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OSSERMAN METRICS ON WALKER 4-MANIFOLDS EQUIPPED WITH A PARA-HERMITIAN STRUCTURE *

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Abstract

The Einstein equations on 4-dimensional Walker manifolds equipped with a natural para-Hermitian structure are analyzed, providing new examples of Osserman metrics whose Jacobi operators are neither diagonalizable nor nilpotent.

1 Introduction

Let (M, g) be an n-dimensional pseudo-Riemannian manifold with Riemann curvature tensor R. The difficulty of working with the whole curvature tensor in full generality leads to the analysis of other objects which allow to recover the curvature tensor of the manifold. Among these objects the Jacobi operator is one of the most natural and widely investigated (cf. [14]). For a point $p \in M$ and a unit vector $X \in T_pM$, the Jacobi operator is defined by $R_X = R(X, \cdot)X$.

Definition 1 *(M, g) is called pointwise Osserman if, for every $p \in M$, the eigenvalues of the Jacobi operators R_X are independent of the choice of X, and it is said to be (globally) Osserman if such eigenvalues are also independent of the point p considered.*

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Note that since the Ricci tensor is obtained by tracing the Jacobi operators, any Osserman metric is Einstein.

In the Riemannian setting, Osserman conjectured that globally Osserman manifolds should be locally two-point homogeneous [17] and this was proved by Chi for $n \neq 4k$, $k \geq 2$ [6]. Recent work of Nikolayevsky has given an almost complete answer to the Osserman Conjecture in the Riemannian case. More precisely, it is shown in [15], [16] that a globally Osserman Riemannian manifold of dimension $n \neq 16$ or a pointwise Osserman Riemannian manifold of dimension $n \neq 2, 4, 16$ is locally isometric to a two-point homogeneous space. Note that in dimension two any Riemannian manifold is pointwise Osserman, while globally Osserman manifolds are those with constant Gauss curvature. In dimension four, pointwise Osserman spaces are those Einstein and self-dual (or anti-self-dual) manifolds, and thus there exist pointwise Osserman four-dimensional manifolds which are not locally symmetric. Therefore, in the Riemannian case, only the 16-dimensional case is still open.

The Osserman Conjecture has also a positive answer in the Lorentzian case (cf. [2], [10]). However, when the Osserman condition is considered in other signatures the situation is completely different and more complicated, with many examples of Osserman spaces which are not locally symmetric and even not locally homogeneous (cf. [11], [12], [14]). Indeed, globally Osserman manifolds are not even classified in signature $(--++)$ where, besides the results in [3] and [13], a description of all $(--++)$-Osserman spaces is not yet complete. In this sense, the fact that all previously known examples of (pointwise) Osserman metrics had either diagonalizable or nilpotent Jacobi operators (see [3], [11], [14] and the references therein) suggested that this should be true in the general case. However, in [8], [9] explicit examples of Osserman metrics whose Jacobi operators are neither diagonalizable nor nilpotent are constructed. Such examples were motivated by an Einstein para-Hermitian structure on Walker manifolds and the purpose of the present work is to develop a detailed study of such Einstein para-Hermitian Walker manifolds, showing that the examples in [8] are a particular situation of a more general family of Osserman metrics with neither diagonalizable nor nilpotent Jacobi operators. More precisely, the following is the main result in this paper:

Theorem *A Walker metric*

$$\begin{aligned} g &= dx^1 \otimes dx^3 + dx^3 \otimes dx^1 + dx^2 \otimes dx^4 + dx^4 \otimes dx^2 \\ &\quad + a\, dx^3 \otimes dx^3 + b\, dx^4 \otimes dx^4 + c(dx^3 \otimes dx^4 + dx^4 \otimes dx^3) \end{aligned}$$

equipped with the almost para-Hermitian structure

$$J\partial_1 = -\partial_1, \qquad J\partial_2 = \partial_2, \qquad J\partial_3 = -a\partial_1 + \partial_3, \qquad J\partial_4 = b\partial_2 - \partial_4,$$

is Osserman para-Hermitian with non-nilpotent Jacobi operators if and only if the defining functions $a(x_1, x_2, x_3, x_4)$, $b(x_1, x_2, x_3, x_4)$ and $c(x_1, x_2, x_3, x_4)$ satisfy

$$\begin{aligned} a &= \tfrac{\tau}{6} x_1^2 + x_1 P + \tfrac{6}{\tau}\left(PT - T^2 + 2T_3\right), \\ b &= \tfrac{\tau}{6} x_2^2 + x_2 Q + \tfrac{6}{\tau}\left(QS - S^2 + 2S_4\right), \\ c &= \tfrac{\tau}{6} x_1 x_2 + x_1 S + x_2 T + \tfrac{6}{\tau}\left(ST + Q_3 - S_3 + P_4 - T_4\right), \end{aligned}$$

for any smooth functions $P(x_3, x_4)$, $Q(x_3, x_4)$, $S(x_3, x_4)$ and $T(x_3, x_4)$.

The paper is structured as follows. In Section 2 we motivate the study of para-Hermitian structures on Walker 4-manifolds in the search of new examples of Osserman manifolds. In §3 a natural para-Hermitian structure is defined on any Walker metric and a characterization of the solutions of the Einstein equation for such a para-Hermitian Walker manifold is obtained. As a consequence, three families of Walker metrics are obtained. Finally, in §4 we investigate the Osserman condition for each one of the three families of metrics, showing that exactly one of them generalizes the examples in [8].

2 Motivation: Four-dimensional Osserman and Walker metrics

Four-dimensional Osserman metrics

For any non-null vector X in the $(--++)$-setting, the induced metric on $X^\perp$ is of Lorentzian signature and hence the eigenvalue-structure does not completely characterize the Jacobi operators. This led to the consideration of the Jordan normal form of the Jacobi operators and in [3] the following four different possibilities were considered:

$$\underset{\textit{Type Ia}}{\begin{pmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{pmatrix}}, \quad \underset{\textit{Type Ib}}{\begin{pmatrix} \alpha & -\beta & \\ \beta & \alpha & \\ & & \gamma \end{pmatrix}}, \quad \underset{\textit{Type II}}{\begin{pmatrix} \beta & & \\ & \alpha & \\ & 1 & \alpha \end{pmatrix}}, \quad \underset{\textit{Type III}}{\begin{pmatrix} \alpha & & \\ 1 & \alpha & \\ & 1 & \alpha \end{pmatrix}}.$$

Type Ia corresponds to diagonalizable Jacobi operators and it is known that any such Osserman metric is locally a real, complex or paracomplex space form [3]. Type Ib corresponds to Jacobi operators with a complex eigenvalue and it is proved in [3] that no four-dimensional Osserman metric may have such structure. Finally, Type II (respectively, Type III) corresponds to a double (respectively, triple) root of the minimal polynomial of the Jacobi operators.

As a starting point in the search of Osserman spaces with nondiagonalizable Jacobi operators we recall the known fact that in the case of two different eigenvalues α and β (α of multiplicity two), the relation $\beta = 4\alpha$ holds and, in such a case, the metric admits a local parallel field of null two-planes, i.e., it is a Walker metric (cf. [3]). Also recall that an important difference between complex and paracomplex space forms from the point of view of their Jacobi operators is that the restriction of the metric to the subspace $E_\beta(X) = span\{X\} \oplus ker(\varepsilon_X R_X - \beta Id)$, $\varepsilon_X = g(X,X)$, is definite in the complex case and indefinite in the paracomplex setting [4]. Further, note that in the case of two distinct eigenvalues the nondiagonalizability of the Jacobi operators implies that the metric induces a Lorentzian inner product on $E_\beta(X)$. This fact leads our attention to para-Kähler structures and, by extension, to Walker manifolds.

Walker metrics

Definition 2 *A Walker manifold is a triple $(M, g, \mathcal{D})$, where M denotes an n-dimensional manifold, g an indefinite metric and $\mathcal{D}$ an r-dimensional parallel null distribution.*

Of special interest are those manifolds admitting a field of null planes of maximum dimensionality ($r = \frac{n}{2}$). Since the dimension of a null plane is $r \leq \frac{n}{2}$, the lowest possible case of a Walker metric is that of $(- - ++)$-manifolds admitting a field of parallel null two-planes.

For such metrics a canonical form has been obtained by Walker [18], showing the existence of suitable coordinates $(x_1, \dots, x_4)$ where the metric expresses as

$$g_{(x_1,x_2,x_3,x_4)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix} \tag{1}$$

for some functions a, b and c depending on the coordinates $(x_1, \dots, x_4)$.

Para-Kähler manifolds, being symplectic manifolds locally diffeomorphic to a product of Lagrangian submanifolds, admit a pair of parallel null distributions of maximal dimensionality. Moreover, if a 4-dimensional manifold is assumed to be Osserman para-Kähler, then it is either Ricci flat or a paracomplex space form [7], and hence this kind of manifolds does not provide the new desired examples of Osserman spaces (i.e., Osserman manifolds whose Jacobi operators are neither diagonalizable nor nilpotent).

The above remark motivates the study of a more general situation: Walker 4-manifolds equipped with a para-Hermitian structure, which we tackle in the following section.

3 Einstein para-Hermitian Walker metrics

3.1 The para-Hermitian structure

An almost para-Hermitian manifold is an almost symplectic manifold (M, Ω) whose tangent bundle splits into a Whitney sum $TM \cong L \oplus L'$ of Lagrangian subbundles. Such decomposition induces an almost paracomplex structure J (i.e., a $(1,1)$-tensor field satisfying $J^2 = Id$) on M such that $g(JX, JY) = -g(X, Y)$ for all vector fields X, Y on M. The pair (g, J) is said to be an almost para-Hermitian structure. Further, it is called *para-Hermitian* if the almost paracomplex structure J is integrable and it is said to be *almost para-Kähler* if the fundamental form $\Omega(X,Y)=g(JX,Y)$ is closed. Finally, (g,J) is said to be *para-Kähler* if both conditions above are satisfied (equivalently, $\nabla J=0$, where ∇ is the Levi-Civita connection of g).

Now, associated to any Walker metric (1) we consider a *natural* almost para-Hermitian structure J defined by

$$J\partial_1=-\partial_1,\qquad J\partial_2=\partial_2,\qquad J\partial_3=-a\partial_1+\partial_3,\qquad J\partial_4=b\partial_2-\partial_4,\tag{2}$$

where here and henceforth $\{\partial_i\}$ denotes the coordinate basis. Also, from now on we use subscripts for partial derivatives, i.e., $h_{i_1\ldots i_r}=\frac{\partial h}{\partial x_{i_1}\ldots\partial x_{i_r}}$, for any function h depending on $(x_1,\ldots,x_4)$.

The Levi-Civita connection of a Walker metric is given by (cf. [9])

$$\begin{aligned}
&\nabla_{\partial_1}\partial_3=\tfrac12 a_1\partial_1+\tfrac12 c_1\partial_2, \qquad \nabla_{\partial_1}\partial_4=\tfrac12 c_1\partial_1+\tfrac12 b_1\partial_2,\\
&\nabla_{\partial_2}\partial_3=\tfrac12 a_2\partial_1+\tfrac12 c_2\partial_2, \qquad \nabla_{\partial_2}\partial_4=\tfrac12 c_2\partial_1+\tfrac12 b_2\partial_2,\\
&\nabla_{\partial_3}\partial_3=\tfrac12(aa_1+ca_2+a_3)\partial_1+\tfrac12(ca_1+ba_2-a_4+2c_3)\partial_2-\tfrac{a_1}{2}\partial_3-\tfrac{a_2}{2}\partial_4,\\
&\nabla_{\partial_3}\partial_4=\tfrac12(a_4+ac_1+cc_2)\partial_1+\tfrac12(b_3+cc_1+bc_2)\partial_2-\tfrac{c_1}{2}\partial_3-\tfrac{c_2}{2}\partial_4,\\
&\nabla_{\partial_4}\partial_4=\tfrac12(ab_1+cb_2-b_3+2c_4)\partial_1+\tfrac12(cb_1+bb_2+b_4)\partial_2-\tfrac{b_1}{2}\partial_3-\tfrac{b_2}{2}\partial_4.
\end{aligned}\tag{3}$$

Analyzing the almost para-Hermitian structure J we obtain the following:

Theorem 3 *A Walker metric (1) equipped with the almost para-Hermitian structure (2) is para-Hermitian if and only if*

$$a_2=b_1=0.\tag{4}$$

Moreover, the almost para-Kähler condition holds if and only if $c_1=c_2=0$ and hence the para-Kähler condition is equivalent to $a_2=b_1=c_1=c_2$.

Proof. For the Nijenhuis tensor $N(X,Y)=[JX,JY]-J[JX,Y]-J[X,JY]+[X,Y]$ associated with J, put $N_{ij}=N(\partial_i,\partial_j)$. N is determined by

$$N_{14}=-2b_1\partial_2,\qquad N_{23}=-2a_2\partial_1,\qquad N_{34}=ba_2\partial_1-ab_1\partial_2.$$

Hence, the integrability of J is characterized by $a_2=b_1=0$. The second part of the result is obtained after a direct calculation from (3). □

3.2 The Einstein equation

In the rest of this section we center our study on the case of (g, J) being a para-Hermitian structure, obtaining a classification of Einstein para-Hermitian Walker metrics (1)-(2) as a first step to analyze the Osserman condition for these Walker metrics.

Let R denote the Riemann curvature tensor taken with the sign convention $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$. From (3) we get the nonzero components of the curvature tensor of any Walker metric (1), which are given as follows (cf. [9]):

$$\begin{aligned}
&R_{1313} = -\tfrac{1}{2}a_{11}, \quad R_{1314} = -\tfrac{1}{2}c_{11}, \quad R_{1323} = -\tfrac{1}{2}a_{12}, \quad R_{1324} = -\tfrac{1}{2}c_{12},\\
&R_{1334} = \tfrac{1}{4}\left(-a_2b_1 + c_1c_2 + 2a_{14} - 2c_{13}\right),\\
&R_{1414} = -\tfrac{1}{2}b_{11}, \quad R_{1423} = -\tfrac{1}{2}c_{12}, \quad R_{1424} = -\tfrac{1}{2}b_{12},\\
&R_{1434} = \tfrac{1}{4}\left(-c_1^2 + a_1b_1 - b_1c_2 + b_2c_1 - 2b_{13} + 2c_{14}\right),\\
&R_{2323} = -\tfrac{1}{2}a_{22}, \quad R_{2324} = -\tfrac{1}{2}c_{22},\\
&R_{2334} = \tfrac{1}{4}\left(c_2^2 - a_2b_2 - a_1c_2 + a_2c_1 + 2a_{24} - 2c_{23}\right),\\
&R_{2424} = -\tfrac{1}{2}b_{22}, \quad R_{2434} = \tfrac{1}{4}\left(a_2b_1 - c_1c_2 - 2b_{23} + 2c_{24}\right),\\
&R_{3434} = \tfrac{1}{4}\big(-ac_1^2 - bc_2^2 + aa_1b_1 + ca_1b_2 - a_1b_3 + 2a_1c_4\\
&\qquad\qquad +ca_2b_1 + ba_2b_2 + a_2b_4 + a_3b_1 - a_4b_2 - 2a_4c_1\\
&\qquad\qquad +2b_2c_3 - 2b_3c_2 - 2cc_1c_2 - 2a_{44} - 2b_{33} + 4c_{34}\big).
\end{aligned} \tag{5}$$

As a matter of notation, let ρ and τ denote the Ricci tensor and the scalar curvature of a Walker metric (1), where ρ is the contraction of the curvature tensor given by $\rho(X,Y) = \operatorname{trace}\{U \rightsquigarrow R(X,U)Y\}$ and τ is obtained by contracting the Ricci tensor, $\tau = \operatorname{trace}\rho$. Further, we denote by $\mathcal{F}$ the Einstein tensor, i.e., $\mathcal{F} = \rho - \frac{\tau}{4}g$.

Lemma 4 *The scalar curvature of a Walker metric (1) is given by*

$$\tau = a_{11} + b_{22} + 2c_{12}.$$

Proof. From (5) we get

$$\begin{aligned}
&\rho_{13} = \tfrac{1}{2}\left(a_{11} + c_{12}\right), \quad \rho_{14} = \tfrac{1}{2}\left(b_{12} + c_{11}\right), \\
&\rho_{23} = \tfrac{1}{2}\left(a_{12} + c_{22}\right), \quad \rho_{24} = \tfrac{1}{2}\left(b_{22} + c_{12}\right), \\
&\rho_{33} = \tfrac{1}{2}\left(-c_2^2 + a_1c_2 + a_2b_2 - a_2c_1 + aa_{11} + 2ca_{12} + ba_{22} + 2c_{23} - 2a_{24}\right), \\
&\rho_{34} = \tfrac{1}{2}\left(-a_2b_1 + c_1c_2 + a_{14} + b_{23} + ac_{11} + 2cc_{12} - c_{13} + bc_{22} - c_{24}\right), \\
&\rho_{44} = \tfrac{1}{2}\left(-c_1^2 + a_1b_1 - b_1c_2 + b_2c_1 + ab_{11} + 2cb_{12} - 2b_{13} + bb_{22} + 2c_{14}\right),
\end{aligned} \tag{6}$$

and hence the result follows after a straightforward calculation. □

Next we prove the main result in this section.

Theorem 5 *A Walker metric (1) equipped with the almost para-Hermitian structure (2) is Einstein para-Hermitian if and only if the defining functions* $a(x_1, x_2, x_3, x_4)$, $b(x_1, x_2, x_3, x_4)$ *and* $c(x_1, x_2, x_3, x_4)$ *are as follows:*

Type A: the scalar curvature τ *vanishes and* a *is a linear function with respect to* x_1 *and is independent of* x_2, b *is a linear function with respect to* x_2 *and is independent of* x_1, *and* c *is a linear function with respect to* x_1 *and* x_2, *i.e.,*

$$\begin{aligned}
a(x_1, x_3, x_4) &= x_1P(x_3, x_4) + \xi(x_3, x_4), \\
b(x_2, x_3, x_4) &= x_2Q(x_3, x_4) + \eta(x_3, x_4), \\
c(x_1, x_2, x_3, x_4) &= x_1S(x_3, x_4) + x_2T(x_3, x_4) + \gamma(x_3, x_4),
\end{aligned} \tag{7}$$

where ξ, η *and* γ *are arbitrary smooth functions, while* P, Q, S, T *are smooth functions satisfying*

$$PT - T^2 + 2T_3 = 0, \quad QS - S^2 + 2S_4 = 0, \quad ST + Q_3 - S_3 + P_4 - T_4 = 0, \tag{8}$$

or

Type B: the scalar curvature τ *is nonzero and* a *is a quadratic function with respect to* x_1 *and is independent of* x_2, b *is a quadratic function with respect to* x_2 *and is independent of* x_1, *and* c *only depends on* x_3 *and* x_4 *as follows:*

$$\begin{aligned}
a(x_1, x_3, x_4) &= \tfrac{\tau}{4}x_1^2 + x_1P(x_3, x_4) + \xi(x_3, x_4), \\
b(x_2, x_3, x_4) &= \tfrac{\tau}{4}x_2^2 + x_2Q(x_3, x_4) + \eta(x_3, x_4), \\
c(x_3, x_4) &= \tfrac{2}{\tau}\left(P_4(x_3, x_4) + Q_3(x_3, x_4)\right),
\end{aligned} \tag{9}$$

for any smooth functions $P(x_3,x_4)$, $Q(x_3,x_4)$, $\xi(x_3,x_4)$, $\eta(x_3,x_4)$, *or otherwise*

Type C: the scalar curvature τ *is nonzero and* a *is a quadratic function with respect to* x_1 *and is independent of* x_2, b *is a quadratic function with respect to* x_2 *and is independent of* x_1, *and* c *is a linear function with respect to* x_1 *and* x_2 *as follows:*

$$\begin{aligned}
&a(x_1,x_3,x_4)=\tfrac{\tau}{6}x_1^2+x_1P+\tfrac{6}{\tau}\left(PT-T^2+2T_3\right),\\
&b(x_2,x_3,x_4)=\tfrac{\tau}{6}x_2^2+x_2Q+\tfrac{6}{\tau}\left(QS-S^2+2S_4\right),\\
&c(x_1,x_2,x_3,x_4)=\tfrac{\tau}{6}x_1x_2+x_1S+x_2T+\tfrac{6}{\tau}\left(ST+Q_3-S_3+P_4-T_4\right),
\end{aligned}\tag{10}$$

for any smooth functions $P(x_3,x_4)$, $Q(x_3,x_4)$, $S(x_3,x_4)$ *and* $T(x_3,x_4)$.

Proof. The Einstein equations for a Walker metric (1) are as follows:

$$\begin{aligned}
&\mathcal{F}_{13}=-\mathcal{F}_{24}=\mathcal{F}_{31}=-\mathcal{F}_{42}=\tfrac{1}{4}\left(a_{11}-b_{22}\right)=0,\\
&\mathcal{F}_{14}=\mathcal{F}_{41}=\tfrac{1}{2}\left(b_{12}+c_{11}\right)=0,\\
&\mathcal{F}_{23}=\mathcal{F}_{32}=\tfrac{1}{2}\left(a_{12}+c_{22}\right)=0,\\
&\mathcal{F}_{33}=\tfrac{1}{4}\left(2a_1c_2+2a_2b_2-2a_2c_1-2c_2^2+a(a_{11}-b_{22})\right.\\
&\qquad\qquad\left.+4ca_{12}+2ba_{22}-4a_{24}-2ac_{12}+4c_{23}\right)=0,\\
&\mathcal{F}_{44}=\tfrac{1}{4}\left(2a_1b_1-2b_1c_2+2b_2c_1-2c_1^2-b(a_{11}-b_{22})\right.\\
&\qquad\qquad\left.+2ab_{11}+4cb_{12}-4b_{13}-2bc_{12}+4c_{14}\right)=0,\\
&\mathcal{F}_{34}=\mathcal{F}_{43}=\tfrac{1}{4}\left(-2a_2b_1+2c_1c_2-ca_{11}+2a_{14}-cb_{22}\right.\\
&\qquad\qquad\qquad\left.+2b_{23}+2ac_{11}+2cc_{12}-2c_{13}+2bc_{22}-2c_{24}\right)=0.
\end{aligned}\tag{11}$$

First of all note that if (g,J) is assumed to be para-Hermitian then $a_2=b_1=0$ by (4), i.e., $a=a(x_1,x_3,x_4)$ and $b=b(x_2,x_3,x_4)$. Hence (11) reduces to

$$\begin{aligned}
&a_{11}-b_{22}=0,\quad c_{11}=0,\quad c_{22}=0,\\
&a_1c_2-c_2^2-ac_{12}+2c_{23}=0,\\
&b_2c_1-c_1^2-bc_{12}+2c_{14}=0,\\
&c_1c_2-ca_{11}+a_{14}+b_{23}+cc_{12}-c_{13}-c_{24}=0.
\end{aligned}\tag{12}$$

We separate the proof of this theorem in three steps.

The first step. In this step we easily check that a and b are, in general, quadratic functions while c must be a linear function of the coordinates (x_1, x_2). Indeed, on the one hand the first equation in (12) implies that a (resp. b) is a quadratic function of x_1 (resp. x_2), with x_3 and x_4 parameters, as follows:

$$\begin{aligned} a(x_1, x_3, x_4) &= x_1^2\kappa(x_3, x_4) + x_1 P(x_3, x_4) + \xi(x_3, x_4), \\ b(x_2, x_3, x_4) &= x_2^2\kappa(x_3, x_4) + x_2 Q(x_3, x_4) + \eta(x_3, x_4), \end{aligned} \tag{13}$$

for some functions $\kappa(x_3, x_4)$, $P(x_3, x_4)$, $Q(x_3, x_4)$, $\xi(x_3, x_4)$ and $\eta(x_3, x_4)$. On the other hand, the second and third equations in (12) imply that c is a linear function with respect to x_1 and x_2, taking the form

$$c(x_1, x_2, x_3, x_4) = x_1 x_2 \alpha(x_3, x_4) + x_1 S(x_3, x_4) + x_2 T(x_3, x_4) + \gamma(x_3, x_4) \tag{14}$$

for some functions $\alpha(x_3, x_4)$, $S(x_3, x_4)$, $T(x_3, x_4)$ and $\gamma(x_3, x_4)$.

The second step. We show in this step that the coefficient $\alpha(x_3, x_4)$ in the defining function c is related with the *distinguished* function $\kappa(x_3, x_4)$ by means of the scalar curvature of the Walker metric and that, moreover, this last function must be constant. In this sense, observe that Lemma 4 combined with (13) and (14) implies that $\tau = 4\kappa(x_3, x_4) + 2\alpha(x_3, x_4)$ (τ being constant), from where

$$\alpha(x_3, x_4) = \frac{\tau}{2} - 2\kappa(x_3, x_4).$$

Now, differentiating the fourth equation in (12) twice by x_1, a direct calculation leads to

$$\tau^2 - 10\tau\kappa(x_3, x_4) + 24\kappa(x_3, x_4)^2 = 0. \tag{15}$$

Thus, $\kappa(x_3, x_4)$ must be constant, and

$$\begin{aligned} a(x_1, x_3, x_4) &= \kappa x_1^2 + x_1 P(x_3, x_4) + \xi(x_3, x_4), \\ b(x_2, x_3, x_4) &= \kappa x_2^2 + x_2 Q(x_3, x_4) + \eta(x_3, x_4), \\ c(x_1, x_2, x_3, x_4) &= \left(\tfrac{\tau}{2} - 2\kappa\right) x_1 x_2 + x_1 S(x_3, x_4) \\ &\quad + x_2 T(x_3, x_4) + \gamma(x_3, x_4), \end{aligned} \tag{16}$$

where the following three possibilities can occur: $\kappa = \tau = 0$ and, if $\tau \neq 0$, either $\kappa = \frac{\tau}{4}$ or $\kappa = \frac{\tau}{6}$.

The third step. We analyze each one of the three possible cases obtained in the previous step by separate.

Type A: $\kappa = \tau = 0$. This is the simplest case, since (16) reduces to

$$\begin{aligned} a(x_1, x_3, x_4) &= x_1 P(x_3, x_4) + \xi(x_3, x_4), \\ b(x_2, x_3, x_4) &= x_2 Q(x_3, x_4) + \eta(x_3, x_4), \\ c(x_1, x_2, x_3, x_4) &= x_1 S(x_3, x_4) + x_2 T(x_3, x_4) + \gamma(x_3, x_4), \end{aligned} \tag{17}$$

which is nothing but (7). Further, one easily checks that for such functions the last three equations in (12) transform into

$$PT - T^2 + 2T_3 = 0, \quad QS - S^2 + 2S_4 = 0, \quad ST + Q_3 - S_3 + P_4 - T_4 = 0,$$

i.e., (8) is obtained.

Type B: $\kappa = \frac{\tau}{4} \neq 0$. In this case, (16) becomes

$$\begin{aligned} a(x_1, x_3, x_4) &= \tfrac{\tau}{4} x_1^2 + x_1 P(x_3, x_4) + \xi(x_3, x_4), \\ b(x_2, x_3, x_4) &= \tfrac{\tau}{4} x_2^2 + x_2 Q(x_3, x_4) + \eta(x_3, x_4), \\ c(x_1, x_2, x_3, x_4) &= x_1 S(x_3, x_4) + x_2 T(x_3, x_4) + \gamma(x_3, x_4). \end{aligned} \tag{18}$$

Next we show that only the *form* of the defining function c is affected by the last three equations in (12), to obtain (9). Note that the fourth and fifth equations in (12) reduce to

$$\begin{aligned} &(\tau x_1 + 2P(x_3, x_4))T(x_3, x_4) - 2T(x_3, x_4)^2 + 4T_3(x_3, x_4) = 0, \\ &(\tau x_2 + 2Q(x_3, x_4))S(x_3, x_4) - 2S(x_3, x_4)^2 + 4S_4(x_3, x_4) = 0, \end{aligned}$$

which hold if and only if

$$T(x_3, x_4) = S(x_3, x_4) = 0. \tag{19}$$

Using this condition, the last equation in (12) leads to

$$\tau\gamma(x_3, x_4) - 2(P_4(x_3, x_4) + Q_3(x_3, x_4)) = 0$$

and therefore γ is given by

$$\gamma(x_3, x_4) = \frac{2}{\tau}(P_4(x_3, x_4) + Q_3(x_3, x_4)). \tag{20}$$

Hence, by (19) and (20) we conclude that

$$c(x_3,x_4)=\frac{2}{\tau}\left(P_4(x_3,x_4)+Q_3(x_3,x_4)\right),$$

which shows (9).

Type C: $\kappa=\frac{\tau}{6}\neq 0$. For this last case, writing (16) we have

$$\begin{aligned}
a(x_1,x_3,x_4)&=\tfrac{\tau}{6}x_1^2+x_1P(x_3,x_4)+\xi(x_3,x_4),\\
b(x_2,x_3,x_4)&=\tfrac{\tau}{6}x_2^2+x_2Q(x_3,x_4)+\eta(x_3,x_4),\\
c(x_1,x_2,x_3,x_4)&=\tfrac{\tau}{6}x_1x_2+x_1S(x_3,x_4)+x_2T(x_3,x_4)+\gamma(x_3,x_4),
\end{aligned}\tag{21}$$

and a straightforward calculation shows that the last three equations in (12) transform into

$$\begin{aligned}
&\tfrac{\tau}{6}\xi(x_3,x_4)-(P(x_3,x_4)T(x_3,x_4)-T(x_3,x_4)^2+2T_3(x_3,x_4))=0,\\
&\tfrac{\tau}{6}\eta(x_3,x_4)-(Q(x_3,x_4)S(x_3,x_4)-S(x_3,x_4)^2+2S_4(x_3,x_4))=0,\\
&\tfrac{\tau}{6}\gamma(x_3,x_4)-(S(x_3,x_4)T(x_3,x_4)+Q_3(x_3,x_4)\\
&\qquad\qquad -S_3(x_3,x_4)+P_4(x_3,x_4)-T_4(x_3,x_4))=0,
\end{aligned}$$

from where we can determine $\xi(x_3,x_4)$, $\eta(x_3,x_4)$ and $\gamma(x_3,x_4)$ as follows:

$$\begin{aligned}
\xi(x_3,x_4)&=\tfrac{6}{\tau}(P(x_3,x_4)T(x_3,x_4)-T(x_3,x_4)^2+2T_3(x_3,x_4)),\\
\eta(x_3,x_4)&=\tfrac{6}{\tau}(Q(x_3,x_4)S(x_3,x_4)-S(x_3,x_4)^2+2S_4(x_3,x_4)),\\
\gamma(x_3,x_4)&=\tfrac{6}{\tau}(S(x_3,x_4)T(x_3,x_4)+Q_3(x_3,x_4)\\
&\qquad -S_3(x_3,x_4)+P_4(x_3,x_4)-T_4(x_3,x_4)).
\end{aligned}\tag{22}$$

Finally, placing these expressions in (21) we obtain (10), which finishes the proof. □

4 Osserman para-Hermitian Walker metrics

In this section we analyze the Osserman condition for the three families of Einstein para-Hermitian Walker structures determined in Theorem 5. First of all, recall that a four-dimensional pseudo-Riemannian manifold is pointwise Osserman if and only if there is a choice of orientation such that the manifold is

Einstein self-dual (or anti-self-dual) (see [1]). Moreover, there is a 1:1 correspondence between the Jordan normal form of the Jacobi operators and the Jordan normal form of the (anti-)self-dual Weyl curvature operators [5]. In our particular case, note that

$$\begin{aligned} e_1 &= \tfrac{1}{2}(1-a)\partial_1 + \partial_3, & e_2 &= -c\partial_1 + \tfrac{1}{2}(1-b)\partial_2 + \partial_4, \\ e_3 &= -\tfrac{1}{2}(1+a)\partial_1 + \partial_3, & e_4 &= -c\partial_1 - \tfrac{1}{2}(1+b)\partial_2 + \partial_4, \end{aligned} \tag{23}$$

define an orthonormal basis for a Walker metric (1), and local bases of the spaces of self-dual and anti-self-dual two-forms can be constructed as $\Lambda^2_\pm = \langle\{E_1^\pm, E_2^\pm, E_3^\pm\}\rangle$, where

$$E_1^\pm = \frac{e^1\wedge e^2 \pm e^3\wedge e^4}{\sqrt{2}}, \quad E_2^\pm = \frac{e^1\wedge e^3 \pm e^2\wedge e^4}{\sqrt{2}}, \quad E_3^\pm = \frac{e^1\wedge e^4 \mp e^2\wedge e^3}{\sqrt{2}}.$$

Also observe that $\langle E_1^\pm, E_1^\pm\rangle = 1$, $\langle E_2^\pm, E_2^\pm\rangle = -1$, $\langle E_3^\pm, E_3^\pm\rangle = -1$, and therefore with respect to the above bases the operators $W^\pm : \Lambda^2_\pm \longrightarrow \Lambda^2_\pm$ have the following matrix form:

$$W^\pm = \begin{pmatrix} W_{11}^\pm & W_{12}^\pm & W_{13}^\pm \\ -W_{12}^\pm & -W_{22}^\pm & -W_{23}^\pm \\ -W_{13}^\pm & -W_{23}^\pm & -W_{33}^\pm \end{pmatrix}, \tag{24}$$

with $W_{ij}^\pm = W(E_i^\pm, E_j^\pm)$ and $W(e^i\wedge e^j, e^k\wedge e^l) = W(e_i, e_j, e_k, e_l)$, where W denotes the Weyl conformal curvature tensor.

Now, a long calculation shows that

$$W^+ = \begin{pmatrix} W_{11}^+ & W_{12}^+ & W_{11}^+ + \frac{\tau}{12} \\ -W_{12}^+ & \frac{\tau}{6} & -W_{12}^+ \\ -(W_{11}^+ + \frac{\tau}{12}) & -W_{12}^+ & -(W_{11}^+ + \frac{\tau}{6}) \end{pmatrix}, \tag{25}$$

and, therefore, it follows that W^+ has eigenvalues $\left\{\frac{\tau}{6}, -\frac{\tau}{12}, -\frac{\tau}{12}\right\}$ (see [9] for the precise expressions of W_{11}^{+} and W_{12}^{+}). Moreover, W^{-} is determined by

$$\begin{aligned} W_{11}^{-} &= -\tfrac{1}{12}(a_{11}+3a_{22}+3b_{11}+b_{22}-4c_{12}),\\ W_{22}^{-} &= -\tfrac{1}{6}(a_{11}+b_{22}-4c_{12}),\\ W_{33}^{-} &= \tfrac{1}{12}(a_{11}-3a_{22}-3b_{11}+b_{22}-4c_{12}),\\ W_{12}^{-} &= \tfrac{1}{4}(a_{12}+b_{12}-c_{11}-c_{22}),\\ W_{13}^{-} &= \tfrac{1}{4}(a_{22}-b_{11}),\\ W_{23}^{-} &= -\tfrac{1}{4}(a_{12}-b_{12}+c_{11}-c_{22}). \end{aligned} \tag{26}$$

4.1 Type A para-Hermitian Walker structures

Type A Einstein para-Hermitian Walker metrics given by (7)-(8) are Osserman, but they do not provide the *new* desired examples. Indeed, if $X = \sum_{i=1}^{4} \alpha_i \partial_i$ is an arbitrary vector, we get from (5) that the associated Jacobi operator, when expressed in the coordinate basis, takes the form

$$R_X = \begin{pmatrix} A & B \\ 0 & {}^{t}A \end{pmatrix}, \qquad A = \frac{\Psi}{4}\begin{pmatrix} -\alpha_3\alpha_4 & -\alpha_4^2 \\ \alpha_3^2 & \alpha_3\alpha_4 \end{pmatrix}, \tag{27}$$

where $\Psi = Q_3 + S_3 - P_4 - T_4$. Hence the characteristic polynomial of the Jacobi operators is $p_\lambda(R_X) = \lambda^4$ (independently of the 2×2-matrix B) and, therefore, *the Jacobi operators of any Type A metric are either vanishing or nilpotent.*

4.2 Type B para-Hermitian Walker structures

First note that, for any Walker metric (1), $W_{22}^{+} = -\frac{\tau}{6}$ holds (see (25)). Now, for any Type B metric given by (9), a direct calculation using (26) shows that also $W_{22}^{-} = -\frac{\tau}{6}$. Therefore, since $\tau \neq 0$, *Type B Einstein para-Hermitian Walker metrics cannot be Osserman.*

4.3 Type C para-Hermitian Walker structures

This last type of Einstein para-Hermitian Walker metrics provides the desired family of Osserman spaces. In particular, we have the following

Theorem 6 *A Walker metric (1) equipped with the para-Hermitian structure (2) is Osserman with non-nilpotent Jacobi operators if and only if the defining functions $a(x_1, x_2, x_3, x_4)$, $b(x_1, x_2, x_3, x_4)$ and $c(x_1, x_2, x_3, x_4)$ satisfy*

$$\begin{aligned} a &= \tfrac{\tau}{6}x_1^2 + x_1 P + \tfrac{6}{\tau}\left(PT - T^2 + 2T_3\right), \\ b &= \tfrac{\tau}{6}x_2^2 + x_2 Q + \tfrac{6}{\tau}\left(QS - S^2 + 2S_4\right), \\ c &= \tfrac{\tau}{6}x_1 x_2 + x_1 S + x_2 T + \tfrac{6}{\tau}\left(ST + Q_3 - S_3 + P_4 - T_4\right), \end{aligned} \tag{28}$$

for any smooth functions $P(x_3, x_4)$, $Q(x_3, x_4)$, $S(x_3, x_4)$ and $T(x_3, x_4)$.

Proof. Note from (26) that, in this case, $W^- = 0$ and hence a Type C Einstein para-Hermitian Walker metric given by (10) is Osserman (Einstein self-dual). Moreover, the eigenvalues of the self-dual operator W^+ determine the eigenvalues of the Jacobi operators, which coincide with $\left\{0, \frac{\tau}{6}, \frac{\tau}{24}, \frac{\tau}{24}\right\}$. A straightforward calculation leads to

$$\left(W^+ - \frac{\tau}{6}Id\right) \cdot \left(W^+ + \frac{\tau}{12}Id\right) = \frac{\tau^2 + 12\tau W_{11}^+ + 48\left(W_{12}^+\right)^2}{48} \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

and therefore the Jacobi operators are diagonalizable if and only if

$$\tau^2 + 12\tau W_{11}^+ + 48\left(W_{12}^+\right)^2 = 0. \tag{29}$$

Otherwise, we get that $\frac{\tau}{24}$ is a double root of the minimal polynomial of the Jacobi operators. $\square$

Remark 7 In [8] we constructed the first examples of Osserman manifolds whose Jacobi operators are neither diagonalizable nor nilpotent. Let $M = \mathbb{R}^4$ with usual coordinates (x_1, x_2, x_3, x_4) and the metric

$$\begin{aligned} g \;=\; & dx^1 \otimes dx^3 + dx^3 \otimes dx^1 + dx^2 \otimes dx^4 + dx^4 \otimes dx^2 \\ & +(4kx_1^2 - \tfrac{1}{4k}f(x_4)^2)dx^3 \otimes dx^3 + 4kx_2^2 dx^4 \otimes dx^4 \\ & +(4kx_1x_2 + x_2 f(x_4) - \tfrac{1}{4k}f'(x_4))(dx^3 \otimes dx^4 + dx^4 \otimes dx^3), \end{aligned} \tag{30}$$

where k is a nonzero constant and $f(x_4)$ an arbitrary real valued function. We showed in [8] that metrics (30) are Osserman of signature $(2, 2)$ with eigenvalues

$\{0, 4k, k, k\}$. Moreover, the Jacobi operators are diagonalizable if and only if

$$24kf(x_4)f'(x_4)x_2 - 12kf''(x_4)x_1 + 3f(x_4)f''(x_4) + 4f'(x_4)^2 = 0. \tag{31}$$

Otherwise, k is a double root of the minimal polynomial of the Jacobi operators and (M, g) is Type II Osserman on the open set where (31) does not hold.

Note that metrics (30) are a particular case of the general family of Type C Einstein para-Hermitian Walker metrics discussed in Theorems 5 and 6.

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