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LIMIT OF KARCHER'S SADDLE TOWERS

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1 Introduction

In 1835, Scherk [11] showed a singly periodic minimal surface S in $\mathbb{R}^3$, which may be viewed as the desingularization of two vertical planes meeting at a right angle. This surface S was generalized later on to a one-parameter family of singly periodic minimal surfaces S_θ in $\mathbb{R}^3$, where $\theta \in (0, \frac{\pi}{2}]$ is the angle between the asymptotic vertical planes (in particular, $S = S_{\frac{\pi}{2}}$). In the quotient by its shortest period vector, each S_θ has genus zero and four ends asymptotic to flat vertical annuli. Annular ends of this kind are called *Scherk-type ends*. These singly periodic Scherk minimal surfaces have recently been classified in [6] as the only properly embedded singly periodic minimal surfaces with four Scherk-type ends in the quotient.

Karcher [2] generalized the previous Scherk minimal surfaces by constructing, for each natural $n \geq 2$, a $(2n-3)$-parameter family of singly periodic minimal surfaces with genus zero and $2n$ Scherk-type ends in the quotient. These surfaces, called *saddle towers*, are the only properly embedded singly periodic minimal surfaces in $\mathbb{R}^3$ with genus zero and finitely many Scherk-type ends in the quotient, see [7]. Note that for $n = 2$ we obtain the singly periodic Scherk minimal surfaces.

Let us now recall the construction of the saddle towers: consider any convex polygonal domain Ω_n whose boundary consists of $2n$ edges of length one, with $n \geq 2$, and mark its edges alternately by $\pm\infty$. Assume Ω_n is non-special (see definition 1 below). By a theorem of Jenkins and Serrin [1], there exists a function u_n which solves the Jenkins-Serrin problem on Ω_n; i.e. u_n is a minimal graph defined on Ω_n which diverges to $\pm\infty$, as indicated by the marking, when we approach to the edges of Ω_n. The boundary of this minimal graph consists of $2n$ vertical lines above the vertices of Ω_n. Hence the conjugate minimal surface of this graph is bounded by $2n$ horizontal symmetry curves, lying in two horizontal planes at distance 1 from each other. By reflecting about one of the two symmetry planes, we obtain a fundamental domain for a saddle tower M_n with period $T = (0, 0, 2)$ and $2n$ Scherk-type ends in the quotient.

Definition 1. *We say that a convex polygonal domain with $2n$ unitary edges is special if $n \geq 3$ and its boundary is a parallelogram with two sides of length one and two sides of length $n-1$.*

Remark 1. *The bounded convex polygonal domains with edges of length one which fail to satisfy the hypothesis of the theorem of Jenkins and Serrin [1] are precisely the special domains (see [4], Appendix A).*

In this paper we study the possible limits of saddle towers by taking limits of sequences of minimal graphs u_n as above. We normalize so that the segment of vertices $(0, 0), (1, 0)$ is one of the edges of the convex polygonal domain Ω_n where u_n is defined, and such edge is marked by $+\infty$.

Theorem 1. *Let $M_n \subset \mathbb{R}^3$ be a saddle tower with $2n$ Scherk-type ends in the quotient, and Ω_n be its associated normalized convex polygonal domain in the above construction. Suppose $\{\Omega_n\}_n$ does not converge to a straight line nor half a straight line. Then a subsequence of $\{M_n\}_n$ converges uniformly on compact sets of $\mathbb{R}^3$ with multiplicity one to either one of the saddle towers with infinitely many ends described in [4], any singly periodic Scherk minimal surface, a doubly periodic Scherk minimal surface of angle $\frac{\pi}{2}$, or a KMR example $M_{\theta,\alpha,0}$ studied in [9] (also called toroidal halfplane layer by Karcher [2]).*

2 Preliminaries

In this section we present some general results for minimal graphs explained in [4] and based on the ideas of Jenkins and Serrin, developed by Collin and Mazet.

Let $u = u(x_1, x_2)$ be a solution of the minimal graph equation,

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \tag{1}$$

defined on a domain $\Omega \subset \mathbb{R}^2$. By an elementary computation, we obtain that the form $d\psi_u = \frac{u_{x_1}}{\sqrt{1+|\nabla u|^2}}\,dx_2 - \frac{u_{x_2}}{\sqrt{1+|\nabla u|^2}}\,dx_1$ is closed. Hence it defines a function $\psi_u = \psi_u(x_1, x_2)$, called *conjugate function of* u, which is well defined up to an additive constant. In fact, ψ_u coincides with the third coordinate function of the conjugate minimal surface of the graph of u, written as a function on the (x_1, x_2)-parameters (although such conjugate surface does not coincide with the graph of ψ_u). It is straightforward to check that ψ_u is a Lipschitz function. In particular, it can be extended continuously to $\partial\Omega$. Moreover, the following lemma holds.

Lemma 1. *Given a solution u of (1) on a domain $\Omega \subset \mathbb{R}^2$, we have:*

(i) For every domain $D \subset \Omega$, $\int_{\partial D} d\psi_u = 0$.

(ii) Let $T \subset \partial\Omega$ be a bounded arc oriented as ∂D. Then,

$$\left|\int_T d\psi_u\right| \leq |T|,$$

and $\int_T d\psi_u = |T|$ (resp. $-|T|$) if and only if u diverges to $+\infty$ (resp. to $-\infty$) as one approaches T within Ω, in which case T must be a straight segment.

Let u_n be a solution of (1) on a domain $\Omega_n \subset \mathbb{R}^2$. We define the *limit domain* Ω_∞ of the domains Ω_n as the set of points in $\mathbb{R}^2$ that admit a neighborhood contained in every Ω_n, for n large enough, and say that $\{\Omega_n\}_n$ converges to Ω_∞.

Observe that, if $p \in \Omega_\infty$, then $u_n(p)$ is well defined for n large enough. Consider the *convergence domain* of $\{u_n\}_n$, defined as

$$\mathcal{B}(u_n) = \{p \in \Omega_\infty \mid \{|\nabla u_n|(p)\}_n \text{ is bounded } \}.$$

For each component D of $\mathcal{B}(u_n)$ and any point $q \in D$, there is a subsequence of $\{u_n - u_n(q)\}_n$ converging uniformly on compact sets of D to a solution of (1). Moreover,

$$\Omega_\infty - \mathcal{B}(u_n) = \cup_{i \in I} L_i,$$

where each L_i, called a *divergence line*, is a component of the intersection of a straight line with Ω_∞. Clearly, to ensure the convergence of a subsequence of the (vertical translated) u_n on Ω_∞, it suffices to prove there are no divergence lines.

Lemma 2. *Let $\{u_n\}_n$ be a sequence of minimal graphs as above, and denote by ψ_n the conjugate function of u_n, for every $n \in \mathbb{N}$.*

(i) Let T be a straight segment contained in a divergence line ($T \subset \Omega_n$ for n big enough). Then $\int_T d\psi_n \to \pm|T|$.

(ii) A divergence line cannot finish at an interior point of an open straight segment $T \subset \partial\Omega_\infty$, if for each n there exists a straight segment T_n in $\partial\Omega_n$ such that u_n diverges to $+\infty$ when we approach T_n within Ω_n and $T_n \to T$.

Finally, we have the following uniqueness result for the limit u under some constraints.

Lemma 3 ([3]). *Let u, v be two solutions of (1) in a domain Ω, whose conjugate functions ψ_u, ψ_v are bounded in Ω and coincide in $\partial\Omega$. Then $u - v$ is constant in Ω.*

3 Taking limits of saddle towers

For every $n \in \mathbb{N}$, let $\Omega_n \subset \mathbb{R}^2$ be a non-special, convex, bounded polygonal domain with $2n$ unitary edges. Denote its vertices as p_i^n, $i = 0, \cdots, 2n-1$, by following the natural cyclic ordering induced by the positive orientation of $\partial\Omega_n$. Assume that $p_0^n = (0,0)$ and $p_1^n = (1,0)$ for every $n \in \mathbb{N}$. After passing to a subsequence, the sequence of domains $\{\Omega_n\}_n$ converges to either a straight line, half a straight line, or a convex, unbounded, polygonal domain Ω_∞ with unitary edges, whose vertices are obtained as limits of the vertices of the domains Ω_n. Assume that the two first cases do not happen.

Let u_n be the solution to the minimal graph equation (1) on Ω_n which takes boundary values $+\infty$ on the edges (p_{2i}^n, p_{2i+1}^n), and $-\infty$ on the edges (p_{2i-1}^n, p_{2i}^n), $i = 0, \cdots, n-1$. Our aim consists of studying the possible limits for $\{u_n\}_n$.

We denote by ψ_n the conjugate function of u_n verifying $\psi_n(p_0^n) = 0$. From Lemma 1-*(ii)* we have $\int_{p_i^n}^{p_{i+1}^n} d\psi_n = (-1)^i$, which implies that $\psi_n(p_i^n)$ is equal to 0 if i is even, and equal to 1 when i is odd. Moreover, ψ_n is an affine function on each edge of Ω_n, so $0 \leq \psi_n \leq 1$ on $\partial\Omega_n$. And by the maximum principle, $0 \leq \psi_n \leq 1$ in Ω_n.

Finally, we will say that a vertex of Ω_∞ is *even* (resp. *odd*) if it is obtained as limit of vertices p_{2i}^n (resp. p_{2i+1}^n) of the domains Ω_n, where $i = 0, \cdots, 2n-1$ (this is possible because we have fixed the vertices p_0^n, p_1^n). In particular, $p_0 = (0,0)$ is an even vertex of Ω_∞, and $p_1 = (1,0)$ is an odd vertex.

Proposition 1. *If the distance between any two non-consecutive vertices of Ω_∞ with different parity is strictly bigger than one and $q \in \Omega_\infty$, then $\{u_n - u_n(q)\}_n$ converges to a minimal graph u_∞ with boundary values $\pm\infty$ disposed alternately on $\partial\Omega_\infty$ and whose conjugate graph lies in the horizontal slab $\{(x_1, x_2, x_3) \mid 0 \leq x_3 \leq 1\}$.*

Proof. Let us prove there are no divergence lines for $\{u_n\}_n$ in the setting of Proposition 1. Suppose by contradiction that there exists one such divergence line L. We deduce from Lemma 2-*(i)* that L must have length no bigger than one, since $0 \leq \psi_n \leq 1$ for every $n \in \mathbb{N}$. Thus Lemma 2-*(ii)* says that L must be a segment (of length at most one) joining two different, non-consecutive vertices p_i, p_j of Ω_∞. Let p_i^n, p_j^n be vertices of Ω_n such that $p_i^n \to p_i$ and $p_j^n \to p_j$. Since $|\int_{p_i^n}^{p_j^n} d\psi_n| \to |p_i - p_j|$ by using Lemma 2-*(i)*, but $|\int_{p_i^n}^{p_j^n} d\psi_n| = |\psi_n(p_j^n) - \psi_n(p_i^n)|$

can only equal 0 or 1, we deduce that the only possibility is $|p_i - p_j| = 1$, and so $|\psi_n(p_j^n) - \psi_n(p_i^n)| = 1$ for n large. In particular, the vertices p_i, p_j have different parity and satisfy $|p_i - p_j| = 1$, which contradicts the hypothesis in Proposition 1.

Hence there exists a subsequence of $\{u_n - u_n(q)\}_n$ converging on compact subsets of Ω_∞ to a minimal graph u_∞. Moreover, we deduce from Lemma 1-*(ii)* that u_∞ takes boundary values $\pm\infty$ alternately on the unitary edges in $\partial\Omega_\infty$. Since the conjugate function ψ_∞ of u_∞ can be obtained as limit of $\{\psi_n\}_n$, we deduce that $0 \leq \psi_\infty \leq 1$ on Ω_∞.

Finally, we obtain from Lemma 3 that it is not only a subsequence but the whole sequence $\{u_n - u_n(q)\}_n$ what converges to u_∞, and Proposition 1 is proven.

□

Remark 2. *From Lemma 3, we know that the graph u_∞ obtained in Proposition 1 must be half a singly periodic Scherk minimal surface of angle $\frac{\pi}{2}$, if Ω_∞ is a halfplane; one of the graphs studied in [8] (i.e. a piece of a KMR example $M_{\theta,\alpha,\frac{\pi}{2}}$), if the limit domain Ω_∞ is a strip; or one of the graphs constructed in [4], in other case.*

Now suppose we are not in the setting of Proposition 1; this is, suppose there exist two non-consecutive vertices p_i, p_j of Ω_∞ with different parity such that $0 \neq |p_i - p_j| \leq 1$. Let $p_{i_n}^n, p_{j_n}^n$ be two vertices of Ω_n such that $p_{i_n}^n \to p_i$ and $p_{j_n}^n \to p_j$. Assume p_i is an even vertex and p_j is an odd vertex. Then $\psi_n(p_{i_n}^n) = 0$ and $\psi_n(p_{j_n}^n) = 1$, so $|\int_{p_{i_n}^n}^{p_{j_n}^n} d\psi_n| = 1$. Thus we deduce from Lemma 1-*(ii)* that $|p_i - p_j| = 1$ and that the straight segment (p_i, p_j) is a divergence line.

Moreover, we can similarly prove that, if $p_{i_n+1}^n, p_{j_n-1}^n$ (resp. $p_{i_n-1}^n, p_{j_n+1}^n$) are not consecutive vertices and for each index α we denote by p_α the vertex of Ω_∞ so that $p_{\alpha_n}^n \to p_\alpha$, then the straight segment (p_{i+1}, p_{j-1}) (resp. (p_{i-1}, p_{j+1})) is also a divergence line of length one. Following this reasoning we obtain that Ω_∞ is either a strip or a special unbounded domain (see Definition 2 below), and the convergence domain $\mathcal{B}(u_n)$ consists of consecutive translated rhombi with

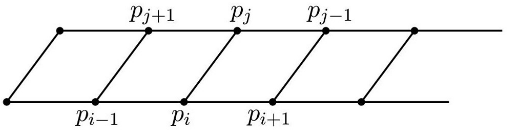

Figure 1: An example of special, unbounded, convex polygonal domain.

unitary edges (see Figure 1).

Definition 2. *An unbounded convex polygonal domain is said to be special when its boundary is made of two parallel half lines and one edge of length one (such a domain may be seen as a limit of special domains with* $2n$ *edges, when* $n \to \infty$*), see Figure 1.*

By uniqueness we know that, after a suitable vertical translation, the graphs u_n converge on each such rhombus to a fundamental piece of a doubly periodic Scherk minimal surface. We then obtain the following lemma.

Lemma 4. *If there exist two non-consecutive vertices* p_i, p_j *of* Ω_∞ *with different parity such that* $|p_i - p_j| \le 1$*, then* Ω_∞ *is either a strip or a special domain, and the convergence domain of* $\{u_n\}_n$ *consists of consecutive translated rhombi* $\{R_k\}_k$ *with unitary edges. Moreover, fixed* $q \in R_k$*, for any* k*, the sequence* $\{u_n - u_n(q)\}_n$ *converges uniformly on compact sets of* R_k *to a fundamental piece of a doubly periodic Scherk minimal surface.*

For each $n \in \mathbb{N}$, let M_n be the saddle tower obtained from the graph u_n. Translate each M_n so that it contains the origin of $\mathbb{R}^3$, and let us now prove that $\{M_n\}_n$ converges with finite multiplicity on compact sets of $\mathbb{R}^3$ to a minimal surface M_∞. Since we are assuming that Ω_∞ is not half a straight line nor a straight line, there exists a uniform radius $r_0 > 0$ such that, for each vertex p_i^n of Ω_n, the disk $D(p_i^n, r_0)$ intersects $\partial\Omega_n$ only along its two edges with common endpoint p_i^n. We can then prove as in [4], Section 4, that there exists a constant C (independent of n) such that the Gauss curvature of M_n is bounded by C. By the Regular Neighborhood Theorem, or Rolling Lemma [10, 5], M_n has an embedded tubular neighborhood of radius $1/\sqrt{C}$. In particular, we have local area bounds (more precisely, the area of M_n inside balls of radius $1/\sqrt{C}$ is bounded by some constant). By standard result, a subsequence of $\{M_n\}_n$ converges with finite multiplicity on compact subsets of $\mathbb{R}^3$ to a minimal surface M_∞.

From Proposition 1, Remark 2 and Lemma 4, we obtain that M_∞ must be the conjugate surface of the corresponding limit of the graphs u_n, i.e. a KMR example $M_{\theta,\alpha,0}$, a doubly periodic Scherk minimal surface of angle $\frac{\pi}{2}$, any singly periodic Scherk minimal surface, or one of the saddle towers with infinitely many ends constructed in [4], which are singly periodic minimal surfaces with genus zero and one limit end in the quotient by all their periods. Furthermore, since M_∞ is not a plane, the multiplicity of convergence is one. This completes the proof of Theorem 1.

Remark 3. *When the domains Ω_n converges to either a straight line or half a straight line, it may be proven that the Gauss curvature of the saddle towers M_n blows-up. And, after scaling to have curvature estimates, the graphs u_n converge to half a helicoid (see [8] for a description of this limit). In fact, this is the same helicoidal limit we obtain by taking limits from fundamental pieces of doubly periodic Scherk minimal surfaces. Hence the scaled saddle towers converge to a catenoid.*

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