text{on } [0,1] \\ r\left(\log r\right)^{1+\mu} e^{r^2(\log r)^{1+\mu}} & \text{on } [3,+\infty), \end{cases} $$

for some $\mu > 0$. The above metric extends to a geodesically complete metric on $M_g = \mathbb{R}^2$. Note that its Gaussian curvature satisfies

$$K_{\langle\,,\rangle}(x) \asymp -r(x)^2 (\log r(x))^{2(1+\mu)}, \text{ as } r(x) \to +\infty.$$

Therefore the growth of the Gaussian curvature barely fails to meet the requirements in Example 6. On the other hand, the function

$$u(x) = \int_0^{r(x)} \frac{1}{g(s)} \left(\int_0^s g(t)\, dt \right) ds$$

is of class C^2 on M_g and bounded above because $\mu > 0$. A simple computation yields

$$\Delta u \equiv 1 \text{ on } M_g. \tag{12}$$

Thus, the Omori-Yau maximum principle does not hold on M_g.

Also note that M_g is an example of a geodesically complete manifold which is not stochastically complete because (12) prevents the validity of weak maximum principle too. Here, the fast decay of the curvature to $-\infty$ produces a drift which sweeps a Brownian particle to infinity in a finite time. On the other hand, stochastic completeness does not imply geodesic completeness. For instance, $\mathbb{R}^m \backslash \{0\}, m \geq 3$, with the Euclidean metric is geodesically incomplete, and yet it is stochastically complete. Indeed, consider the smooth function on $\mathbb{R}^m \backslash \{0\}$

$$\gamma(x) = |x|^2 + |x|^{2-m}.$$

A direct computation gives

$$\Delta \gamma(x) = 2m.$$

Hence, we can apply the Khas'minskii test with (4) replaced by (5) and $f(t) = 2m$, to conclude that $\mathbb{R}^m \backslash \{0\}$ is stochastically complete.

Thus, stochastic completeness and geodesic completeness are two independent concepts.

The above example also shows that the "function theoretic" requests of the Khas'minskii test do not imply geodesic completeness. This is in contrast with those of Theorem 4 which force the metric to be complete. We shall come back to this point in Section 3.

It is clear that assumptions that guarantee the validity of the Omori-Yau maximum principle also yield that of the weak maximum principle. We now look for a more specific condition which implies the validity of the weak maximum principle, or equivalently, that of stochastic completeness.

Towards this aim, we consider an m-dimensional model manifold M_g with metric in polar coordinates given by (11). Then,

$$\mathrm{vol}\partial B_r = \omega_m g(r)^{m-1}, \qquad \mathrm{vol} B_r = \omega_m \int_0^r g(t)^{m-1}\, dt,$$

where ω_m denotes the volume of the unite sphere in $\mathbb{R}^m$. We define , similarly to what we did above,

$$\gamma(x) = \int_0^{r(x)} \frac{\mathrm{vol} B_t}{\mathrm{vol}\partial B_t} dt. \tag{13}$$

Since, outside the origin, $\Delta r = (m-1) g'(r)/g(r)$, a computation gives

$$\Delta\gamma(x) \equiv 1 \text{ on } M_g, \tag{14}$$

and if

$$\frac{\mathrm{vol} B_t}{\mathrm{vol}\partial B_t} \notin L^1(+\infty), \tag{15}$$

then, by the Khas'minskii test, M_g is stochastically complete. On the other hand, if (15) is false, $\gamma(x)$ is bounded and (14) shows that the maximum principle at infinity does not hold. We have thus proved the following

Proposition 8 *Let M_g be a model manifold. Then M_g is stochastically complete if and only if (15) holds.*

It is an open conjecture that (15) is a sufficient condition for a general complete Riemannian manifold $(M, \langle\,,\rangle)$ to be stochastically complete. In this respect, it had been shown by A. Grigor'yan, [10], that the slightly stronger assumption

$$\frac{r}{\log \operatorname{vol} B_r} \notin L^1(+\infty) \tag{16}$$

forces a complete manifold $(M, \langle\,,\rangle)$ to be stochastically complete. This can be seen as a consequence of the following result proved in [22] page 52.

Theorem 9 *Let $(M, \langle\,,\rangle)$ be a geodesically complete manifold satisfying*

$$\frac{r^{1-\mu}}{\log \operatorname{vol} B_r} \notin L^1(+\infty), \tag{17}$$

for some $\mu \in \mathbb{R}$. Then, for every $u \in C^2(M)$ with $u^* = \sup_M u < +\infty$, and for every $\gamma < u^*$,

$$\inf_{\Omega_\gamma} (1+r)^\mu \Delta u \le 0, \tag{18}$$

where

$$\Omega_\gamma = \{x \in M : u(x) > \gamma\}.$$

In order to prove stochastic completeness, we can also use comparison with a suitable model, and the aid of the next theorem. For a proof we refer to [9] and [23].

Theorem 10 *Let $(M, \langle\,,\rangle)$ be a geodesically complete manifold of dimension $m = \dim M$, $r(x) = dist_{(M,\langle\,,\rangle)}(x, o)$, $o \in M$ a fixed origin. Let $g \in C^\infty([0, +\infty))$,*

$g^{(2k)}(0) = 0$ *for every* $k = 0, 1, 2, \ldots$, $g'(0) = 1$, $g(t) > 0$ *on* $(0, +\infty)$ *and consider the corresponding model* M_g *of the same dimension* m. *Assume that*

$$\Delta r(x) \leq (m-1)\frac{g'(r(x))}{g(r(x))} \tag{19}$$

holds on $M \backslash (\{o\} \cup cut(o))$. *If* M_g *is stochastically complete, then* $(M, \langle\, , \rangle)$ *is stochastically complete.*

As a consequence we have the following result which detects the "maximum amount" of negative curvature that one can allow without destroying stochastic completeness.

Theorem 11 *Let* $(M, \langle\, , \rangle)$ *be a geodesically complete manifold with radial Ricci curvature satisfying*

$$Ricci(\nabla r, \nabla r) \geq -(m-1)G(r(x)) \ \text{ on } M \tag{20}$$

for some positive, non-decreasing function G *with*

$$\frac{1}{\sqrt{G(t)}} \notin L^1(+\infty). \tag{21}$$

Then, $(M, \langle\, , \rangle)$ *is stochastically complete.*

Proof: Without loss of generality, we can assume that G is smooth and odd at the origin, i.e., $G^{(2k+1)}(0) = 0$, $k = 0, 1, 2, \ldots$.We let g be the solution of the Cauchy problem

$$\begin{cases} g''(t) - G(t)g(t) = 0 & \text{on } [0, +\infty) \\ g(0) = 0,\ g'(0) = 1. \end{cases} \tag{22}$$

Thus, g is smooth, positive outside zero and even at the origin, since G is odd. We construct the model M_g. Letting $r(x) = dits_{(M,\langle\, , \rangle)}(x, o)$ by the Laplacian comparison theorem

$$\Delta r(x) \leq (m-1)\frac{g'(r(x))}{g(r(x))} \text{ on } M \backslash (\{o\} \cup cut(o)).$$

To obtain the desired conclusion, it is enough to show that the model M_g is stochastically complete. Towards this end, we define

$$h(r) = \frac{1}{D\sqrt{G(0)}} \left\{ e^{D\int_0^r \sqrt{G(s)}ds} - 1 \right\}$$

and we observe that, since $G' \geq 0$, we can choose $D > 0$ sufficiently large that h is a sub-solution of (22). By the Sturm comparison theorem we have

$$^{M_g}\Delta r = (m-1)\frac{g'(r)}{g(r)} \leq (m-1)\frac{h'(r)}{h(r)}$$

on M_g, and since

$$\frac{h'(r)}{h(r)} \asymp \sqrt{G(r)}, \text{ as } r \to +\infty$$

we conclude that

$$\frac{g'(r)}{g(r)} \leq C\sqrt{G(r)}$$

for $r > 1$ and for some constant $C > 0$. Now apply the Khas'minskii test on M_g with the γ defined as in (13) with $\mathrm{vol}_{M_g}\partial B_t = c_m g(t)^{m-1}$ and $\mathrm{vol}_{M_g} B_t = c_m \int_0^t g(s)^{m-1}ds$. Note that since $G(t) \geq 0$, the function $g(t)$ diverges to infinity as $t \to +\infty$, and one shows that $\gamma(t) \to +\infty$ using (21) and de L'Hospital's rule.

□

We note that, in [23], we proved that for a stochastically complete model M_g we can find γ satisfying (3) and (4) of Theorem 2, i.e., in a model manifold the requirements of the Khas'minskii test are necessary and sufficient for stochastic completeness.

Question 12 Is this fact true on a general geodesically complete manifold $(M, \langle\,,\rangle)$?

We have no answer, but we doubt that this is the case. As we shall see in Theorem 21 below, this question is motivated by the fact that an analogous characterization of parabolicity due to Kuramochi-Nakai, [17], [19], holds.

2 Some applications of the maximum principle at infinity

The aim of this section is to show the usefulness of the maximum principle at infinity both in the weak and in the Omori-Yau form, and to justify a possible extension, with examples in different settings.

Estimates of the extrinsic diameter of a bounded immersion. Let $(N,(\,,))$ be a Riemannian manifold and $q \in N$. A geodesic ball $B_R(q) \subset N$ is said to be regular if the following two conditions are satisfied: (a) $B_R(q) \cap cut(q) = \emptyset$ and (b) denoting by ${}^N K_p$ the supremum of the sectional curvatures of N at p, one has

$$\max\left\{0, \sup_{B_R(q)} {}^N K_p\right\} < \frac{\pi}{2R}.$$

The following result generalizes and extends previous work of many authors. We limit ourselves to quote papers by T. Hasanis and D. Koutroufiotis, [13], L.P. Jorge and F. Xavier, [14], and L. Karp, [16].

Theorem 13 *Let $B_R(q)$ be a regular geodesic ball of $(N,(\,,))$ such that, for some $k \in \mathbb{R}$,*

$$\sup_{B_R(q)} {}^N K_p \leq k.$$

Let $f : (M, \langle\,,\rangle) \to B_R(q)$ be a smooth map with tension field $\tau(f)$ satisfying

$$|\tau(f)| \leq \tau_0,$$

for some $\tau_0 > 0$. *Assume that* $(M, \langle\,,\rangle)$ *is stochastically complete. Then, denoting by* $e(f) = \frac{1}{2}|df|^2$ *the energy density of* f, *we have*

$$R \geq \begin{cases} k^{-\frac{1}{2}} \arctan\left(2\tau_0^{-1}\sqrt{k}\inf_M e(f)\right) & \text{if } k > 0 \\ 2\tau_0^{-1}\inf_M e(f) & \text{if } k = 0 \\ (-k)^{-\frac{1}{2}} \operatorname{arcth}\left(2\tau_0^{-1}\sqrt{-k}\inf_M e(f)\right) & \text{if } k < 0. \end{cases} \tag{23}$$

Proof: We consider the case $k < 0$, the other cases being similar. Let $\rho(y) = dist_{(N,(,))}(y, q)$. Since ${}^N K_p \leq k$ on $B_R(q)$, the Hessian comparison theorem yields

$$Hess(\rho) \geq \sqrt{-k}\coth\left(\sqrt{-k}\rho\right)\{(,) - d\rho \otimes d\rho\}. \tag{24}$$

To simplify the writings we set $k = -1$ and let $u = \frac{1}{2}\cosh(\rho \circ f)$. Computations give

$$\Delta u = \sum_{i=1}^{m} Hess\left(\frac{1}{2}\cosh\rho\right)(df(e_i), df(e_i)) + d\left(\frac{1}{2}\cosh\rho\right)(\tau(f)),$$

where $\{e_i\}$ is a local orthonormal frame on M. From (24) we then deduce

$$\Delta u \geq u\{2e(f) + \tanh(\rho \circ f)(\nabla\rho, \tau(f))\}. \tag{25}$$

Since $u \geq 1/2$ and

$$-\tanh(R) \leq \tanh(\rho \circ f)(\nabla\rho, \tau(f)),$$

using Schwarz inequality we have:

$$\Delta u \geq \inf_M e(f) - \frac{1}{2}\tanh(R)\tau_0. \tag{26}$$

Since $(M, \langle\, ,\rangle)$ is stochastically complete we apply the weak maximum principle to obtain

$$\inf_M e(f) \leq \frac{1}{2} \tanh(R)\, \tau_0.$$

□

The above theorem can be used to give, in the negative, an answer to the following question of E. Calabi: does there exist a geodesically complete minimal hypersurface of $\mathbb{R}^{m+1}$ with bounded image?

Indeed, let $f : M^m \to \mathbb{R}^{m+1}$ be a minimal isometric immersion. Then $\tau(f) = 0$ and $e(f) = m/2$. Thus, if $(M, \langle\, ,\rangle)$ is stochastically complete, Theorem 13 applies to infer that $f(M)$ is unbounded.

We point out that, in case of surfaces $m = 2$, in a recent paper, [20], N. Nadirashvili has exhibited an example solving in the affirmative the above standing problem; see also [5]. However, stochastic completeness is a really mild geometric assumption on $(M, \langle\, ,\rangle)$ which does not even require geodesic completeness and, in our opinion, the search of "the best geometric conditions" on a minimal immersion $f : M^m \to \mathbb{R}^{m+1}$ in order that $f(M)$ be unbounded remains a challenging task. In this respect we quote the striking paper [4] by T. Colding a W. Minicozzi where (a more "ambitious" version of) the Calabi conjecture is proved to be true for embedded minimal surfaces of finite topology.

In the same vein we quote the next consequence of Theorem 13 (compare with [8]).

Theorem 14 *Let $f : (M, \langle\, ,\rangle) \to \mathbb{R}^n$ be an m-dimensional submanifold with parallel mean curvature and suppose that $(M, \langle\, ,\rangle)$ is stochastically complete.*

Let $R = \pi/2$ if $n - m = 1$ or $R = \pi/2\sqrt{2}$ if $n - m > 1$. Finally, let $\gamma_f : M \to G_m(\mathbb{R}^n)$ be the Gauss map of f and assume that there exists a decomposable n-vector q such that

$$\langle \gamma_f, q \rangle \geq \left\{ \cos\left(\frac{R}{\sqrt{n-m}} \right) \right\}^{n-m} \quad \text{on } M.$$

Then f is minimal.

Comparison and Liouville-type theorems for solutions of differential inequalities. We now discuss an extension of the weak maximum principle and we give an application to the uniqueness problem for the Yamabe equation and to some Liouville problems for general differential inequalities.

Towards this aim, we consider the following example. On $\mathbb{R}^m$ with its Euclidean metric consider the function $u(x) = |x|^{\alpha}$ where $\alpha \geq 0$. Then,

$$\Delta u = \alpha (m - 2 + \alpha) |x|^{\alpha - 2}.$$

Thus

$$\begin{cases} \text{if } \alpha > 2, & \Delta u \text{ is unbounded} \\ \text{if } \alpha = 2, & \Delta u = 2m \\ \text{if } \alpha < 2, & \inf_{\mathbb{R}^m} \Delta u = 0. \end{cases}$$

This suggests that we should look for "a weak maximum principle" in the following form:

Let $\Lambda \geq 0$, $\delta, \mu \in \mathbb{R}$. For each $u \in C^2(M)$ non-constant and such that

$$\hat{u} = \limsup_{r(x) \to +\infty} \frac{u(x)}{r(x)^{\sigma}} < +\infty, \qquad \Omega_{\gamma} = \{x \in M : u(x) > \gamma\} \neq \emptyset \tag{27}$$

then

$$\inf_{\Omega_{\gamma}} (1 + r)^{\mu} \Delta u \leq \Lambda \max\{\hat{u}, 0\}. \tag{28}$$

Note that, for $\Lambda = \sigma = 0$, this is precisely the conclusion of Theorem 9 and in the further assumption $\mu = 0$ we obtain the original form of the weak maximum principle given before.

In [22] (see also [28]) we proved

Theorem 15 *Let $(M, \langle\, ,\rangle)$ be a complete Riemannian manifold, and let $\mu, \sigma \in \mathbb{R}$ satisfy*

$$\sigma \geq 0, \qquad \sigma + \mu < 2. \tag{29}$$

Assume that

$$\liminf_{r\to+\infty} \frac{\log \mathrm{vol} B_r}{r^{2-\sigma-\mu}} = d_o < +\infty. \tag{30}$$

Then, for every $u \in C^2(M)$ such that

$$\hat{u} = \limsup_{r(x)\to+\infty} \frac{u(x)}{r(x)^\sigma} < +\infty \tag{31}$$

and $\gamma \in \mathbb{R}$ with

$$\Omega_\gamma = \{x \in M \,:\, u(x) > \gamma\} \neq \emptyset,$$

we have

$$\inf_{\Omega_\gamma} [1 + r(x)]^\mu \Delta u(x) \leq d_o \max\{\hat{u}, 0\} C(\sigma, \mu), \tag{32}$$

with

$$C(\sigma,\mu) = \begin{cases} 0 & \text{if } \sigma = 0 \\ (2-\sigma-\mu)^2 & \text{if } 2(1-\sigma) > \mu,\ \sigma > 0 \\ \sigma(2-\sigma-\mu) & \text{if } 2(1-\sigma) \leq \mu,\ \sigma > 0. \end{cases}$$

Observe that for $\sigma = 0$ we obtain the conclusion of Theorem 9 under assumption (30) which is slightly stronger that the corresponding (17). However since d_0 appears in (32), when $\sigma > 0$ there is no hope to relax (30) to a condition of the type (17).

A direct application of Theorem 15 yields Liouville-type properties for entire solutions of certain differential inequalities on the complete manifold $(M, \langle\, ,\rangle)$. For instance, we point out the following

Corollary 16 *Let $b(x) \in C^0(M)$ satisfy*

$$b(x) \geq \frac{C}{(1+r(x))^{\mu}} \text{ on } M$$

for some constants $C \geq 1$ and $\mu \in \mathbb{R}$. Let $f \in C^0(\mathbb{R})$ be a non-decreasing function and suppose that, for some $\sigma \geq 0$ with $\sigma + \mu < 2$, we have

$$\liminf_{r \to +\infty} \frac{\log \mathrm{vol} B_r}{r^{2-\sigma-\mu}} < +\infty. \tag{33}$$

Then, any entire solution $u \in C^2(M)$ of the equation

$$\Delta u = b(x) f(u) \tag{34}$$

such that

$$u(x) = o\left(r(x)^{\sigma}\right), \text{ as } r(x) \to +\infty \tag{35}$$

satisfies

$$f(u(x)) \equiv 0 \text{ on } M.$$

Note that, as a consequence, the function $u(x)$ in the above statement is harmonic. Thus, in particular, if $(M, \langle\, ,\rangle)$ satisfies the strong Liouville property, namely every positive harmonic function on M is constant, and (33) holds with $\sigma = 1$, then we conclude that any entire sublinear solution of (34) is constant, generalizing a well known result of $\mathbb{R}^m$.

Proof: Clearly, it suffices to prove that, if u solves

$$\Delta u \geq b(x) f(u), \tag{36}$$

then

$$f\left(u\left(x\right)\right) \leq 0 \text{ on } M.$$

Indeed, we can always reduce to this situation for then, replacing $u\left(x\right)$ and $f\left(t\right)$ with $\tilde{u}\left(x\right) = -u\left(x\right)$ and $\tilde{f}\left(t\right) = -f\left(-t\right)$, we conclude that $f(u(x)) = 0$ on M, as required.

Now, in the case where $u^{*} = \sup_{M} u < +\infty$, by Theorem 15, $f\left(u^{*}\right) \leq 0$. Since f is non-decreasing, and $u(x) \leq u^{*}$, we deduce that, for each $x \in M$, $f\left(u\left(x\right)\right) \leq 0$. Suppose then that $u^{*} = +\infty$, so that for every fixed $\gamma > 0$, we have $\Omega_{\gamma} \neq \emptyset$. By Theorem 15 with $\hat{u} = 0$, for every $\varepsilon > 0$ there exists $x_{0} \in \Omega_{\gamma}$ such that $f\left(u\left(x_{0}\right)\right) < \varepsilon$. Since f is non-decreasing and $u\left(x_{0}\right) > \gamma$, we deduce that $f\left(x\right) < \varepsilon$ for every $x \in M \backslash \Omega_{\gamma} = \left\{x \in M : u\left(x\right) \leq \gamma\right\}$. Letting $\gamma \nearrow +\infty$ gives $M \backslash \Omega_{\gamma} \nearrow M$ and $f\left(u\left(x\right)\right) \leq \varepsilon$ on M. Now let $\varepsilon \searrow 0$ to conclude that $f\left(u\left(x\right)\right) \leq 0$ on M, as desired.

□

To show why a weak maximum principle of the general form of Theorem 15 is interesting we also consider the Yamabe type equation

$$\Delta u + a\left(x\right) - b\left(x\right) u^{p} = 0, \tag{37}$$

where $p > 1$, $a\left(x\right), b\left(x\right) \in C^{0}\left(M\right)$. Extending on [27] we shall prove the following result; see [29].

Theorem 17 *Let $(M, \langle\,,\rangle)$ be a complete manifold, let $a\left(x\right), b\left(x\right) \in C^{0}\left(M\right)$, $p > 1$, $\tau \geq 0$, $\beta + \tau\left(p-1\right) > -2$ and suppose that $b\left(x\right) \geq a\left(x\right)$ and*

$$i)\ \liminf_{r(x)\to+\infty} \frac{b\left(x\right)}{r\left(x\right)^{\tau}} > 0, \qquad ii)\ \limsup_{r(x)\to+\infty} \frac{a_{-}\left(x\right)}{b\left(x\right)} r\left(x\right)^{\tau(1-p)} < +\infty. \tag{38}$$

Let $u, v \in C^{2}\left(M\right)$ be non-negative solutions of

$$\Delta u + a\left(x\right) u - b\left(x\right) u^{p} \geq 0 \geq \Delta v + a\left(x\right) v - b\left(x\right) v^{p} \tag{39}$$

on M, satisfying

$$i)\ \liminf_{r(x)\to+\infty}\frac{v(x)}{r(x)^{\tau}}>0,\qquad ii)\ \limsup_{r(x)\to+\infty}\frac{u(x)}{r(x)^{\tau}}<+\infty. \tag{40}$$

If

$$\liminf_{r\to+\infty}\frac{\log \mathrm{vol} B_r}{r^{2+\beta+\tau(p-1)}}<+\infty, \tag{41}$$

then $u(x)\le v(x)$ on M.

Proof: We suppose $u(x)\not\equiv 0$ otherwise there is nothing to prove. Next, we observe that, by (39) and the strong maximum principle, $v(x)>0$ on M. This fact, $u(x)\not\equiv 0$ and (40) imply

$$0<\xi=\sup_M\frac{u(x)}{v(x)}<+\infty.$$

If $\xi\le 1$ then $u(x)\le v(x)$. We assume by contradiction $\xi>1$ and define

$$\varphi=u-\xi v.$$

Note that $\varphi\le 0$ on M. We claim that

$$\sup_M\frac{\varphi(x)}{r(x)^{\tau}}=0. \tag{42}$$

Indeed, let $\{x_n\}\subset M$ be a sequence along which u/v tends to ξ. Then

$$\frac{\varphi(x_n)}{r(x_n)^{\tau}}=\frac{v(x_n)}{r(x_n)^{\tau}}\left\{\frac{u(x_n)}{v(x_n)}-\xi\right\}. \tag{43}$$

Now, we observe that $v(x_n)/r(x_n)^{\tau}$ is bounded, because otherwise (40) ii) would imply $\xi=0$. From (43) it then follows $\varphi(x_n)/r(x_n)^{\tau}\to 0$ as $n\to+\infty$ proving (42). The next step is to use (39) to obtain

$$\Delta\varphi\ge -a(x)\varphi+b(x)\left(u^p-(\xi v)^p\right)+b(x)v^p\xi\left(\xi^{p-1}-1\right). \tag{44}$$

We have

$$u^p - (\xi v)^p = h\varphi \tag{45}$$

where

$$h(x) = \begin{cases} pu(x)^{p-1} & \text{if } u(x) = \xi v(x) \\ \frac{p}{u(x)-\xi v(x)} \int_{\xi v(x)}^{u(x)} t^{p-1} dt & \text{if } u(x) \neq \xi v(x) \end{cases}$$

is continuous and non-negative on M. Furthermore, since $v(x) r(x)^{-\tau}$ is bounded above on the set $\{x : \varphi(x) > -1\}$ then so is $u(x)r(x)^{-\tau}$, and, changing variables in the integral, it is not hard to show that

$$h(x) \leq Cr(x)^{\tau(p-1)} \tag{46}$$

on $\{x : \varphi(x) > -1\}$, for some constant $C > 0$. Note also that, since $b(x) > 0$ on M, we can rewrite (38) i) as

$$b(x) \geq B(1 + r(x))^{\beta} \text{ on } M, \tag{47}$$

for some appropriate $B > 0$. Using $b(x) > 0$ and (45), from (44) we deduce

$$\frac{1}{b(x)} \Delta\varphi \geq \left(\frac{a_-(x)}{b(x)} + h(x) \right) \varphi(x) + v^p(x) \xi \left(\xi^{p-1} - 1 \right)$$

and, therefore, from $\xi > 1$, (38) ii), (46), (40) i) and $\varphi(x) \leq 0$,

$$\frac{(1 + r(x))^{-p\tau}}{b(x)} \Delta\varphi \geq C(1 + r(x))^{-\tau} \varphi(x) + D\xi \left(\xi^{p-1} - 1 \right) \text{ on } \{x : \varphi(x) > -1\}$$

for some appropriate $C, D > 0$. Next, we may choose $\varepsilon > 0$ sufficiently small so that

$$C(1 + r(x))^{-\tau} \varphi(x) \geq -\frac{1}{2} D\xi \left(\xi^{p-1} - 1 \right) \tag{48}$$

on

$$\Omega_\varepsilon = \{x \in M : \varphi(x) > -\varepsilon\}.$$

Then, on $\Omega_\varepsilon, \Delta\varphi \geq 0$ so that (47) implies

$$(1+r(x))^{-\beta-p\tau}\,\Delta\varphi \geq \frac{(1+r(x))^{-p\tau}}{b(x)}\Delta\varphi,$$

and therfore, since $\xi > 1$,

$$\inf_{\Omega_\varepsilon}(1+r(x))^{-\beta-p\tau}\,\Delta\varphi \geq \frac{1}{2}D\xi\left(\xi^{p-1}-1\right) > 0.$$

This fact together with (41) contradicts Theorem 15.

□

Applications of this result are given, e.g., in the non compact Yamabe problem; see [29].

More differential inequalities: the need of the full Omori-Yau maximum principle. As in Corollary 16 above, let $(M, \langle\, ,\rangle)$ be a complete manifold such that

$$\liminf_{r\to+\infty}\frac{\log \mathrm{vol}B_r}{r^{2-\sigma-\mu}} < +\infty, \tag{33}$$

for some $\sigma \geq 0$, $\mu \in \mathbb{R}$ with $\sigma+\mu < 2$.Assume that $b(x) \in C^0(M)$ satisfies

$$b(x) \geq \frac{C}{(1+r(x))^{\mu}} \text{ on } M,$$

where $C > 0$ and $r(x) = dist_{(M,\langle\, ,\rangle)}(x, o)$ for some origin $o \in M$. Then, the equation

$$\Delta u = b(x)\left(|\nabla u| + a\right),\; a \geq 0, \tag{49}$$

has no entire solutions satisfying (35), namely

$$|u(x)| = o\left(r(x)^{\sigma}\right), \text{ as } r(x) \to +\infty. \tag{35}$$

Indeed, in the above assumptions Theorem 15 implies

$$|\nabla u| + a \leq 0$$

on each superset

$$\Omega_\gamma = \{x \in M : u(x) > \gamma\} \neq \emptyset.$$

Since $a \geq 0$ we immediately obtain a contradiction.

Maintaining the same assumptions on $(M, \langle\,,\rangle)$ and $b(x)$ we now suppose that $a < 0$. Setting $v = -u$,

$$v(x) = o\left(r(x)^\sigma\right), \text{ as } r(x) \to +\infty$$

and

$$\Delta v = b(x)\left(-|\nabla v| - a\right).$$

In order to reach a contradiction as above, we need (32) to be satisfied on a set of the form

$$\Omega_{\gamma,\varepsilon} = \{x \in M : v(x) > \gamma, |\nabla v| < \varepsilon\} \neq \emptyset,$$

for some γ and $0 < \varepsilon < -a$. Thus, in this case, a control on $|\nabla v|$ seems to be necessary. This fact appears difficult when $v^* = \sup_M v = +\infty$. Therefore, we assume $v^* < +\infty$ and we make the further simplification $b(x) \equiv 1$. In this case, if we assume, for instance, that

$$Ricci \geq -G(r(x)) \text{ on } M \tag{50}$$

with

$$G(r) = r^2, \text{ for } r >> 1, \tag{51}$$

then, applying Theorem 4 we have

$$0 \geq \inf_{\Omega_{\gamma,\varepsilon}} \Delta v \geq -\varepsilon - a > 0,$$

obtaining the desired contradiction. Hence, we can conclude that on a complete manifold $(M, \langle\,,\rangle)$ with Ricci tensor satisfying (50) and (51), the equation

$$\Delta u = |\nabla u|^2 + a, \; a < 0$$

has no entire solutions bounded below. We observe that the presence of the factor $b(x)$ in (49) in case $a < 0$ is not a serious problem. Indeed, the above conclusion can now be reached using Theorem A in [25]. The relevant request remains $u_* = \inf_M u > -\infty$, or equivalently, $v^* = \sup_M (-u) < +\infty$. Nevertheless we can argue with the help of the following

Theorem 18 *Let $(M, \langle\, , \rangle)$be a complete manifold with Ricci tensor satisfying*

$$Ricci \geq -(m-1) H^2, \tag{52}$$

for some $H > 0$, where $r(x) = dist_{(M,\langle\, ,\rangle)}(x, o)$ for some origin $o \in M$. Let $b(x) \in C^0(M)$ satisfy

$$b(x) \geq \tilde{b}(r(x)) > 0 \text{ on } M \tag{53}$$

where $\tilde{b}(r) \in C^0([0, +\infty))$, $\tilde{b}(r) > 0$, and $\tilde{b}(r) = Cr^\beta$ for some constants $-1 \leq \beta \leq 0$ and sufficiently large r. Let $g \in C^0([0, +\infty))$ satisfy

$$\liminf_{s \to 0^+} g(s) > 0. \tag{54}$$

Finally, let f be a non-decreasing function. Consider an entire solution $u \in C^2(M)$ of the differential inequality

$$\Delta u \geq b(x) f(u) g(|\nabla u|) \tag{55}$$

satisfying

$$u(x) = \begin{cases} o\left(r(x)^{\beta+1}\right) & as\ r(x) \to +\infty \quad if\ -1 < \beta \leq 0 \\ o(\log r(x)) & as\ r(x) \to +\infty \quad if\ -1 = \beta. \end{cases} \tag{56}$$

Then, for each $x \in M$,

$$f(u(x)) \leq 0. \tag{57}$$

Proof: We reason by contradiction and we assume the existence of $x_0 \in M$ such that $f(u(x_0)) > 0$. As it will become clear from the argument below, there is no loss of generality if we suppose $x_0 = o$. To simplify the exposition we assume that o is a pole of M. The general case can be handled using a classical trick by E. Calabi, [3].

For ease of notation, we set $u_o = u(o)$ and we let $A = f(u_o) > 0$. By the Laplacian comparison theorem and (52),

$$\Delta r(x) \leq (m-1) \frac{h'(r(x))}{h(r(x))}, \tag{58}$$

where $h(r) = H^{-1} \sinh(Hr)$. We define

$$\alpha(r) = \alpha_0 + \int_0^r h^{1-m}(t) \int_0^t h(s)^{m-1} \tilde{b}(s)\, ds dt \tag{59}$$

where $\alpha_0 = \alpha(0) > 0$ and $\alpha'(0) = 0$. It is immediate to verify that $\alpha' \geq 0$ on $[0, +\infty)$ and α solves the equation

$$\alpha'' + (m-1) \frac{h'}{h}(r)\, \alpha' = \tilde{b}(r), \tag{60}$$

Since $\alpha' \geq 0$, (60) and (58) imply

$$\alpha''(r(x)) + \alpha'(r(x))\, \Delta r(x) \leq \tilde{b}(r(x)),$$

and therefore, having defined

$$\varphi(x) = \alpha(r(x)) - \alpha_0, \tag{61}$$

we have

$$\begin{cases} \Delta\varphi \leq b(x) & \text{on } M \\ \varphi \geq 0 \text{ on } M, & \varphi(o) = 0. \end{cases} \tag{62}$$

It is easy to check that

$$\sup_{\mathbb{R}_{\geq 0}} \alpha' < +\infty \tag{63}$$

and

$$\alpha\left(r\right)\geq C\begin{cases} r^{\beta+1} & \text{if } -1<\beta\leq 0\\ \log r & \text{if } -1=\beta,\end{cases} \tag{64}$$

for some appropriate constant $C>0$ and $r>>1$. It follows that

$$\sup_{M}\left|\nabla\varphi\right|=S<+\infty \tag{65}$$

and (56) together with (64) imply that

$$u\left(x\right)=o\left(\varphi\left(x\right)\right),\text{ as } r\left(x\right)\rightarrow+\infty. \tag{66}$$

Now, because of (54), there exist δ, $B>0$ sufficiently small such that, if $|t|<\delta$,

$$g\left(t\right)\geq B. \tag{67}$$

We choose

$$0<\varepsilon<\frac{1}{2}\min\left\{AB,\frac{\delta}{2S}\right\} \tag{68}$$

and we define the function $v\left(x\right)$ on M by setting

$$v\left(x\right)=u\left(x\right)-u_o-\varepsilon\varphi\left(x\right).$$

Then $v\left(o\right)=0$ and (66) implies that $v\left(x\right)$ takes its non-negative maximum v^* at some point $\bar{x}\in M$:

$$v\left(\bar{x}\right)=\max_{M}v=v^*\geq 0.$$

Note that, at $\bar{x}$,

$$u\left(\bar{x}\right)\geq u_o+\varepsilon\varphi\left(\bar{x}\right)\geq u_o \tag{69}$$

and

$$\left|\nabla u\left(\bar{x}\right)\right|=\varepsilon\alpha'\left(r\left(\bar{x}\right)\right)=\varepsilon\left|\nabla\varphi\left(\bar{x}\right)\right|\leq\varepsilon S<\frac{\delta}{2}. \tag{70}$$

Since f is non-decreasing, we deduce from (69) that

$$f\left(u\left(\bar{x}\right)\right)\geq f\left(u_o\right)=A, \tag{71}$$

while (70), together with (67), gives

$$g\left(\left|\nabla v\left(\bar{x}\right)\right|\right) \geq B. \tag{72}$$

Now by (53), (55), (62), (71), (72) we obtain

$$\begin{aligned} 0 &\geq \Delta v\left(\bar{x}\right) = \Delta u\left(\bar{x}\right) - \varepsilon\Delta\varphi\left(\bar{x}\right) \\ &\geq b\left(\bar{x}\right)\left\{f\left(u\left(\bar{x}\right)\right) g\left(\left|\nabla u\left(\bar{x}\right)\right|\right) - \varepsilon\right\} \\ &\geq b\left(\bar{x}\right)\left\{AB - \varepsilon\right\} \\ &> 0, \end{aligned}$$

a contradiction.

□

Applying Theorem 18 to $-u$ with $f(-u) = 1$ and $g(t) = -a - |\nabla(-u)|$, we immediately conclude that in the assumptions on *Ricci* and $b\left(x\right)$ listed in the statement, equation (49) with $a < 0$ has no solutions u on M satisfying (56).

As the reader has certainly noted, in the previous proof the linearity of was crucial in dealing with Δv, and therefore comparing u and φ.

Question 19 Can we obtain the same kind of result for a more general class of non-necessarily linear differential operators? For instance, for the p-Laplace operator Δ_p (with$p > 1$)?

3 Parabolicity

A Riemannian manifold $(M, \langle\, ,\rangle)$ is said to be parabolic if the only subharmonic functions bounded above are the constants.

It is not hard to see, specifying the regularity of the solutions u of $\Delta u \geq 0$, that we can deal equivalently with either weak $W^{1,2}_{loc} \cap C^0$ or classical C^2 solutions.

Suppose now that $(M, \langle\,,\rangle)$ is parabolic. Let $u \in C^2(M)$ be a non-constant function with $u^* = \sup_M u < +\infty$, and choose $\eta < u^*$. Let

$$\Omega_\eta = \{x \in M : u(x) > \eta\}$$

and suppose that

$$\Delta u \geq 0 \text{ on } \Omega_\eta.$$

Pick $0 < \varepsilon < u^* - \eta$ and define

$$v(x) = \max\left\{u(x), \eta + \frac{\varepsilon}{2}\right\}.$$

Then, obviously, $v^* = \sup_M v = u^* < +\infty$ and $v(x)$ is again subharmonic (in the weak sense). Therefore, parabolicity implies that $v \equiv \eta + \varepsilon/2$ on M, contradicting the fact that $v^* = u^* > \eta + \varepsilon/2$. We have thus proved that if $(M, \langle\,,\rangle)$ is parabolic then, for every non-constant $u \in C^2(M)$ with $u^* < +\infty$, and for every $\eta < u^*$ we have

$$\inf_{\Omega_\eta} \Delta u < 0. \tag{73}$$

Since the converse is obviously true we can conclude that $(M, \langle\,,\rangle)$ is parabolic if and only if it satisfies the somewhat stronger version of the weak maximum principle given above. This explains why we expect that parabolicity and stochastic completeness should exhibit strong analogies. We give a few results to illustrate this point. We begin with an analogous of the Khas'minskii test of Theorem 2 providing a simple proof.

Theorem 20 *Let $(M, \langle\,,\rangle)$ be a Riemannian manifold and assume that M supports a C^2 function γ such that*

$$\gamma(x) \to +\infty, \text{ as } x \to \infty \tag{74}$$

and

$$\Delta\gamma \leq 0 \ on \ M\backslash K \tag{75}$$

for some compact set $K \subset M$. Then, $(M, \langle \, , \rangle)$ is parabolic.

Proof: We reason by contradiction and assume that there exists a non-constant function $u \in C^2(M)$ with $u^* = \sup_M u < +\infty$ and such that

$$\Delta u \geq 0, \text{ on } \Omega_\eta$$

for some $\eta < u^*$. By the strong maximum principle u cannot attain its maximum in Ω_η. Thus, $\overline{\Omega_\eta}$ cannot be compact and, by choosing η closer to u^*, if needed, we may assume that $\Omega_\eta \cap K = \emptyset$. Pick $x_0 \in \Omega_\eta$ such that

$$u(x_0) > \eta + \frac{1}{2}(u^* - \eta) = \frac{1}{2}(u^* + \eta)$$

and, by adding and multiplying γ by appropriate constants suppose that

$$\gamma(x_0) < \frac{1}{2}(u^* - \eta), \qquad \gamma(x) > 0$$

on M. Finally, let A be the connected component containing x_0 of the set $\{x \in M : u(x) > \eta + \gamma(x)\}$. Since $\gamma > 0$ and $\gamma(x) \to +\infty$ as $x \to \infty$, $\bar{A}$ is compact subset of Ω_η and

$$\begin{cases} \Delta u \geq 0 \geq \Delta(\gamma + \eta) & \text{on } A \\ u = \eta + \gamma & \text{on } \partial A. \end{cases}$$

Thus, by comparison, $u \leq \eta + \gamma$ on A, contradicting the definition of A.

$\square$

Contrary to the case of stochastic completeness, the validity of the converse of this result is known. For Riemann surfaces it has been established long ago by K. Kuramochi, [17]. The proof of this result has been later simplified by M. Nakai, [19], and works for arbitrary Riemannian manifolds. In fact Kuramochi-Nakai result is even stronger than this.

Theorem 21 *If $(M, \langle\, ,\rangle)$ is parabolic, then there exists $\gamma \in C^2(M)$ satisfying (74) and harmonic outside a compact set.*

We observe that (74) forces us to require the validity of (75) only outside a compact set. For, otherwise, we would get a non-constant , bounded below, superharmonic function, against the parabolicity of M. We note however that when the manifold M has at least two ends, there exist functions γ which are (super-)harmonic on all of M provided condition (74) is replaced with the assumption that γ is proper, i.e.,

$$|\gamma(x)| \to +\infty, \text{ as } x \to \infty.$$

We also point out the following

Proposition 22 *Let $(M, \langle\, ,\rangle)$ and $(N, (\, ,))$ be non compact Riemannian manifolds and assume that there exist compact sets $H \subset M$ and $K \subset N$, and an isometry $\varphi : M\backslash H \to N\backslash K$ which preserves divergent sequences in the ambient spaces. Then $(M, \langle\, ,\rangle)$ is parabolic if and only if so is $(N, (\, ,))$.*

In particular, parabolicity of $(M, \langle\, ,\rangle)$ is not affected by modifying $\langle\, ,\rangle$ inside a compact set of M.

Results similar to those presented for stochastic completeness can be considered for parabolicity. We refer to [22] and [23] for a more complete treatment.

We would like to focus our attention here on some alternative approaches to parabolicity related to vector fields. First, let us recall the following version of the Kelvin-Nevanlinna-Royden criterion, due to T. Lyons and D. Sullivan, [18].

Theorem 23 *The manifold $(M, \langle\, ,\rangle)$ is parabolic if and only if the following holds. Let X be a vector field on M such that* *i)* $|X| \in L^2(M)$,

ii) $\operatorname{div} X \in L^1_{loc}(M)$ *and* $(\operatorname{div} X)_- = \min(\operatorname{div} X, 0) \in L^1(M)$,

iii) $0 \leq \int_M \operatorname{div} X \leq +\infty$.

Then

$$\int_M \operatorname{div} X = 0.$$

We get rid of integrals in the above statements using a minor modification of a proof of Theorem 23 due to V. Gol'dshtein and M. Troyanov, [11].

Theorem 24 *The Riemannian manifold* $(M, \langle\,,\rangle)$ *is parabolic if and only if for each vector field* X *with* $|X| \in L^2(M)$ *and* $\operatorname{div} X \geq 0$*, we have* $\operatorname{div} X = 0$*, on* M*.*

We now go back to functions. Toward this aim, let $(M, \langle\,,\rangle)$ be an oriented, geodesically complete, non-compact manifold. Set

$$W^{1,2}\Lambda^k(M) = \overline{C_c^\infty \Lambda^k(M)}$$

where $\Lambda^k(M)$ is the fiber bundle of k-forms on M and where the closure is intended with respect to the norm

$$\|\omega\|^2 = \int_M |\omega|^2 + \int_M |d\omega|^2.$$

Note that the space thus defined coincides with the space

$$\{\omega \in L^2\Lambda^k(M) : d\omega \in L^2\Lambda^{k+1}(M)\},$$

where $d\omega$ is the distributional exterior differential of ω. Indeed, it is clear that if ω is in $W^{1,2}\Lambda^k(M)$, then it has a distributional exterior differential in L^2: by definition there exists ω_k in $C_c^\infty\Lambda^k(M)$ such that $\omega_k \to \omega$ and $d\omega_k$ converges in L^2 to a $(k+1)$-form α. It follows that for every $\sigma \in C_c^\infty\Lambda^k(M)$

$$\int \langle \omega, \delta\sigma\rangle = \lim_k \int \langle \omega_k, \delta\sigma\rangle = \lim_k \int \langle d\omega_k, \sigma\rangle \to \int \langle \alpha, \delta\sigma\rangle$$

where δ is the formal adjoint of d with respect to the usual L^2 inner product. This shows that the distributional exterior differential of ω is the L^2 $(k+1)$-form α and therefore that $W^{1,2}\Lambda^k(M) \subseteq \{\omega \in L^2 : d\omega \in L^2\}$. The converse is slightly more involved, and requires an approximation procedure which we only outline. First note that if ω is an L^2 k-form with L^2 distributional differential $d\omega$, then by a cut off argument it may be approximated in the $W^{1,2}$ norm with forms with compact support. Indeed, if f_k is a C_c^∞ cut off function which equals one in B_k vanishes off B_{k+1} and such that $|df_k| \leq C$, then clearly, $||f_k\omega - \omega||_{L^2} \to 0$ and $||df_k \wedge \omega||_{L^2} \to 0$, so that $d(f_k\omega) = f_k d\omega + df_k \wedge \omega \to d\omega$ in L^2. We may therefore suppose that ω is compactly supported. Let $e^{-t\Delta}$ be the heat semigroup generated by the de Rham Laplacian Δ acting on forms. Since ω and $d\omega$ are in L^2, then, as $t \to 0$, $e^{-t\Delta}\omega$ converges to ω in the norm of $W^{1,2}\Lambda^k(M)$. A cutoff procedure then shows that there exists a sequence in $C_c^\infty\Lambda^k(M)$ which tends to ω in the given norm, as required to show that $\{\omega \in L^2 : d\omega \in L^2\} \subseteq W^{1,2}\Lambda^k(M)$. As a consequence of this, the space $W^{1,2}\Lambda^k(M)$ is complete. Indeed, let ω_j be a Cauchy sequence in $W^{1,2}\Lambda^k(M)$. Then ω_j and $d\omega_j$ converge in L^2 to L^2 forms ω and α, respectively, and by definition of weak exterior differential it is easily seen that $d\omega = \alpha$, i.e., $\omega \in W^{1,2}\Lambda^k(M)$.

By the definition of $W^{1,2}\Lambda^k(M)$, and the fact that $d^k \circ d^{k-1} = 0$ on $C_c^\infty(\Lambda^k(M))$, the k-th exterior differential d^k extends to a bounded operator, still denoted by d^k,

$$d^k : W^{1,2}\Lambda^k(M) \to W^{1,2}\Lambda^{k+1}(M)$$

and gives rise to the L^2-de Rham cochain complex. Using again the fact that $d^k \circ d^{k-1} = 0$, one shows that the closure of $Im(d^{k-1})$ with respect to the norm of $W^{1,2}$ coincides with the closure in the L^2 norm. The corresponding unreduced and reduced L^2-cohomologies are therefore defined by

$$L^2 H^k(M) = \ker d^k / \operatorname{Im} d^{k-1}; \qquad \overline{L^2 H^k(M)} = \ker d^k / \overline{\operatorname{Im} d^{k-1}}^{L^2}.$$

Furthermore, by the definition of $W^{1,2}\Lambda^k(M)$ and the continuity of d^{k-1},

$$\overline{\operatorname{Im} d^{k-1}}^{L^2} = \overline{dC_c^\infty \Lambda^{k-1}(M)}^{L^2}.$$

The L^2-cohomology of M is said to be reduced at the order k if the k-th L^2-torsion space vanishes, i.e.,

$$L^2 T^k(M) = \overline{\operatorname{Im} d^{k-1}}^{L^2} / \operatorname{Im} d^{k-1} = 0.$$

This happens whenever the inequality

$$\|d\omega\|^2 \geq C_k \|\omega\|^2 \tag{76}$$

holds for every $\omega \in C_c^\infty \Lambda^{k-1}(M)$ and for some constant $C_k > 0$. In case $k = 1$, (76) says precisely that the bottom of the spectrum of the Laplace-Beltrami operator satisfies

$$\lambda_1^{-\Delta}(M) > 0. \tag{77}$$

We observe that when $k = 1$, and $\operatorname{vol}(M) = +\infty$ then (76) is in fact equivalent to the fact that $L^2 T^1(M) = 0$. Indeed, if the latter holds, then $\ker d^0 = \{0\}$ and using the open mapping theorem one realizes that d^0 has bounded inverse, and therefore (76) holds. This was first observed by P. Pansu, [21], in the more general setting of $L^{p,q}$-cohomology.

Now, the Hodge-de Rham-Kodaira decomposition theorem gives the direct sum

$$L^2\Lambda^k(M) = \overline{dC_c^\infty\Lambda^{k-1}(M)}^{L^2} \bigoplus \overline{\delta C_c^\infty\Lambda^{k+1}(M)}^{L^2} \bigoplus L^2\mathcal{H}^k(M), \tag{78}$$

where $L^2\mathcal{H}^k(M)$ is the vector space of smooth k-forms ω which are closed and co-closed, i.e., $(d+\delta)\omega = 0$. Note that since we may regard d as a closed unbounded operator from $L^2\Lambda^k(M)$ into $L^2\Lambda^{k+1}(M)$ with domain $\{\omega \in L^2\Lambda^k(M) : d\omega \in L^2\Lambda^{k+1}(M)\}$, its adjoint δ is also a closed operator.

Proposition 25 *Let $(M, \langle\,,\rangle)$ be as above. Then, for every vector field $|X| \in L^2(M)$ there exists a function $u \in W^{1,2}_{loc}(M)$ satisfying $|\nabla u| \in L^2(M)$ and*

$$\operatorname{div} X = \Delta u, \text{ weakly on } M.$$

Proof: Given the vector field X consider the differential 1-form $\omega = X^\flat \in L^2\Lambda^1(M)$, where $\flat$ denotes the musical isomorphism. According to (78), there exist sequences $\{u_k\} \subset C_c^\infty(M)$, $\{v_k\} \subset C_c^\infty\Lambda^2(M)$ and a form $\gamma \in L^2\mathcal{H}^k(M)$ such that

$$\text{i) } du_k \overset{L^2}{\to} \alpha, \qquad \text{ii) } \delta v_k \overset{L^2}{\to} \beta \tag{79}$$

and

$$\omega = \alpha + \beta + \gamma.$$

Fix an arbitrary compact domain $\Omega \subset\subset M$ with smooth boundary. Since $\{|du_k|\}$ is bounded in $L^2(\Omega)$ using the local Poincarè inequality we have that $\{u_k\}$ is bounded in $W^{1,2}(\Omega)$. By Rellich-Kondrakov compactness theorem, a subsequence of $\{u_k\}$, which we call again $\{u_k\}$, satisfies

$$u_k \overset{L^2(\Omega)}{\to} u$$

for some $u \in L^2(\Omega)$. In fact, due to (79) i), $\{u_k\}$ is a convergent sequence in $W^{1,2}(\Omega)$. Therefore $u \in W^{1,2}(\Omega)$ and $du = \alpha$. Repeating the argument on a smooth exhaustion of M, and using a diagonalization procedure, we obtain a function $u \in W^{1,2}_{loc}(M)$ satisfying $du = \alpha$.

Next, using the closeness of δ, we note that (79) ii) implies $\delta\beta = 0$. Since, clearly, $\delta\gamma = 0$ we deduce

$$\operatorname{div} X = -\delta\omega = -\delta\alpha = -\delta du = \Delta u.$$

□

Combining Proposition 25 with Theorem 24 gives the following

Theorem 26 *Let $(M, \langle\,,\rangle)$ be a complete manifold. Then $(M, \langle\,,\rangle)$ is parabolic if and only if, for every $u \in C^2(M)$ with $|\nabla u| \in L^2(M)$, then $\Delta u \geq 0$ implies $\Delta u = 0$.*

As a matter of fact, a stronger conclusion holds ([30], [26]):

Corollary 27 *A geodesically complete manifold $(M, \langle\,,\rangle)$ is parabolic if and only if for every $u \in C^2(M)$ with $|\nabla u| \in L^2(M)$ and $\Delta u \geq 0$ we have $u = const.$*

Proof: In view of the previous theorem, we only need to show that, if M is parabolic and u satisfies the conditions in the statement, then u is constant. To this end, let $f = (1+u^2)^{1/2}$. Then $|\nabla f| = |u|(1+u^2)^{-1/2}|\nabla u| \leq |\nabla u| \in L^2$, and $\Delta f = (1+u^2)^{-3/2}|\nabla u|^2 \geq 0$, so by the Theorem f is positive and harmonic, and it follows from the assumed parabolicity of M that f, and therefore u is constant.

□

We note that the statement of Theorem 23 looks like a strict relative of that of L. Karp's version of the divergence theorem which can be expressed in the following form ([15]):

Theorem 28 *Let $(M, \langle\,,\rangle)$ be a geodesically complete manifold and X a vector field on M such that:*

i) $|X| \in L^1(M)$

ii) $\operatorname{div} X \in L^1_{loc}(M)$ *and* $(\operatorname{div} X)_- \in L^1(M)$

iii) $0 \leq \int_M \operatorname{div} X \leq +\infty$.

Then,

$$\int_M \operatorname{div} X = 0.$$

Thus, the only qualifying difference (on infinite volume manifolds) is the integrability class of X.

Question 29 Is there some deep reason behind this analogy?

4 Vector fields and stochastic completeness

In the previous section, we alluded to strong analogies between stochastic completeness and parabolicity of a Riemannian manifold. In this perspective, recalling the important role played by L^2 vector fields in the parabolic setting, it seems quite interesting to investigate the following

Question 30 Have L^2 vector fields something to do with stochastic completeness?

Towards a possible answer to the question, the following brief discussion could be of some interest. Consider the maximum principle viewpoint. In accordance to it, one has that a stochastically complete manifold does not support any bounded above function satisfying $\Delta u \geq c > 0$. Recall also that, according to Theorem 8, the lower bound c can be substantially relaxed by imposing suitable volume growth conditions. In fact, if

$$\frac{r^{1-\mu}}{\log \mathrm{vol} B_r} \notin L^1(+\infty)$$

for some $\mu \in \mathbb{R}$, then a function $u \in C^2(M)$ such that $u^* = \sup_M u < +\infty$ satisfies

$$\inf_{\{u > u^* - \eta\}} (1 + r(x)^\mu)\, \Delta u \leq 0,$$

for every fixed $\eta < u^*$. We shall prove the following

Proposition 31 *Let $(M, \langle\, ,\rangle)$ be a complete manifold satisfying*

$$\frac{r^{1-\mu}}{\log \mathrm{vol} B_r} \notin L^1(+\infty), \tag{80}$$

for some $\mu \in \mathbb{R}$. Suppose also that

$$\lambda_1^{-\Delta}(M) = \lambda > 0. \tag{81}$$

Then, for every vector field $X \in C^{1,\alpha}(M)$, $\alpha > 0$, such that $|X| \in L^2(M)$, it holds

$$\inf_M (1 + r(x))^\mu \operatorname{div} X \leq 0. \tag{82}$$

Observe that conditions (80) and (81) are compatible in situations where the volume growth of balls is at least exponential; see [2]. In fact, by Bishop and Mc Kean comparison theorems, every simply connected, complete manifold of negatively pinched curvature enjoys both (80) and (81).

We need the following

Lemma 32 *Let $(M, \langle\,,\rangle)$ be a complete manifold. Let $u \in C^2(M)$ be a subharmonic function satisfying $u \in L^p(M)$, for some $p > 1$. Then, either u is constant or $u < 0$.*

Proof: Recall that, by a result of S.T. Yau, [32], a subharmonic function $0 \leq w \in C^2(M)$ satisfying $w \in L^p(M)$ for some $p > 1$ must be constant. As a matter of fact, Yau's argument relies only on integration by parts and, therefore, works even for functions with low regularity. Indeed, suppose $w \in W^{1,\infty}_{loc}(M)$ is a weak solution of $\Delta w \geq 0$, i.e.,

$$\int_M \langle \nabla w, \nabla \rho \rangle \leq 0,$$

for every $0 \leq \rho \in Lip_c(M)$. Having fixed $\varepsilon > 0$, choose

$$\rho = \varphi^2 (w + \varepsilon)^{p-1}$$

where $\varphi \in C^\infty_c(M)$. Then, direct computations that use Scwharz's inequality and the elementary inequality $2ab \leq \delta^{-1} a^2 + \delta b^2$, $\delta > 0$, yields

$$\begin{aligned}
0 &\geq \int_M \left\langle \nabla w, \nabla \left(\varphi^2 (w+\varepsilon)^{p-1} \right) \right\rangle \\
&\geq -2 \int_M \varphi (w+\varepsilon)^{p-1} |\nabla w| |\nabla \varphi| + (p-1) \int_M \varphi^2 (w+\varepsilon)^{p-2} |\nabla w|^2 \\
&\geq -\frac{1}{\delta} \int_M (w+\varepsilon)^p |\nabla \varphi|^2 + (p-1-\delta) \int_M \varphi^2 (w+\varepsilon)^{p-2} |\nabla w|^2 .
\end{aligned}$$

If $\delta > 0$ is so small that $(p-1-\delta) > 0$, letting $\varepsilon \searrow 0$ we deduce the Caccioppoli inequality

$$\int_M \varphi^2 w^{p-2} |\nabla w|^2 \leq \frac{1}{\delta(p-1-\delta)} \int_M |\nabla \varphi|^2 w^p.$$

Now, choosing

$$\text{i) } \varphi \equiv 1 \text{ on } B_R;\ \text{ii) } \varphi = 0 \text{ on } M \backslash B_{2R};\ \text{iii) } |\nabla \varphi| \leq \frac{3}{R} \text{ on } M,$$

we get

$$\int_{B_R} w^{p-2} |\nabla w|^2 \leq \frac{3}{\delta(p-1-\delta)} \frac{\int_{B_{2R}} w^p}{R^2},$$

whence, letting $R \to +\infty$,

$$\int_M w^{p-2} |\nabla w|^2 = 0.$$

This latter easily implies that w is constant.

Now, consider $u_+(x) = \max(u(x), 0)$. Then $u_+ \geq 0$ is subharmonic and $u_+ \in L^p(M), p > 1$. According to what we have just proved, u_+ is constant. If $u_+ > 0$ then $u = u_+$ is constant, so if u is not constant $u_+ \equiv 0$ and $u \leq 0$. But then $u \leq 0$ is a non-positive non constant subharmonic function, and by the maximum principle u cannot attain a maximum, so $u < 0$ on M as claimed.

□

Proof: [Proof (of Proposition 31)] By contradiction, suppose there exists a C^1 vector field $X \in L^2(M)$ satisfying

$$\operatorname{div} X \geq \frac{C}{(1 + r(x))^{\mu}}, \text{ on } M,$$

for some constant $C > 0$. As explained in the previous section, assumption (81) implies that the first de Rham group of L^2 cohomology is reduced, namely,

$$\overline{dC_c^{\infty}(M)} = dW^{1,2}(M).$$

Applying the Hodge-de Rham-Kodaira decomposition to the differential 1-form $\omega = X^{\flat}$ we get

$$\omega = du + \beta + \gamma$$

where $u \in W^{1,2}(M)$, $\beta \in \overline{\delta C_c^{\infty} \Lambda^2(M)}$, and γ is smooth and both closed and co-closed. Co-differentiation of both sides of the latter, recalling that $\delta\beta = 0$, since δ is closed, gives

$$0 < \frac{C}{(1 + r(x))^{\mu}} \leq \operatorname{div} X = -\delta\omega = \Delta u. \tag{83}$$

Since $X \in C^{1,\alpha}(M)$, by elliptic regularity, $u \in C^2(M)$. We have thus obtained the existence of a subharmonic function $u \in L^2(M) \cap C^2(M)$. By Lemma 32, u is bounded above and, by the maximum principle, we have

$$0 \geq \inf (1 + r(x))^{\mu} \Delta u \geq C > 0,$$

a contradiction.

□

5 Maximum principle for the Hessian operator

The maximum principles at infinity as stated in Section 1 involve the Laplace-Beltrami operator. Formally, one can extend the definitions to other differential operators both of linear and of non-linear nature. For instance, in the linear setting, one can replace the Laplacian with the divergence form operator $L_A(u) = A(x)^{-1} \operatorname{div}(A(x) \nabla u)$, $0 < A(x) \in C^1(M)$, which is related to a suitable stochastic process X_t, called a symmetric diffusion. Following Dodziuk construction, we can obtain a minimal heat kernel $p_A(x, y, t)$ for the operator L_A. Its total mass $\int_M p_A(x, y, t)\, dy$ turns out to be related to the intrinsic explosion time of the associated diffusion process X_t. Accordingly, the (weak) maximum principle at infinity for the operator L_A holds if and only if the underlying manifold is stochastically complete with respect to X_t; see [22].

In this section we consider a different (linear) extension of the maximum principle at infinity which employs quadratic forms instead of their traces. This should take into account the asymptotic behavior of a large class of stochastic processes generalizing the Brownian motion and, apparently, having no kernel at all.

We say that the weak maximum principle for the Hessian holds on $(M, \langle\, ,\rangle)$ if, for every $u \in C^2(M)$ satisfying $u^* = \sup_M u < +\infty$, and for every $\gamma < u^*$, one has

$$\inf_{\Omega_\gamma} \max_{X \in T_x M \setminus \{0\}} \mathrm{Hess}\,(u) \left(\frac{X}{|X|}, \frac{X}{|X|} \right) \leq 0,$$

where

$$\Omega_\gamma = \{ x \in M : u(x) > \gamma \}.$$

Equivalently, there exists a sequence $\{x_k\} \subset M$ along which

$$\text{i) } u(x_k) > u^* - \frac{1}{k}, \qquad \text{ii) } \mathrm{Hess}\,(u)(x_k) < \frac{1}{k} \langle\, ,\rangle_{x_k}$$

in the sense of quadratic forms. By adding the further request

$$\text{iii) } |\nabla u|(x_k) < \frac{1}{k}$$

we get the original formulation of the full maximum principle at infinity due to H. Omori.

As the weak maximum principle for the Laplacian gives information on the intrinsic explosion time of the Brownian paths, one may ask the following

Question 33 Is there any probabilistic counterpart of the weak maximum principle for the Hessian?

We conjecture that the relevant processes are represented by the whole family of Γ-martingales of M, and that the validity of the maximum principle for the Hessian is related to the explosion time of every process in this family. Thus, we arrive at the notion of stochastic completeness with respect to martingales, or shortly, martingale completeness; see M. Emery book [7] for the relevant definitions. There are some indications that martingale completeness is related to the Hessian maximum principle. The starting point for our considerations is the following result by M. Emery, see [7] Proposition 5.37 on page 68.

Proposition 34 *Suppose that there exists a positive function $f \in C^2(M)$ satisfying the following conditions*

$$i)\ f \text{ is proper;} \qquad ii)\ |\nabla f| \leq C; \qquad iii)\ \operatorname{Hess}(f) \leq C\langle\,,\rangle \tag{84}$$

for some constant $C > 0$. Then $(M, \langle\,,\rangle)$ is martingale complete.

Observe that conditions (84) i) and ii) force divergent paths to have infinite length. Therefore $(M, \langle\,,\rangle)$ is geodesically complete. We shall come back to this fact momentarily.

Some remarks on the above statement are in order. On the one hand, we know from the general theory developed in [22] (see especially Theorem 1.9) that function theoretic properties like those listed in (84) are tightly related with the validity of the maximum principles at infinity and in fact imply the validity of the full maximum principles for the Hessian. Indeed, we have

Theorem 35 *Suppose that $(M, \langle\,,\rangle)$ satisfies the assumptions of Theorem 4 above with condition (9) replaced by*

$$\operatorname{Hess}(\gamma) \leq B\sqrt{\gamma G(\sqrt{\gamma})}\langle\,,\rangle, \text{ on } M\backslash K.$$

Then, the Omori maximum principle at infinity for the Hessian is satisfied.

On the other hand, conditions (84) i), iii) resemble the Khas'minskii test for the completeness with respect to the Brownian motion. The following result is from [22].

Proposition 36 *If* $(M, \langle\, ,\rangle)$ *supports a function* $f \in C^2(M)$ *such that*

$$f(x) \to +\infty \ \textit{as} \ x \to +\infty \quad \textit{and} \quad \mathrm{Hess}(f) \leq \lambda f \ \langle\, ,\rangle \ \textit{off a compact set} \tag{85}$$

in the sense of quadratic forms, for some $\lambda > 0$, *then the weak maximum principle for the Hessian holds on* $(M, \langle\, ,\rangle)$.

Thus, we naturally come to the following question.

Question 37 In the assumptions of the modified Khas'minskii test of Proposition 36, is $(M, \langle\, ,\rangle)$ martingale complete?

So far, we have considered some interplays between stochastic properties and function theoretic properties of the underlying manifold $(M, \langle\, ,\rangle)$. It is time to take into account some geometric properties of $(M, \langle\, ,\rangle)$.

We have already remarked that geodesic completeness and stochastic completeness for Brownian motion are independent concepts. In sharp contrast, Emery proved the following result; see Proposition 5.36 in [7].

Proposition 38 *If* $(M, \langle\, ,\rangle)$ *is martingale complete then* $(M, \langle\, ,\rangle)$ *is geodesically complete.*

Thus, we are led to asking the following

Question 39 Does the weak maximum principle for the Hessian imply geodesic completeness?

In this respect, we note the following partial answer.

Proposition 40 *Suppose that* $(M, \langle\, ,\rangle)$ *satisfies the weak maximum principle at infinity for the Hessian. Then* $(M, \langle\, ,\rangle)$ *is non-extensible, namely, it is not isometric to a proper, open subset of some connected manifold* $(N, (\, ,))$.

Proof: Suppose the contrary. Pick a point $p \in \partial M$, the topological boundary of M in N, and define $r(x) = \mathrm{dist}_N(x, p)$. Next, fix $0 < R < \mathrm{inj}_N(p)$ and let $f \in C^\infty(N \setminus \{p\}) \cap C^0(N)$ be a radial non-increasing function such that

$$f(x) = \begin{cases} e^{-r(x)} & \text{if } r(x) < R/2 \\ 0 & \text{if } r(x) > R \end{cases}$$

Clearly, $f \in C^\infty(M)$ is bounded from above with

$$\sup_M f = f(p) = 1.$$

A computation shows that

$$\mathrm{Hess}(f)(\nabla r, \nabla r) \geq e^{-r} \geq e^{-R/2} > 0,$$

on ${}^N B_{R/2}(p) - \{p\}$. Since any sequence $\{x_n\} \subset M$ along which f reaches its supremum must be eventually contained in ${}^N B_{R/2}(p) \setminus \{p\}$, we conclude that the weak maximum principle for the Hessian is not satisfied.

$\square$

It is well known that geodesic completeness implies non-extensibility and that the converse is false. It is perhaps interesting to observe the following

Proposition 41 *Under the assumptions of the modified Khas'minskii test with the Hessian condition (85), the manifold is geodesically complete.*

Proof: Let f be as in Proposition 36, and suppose condition (85) is satisfied outside the compact set $K \subset M$. Without loss of generality, we can assume $\lambda = 1$. Let $\gamma : [0, l) \to M$ be a maximal geodesic path parametrized by arc-length. We have to show that $l = +\infty$. To this end, note that γ is a divergent path, i.e., it eventually leaves every fixed compact set of M. Pick $t_0 > 0$ such that $\gamma(t) \notin K$ for every $t \geq t_0$ and consider the unit-speed, geodesic

$\Gamma(t) = \gamma(t + t_0) : [0, l - t_0) \to M - K$. Set $\phi = f \circ \Gamma$. A computation shows that $\phi(t) > 0$ is a solution of

$$\phi''(t) \leq \phi(t), \text{ on } [0, l - t_0). \tag{86}$$

Furthermore

$$\phi(l - t_0) = +\infty.$$

Using the classical comparison argument by Sturm, (86) implies that the function

$$\sinh(t)\,\phi'(t) - \cosh(t)\,\phi(t)$$

is non-increasing. As a consequence

$$\frac{\phi'(t)}{\phi(t)} \leq \coth(t)$$

which integrated implies that ϕ cannot explode in a finite time.

□

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