Matemática Contemporânea, Vol. 30, 41–61
http://doi.org/10.21711/231766362006/rmc303

LEGENDRIAN SUBMANIFODS FOLIATED BY $(n-1)$-SPHERES IN $\mathbb{S}^{2n+1}$

Henri Anciaux

Abstract

We give a characterization of those Legendrian submanifolds of $\mathbb{S}^{2n+1}$ which are foliated by $(n-1)$-dimensional spheres. We show that the only minimal submanifolds in this class are the totally geodesic n-spheres and a one-parameter family of $SO(n)$-equivariant submanifolds which are described in terms of some spherical curves. We deduce the existence of a countable family of closed Lagrangian minimal submanifolds in $\mathbb{C}P^n$.

Introduction

Lagrangian (resp. Legendrian) submanifolds constitute a particular class of submanifolds that appears in symplectic (resp. contact) geometry. Among the many inter-relations between these two classes one may mention two instances: firstly, the image of a Legendrian submanifold of $\mathbb{S}^{2n+1}$ by the Hopf projection is a Lagrangian submanifold of the complex projective space $\mathbb{C}P^n$ and conversely, any Lagrangian submanifold of $\mathbb{C}P^n$ admits locally a Legendrian lift in $\mathbb{S}^{2n+1}$ (which is unique up to rotation); secondly, the cone over a submanifold $\mathcal{L}$ of $\mathbb{S}^{2n+1}$ is Lagrangian in $\mathbb{C}^{n+1}$ if and only if $\mathcal{L}$ is Legendrian. Moreover, in both constructions the minimality of the Legendrian submanifold is equivalent to the minimality of the corresponding Lagrangian.

On a Lagrangian submanifolds of $\mathbb{C}^{n+1}$ (resp. a Legendrian submanifold of $\mathbb{S}^{2n+1}$) is defined an angle function β, called the *Lagrangian angle function*

2000 MSC: 53C42(Primary) 53D12 (Secondary)

(resp. *Legendrian angle function*). It has the striking property of being related to the mean curvature vector H of the submanifold by the following formula: $(n+1)H = J\nabla\beta$ (resp. $nH = J\nabla\beta$), where J denotes the complex structure of $\mathbb{C}^{n+1}$ and ∇ is the gradient associated to the induced metric. As a corollary, minimal Lagrangian (resp. minimal Legendrian) submanifolds are characterized as being those Lagrangian (resp. Legendrian) submanifolds with constant Lagrangian (resp. Legendrian) angle. Since a convenient rotation adds a fixed constant to the Lagrangian (resp. Legendrian) angle, there is no loss of generality in restricting ourselves to the study of submanifolds with vanishing angle, which are known in the literature, in the Lagrangian case, as *Special Lagrangian submanifolds.* By analogy, we shall refer to Legendrian submanifolds with vanishing Legendrian angle as *Special Legendrian submanifolds.*

Some attention has been devoted recently to the study of Special Legendrian submanifolds (cf [Jo2], [Ha],[BG],[McI],[CMcI]), motivated by the fact that the cone over a Special Legendrian submanifold is Special Lagrangian. The latter are of great interest because some deep conjectures about mirror symmetry involve fibrations of some 3-dimensional Calabi-Yau manifolds by Special Lagrangian submanifolds, possibly with singularities (cf [SYZ],[Jo1]).

In this paper we give a characterization of those Legendrian submanifolds of the odd-dimensional unit sphere $\mathbb{S}^{2n+1}$ (with $n \geq 2$) which are foliated by $(n-1)$-dimensional spheres and we refine our description in the Special Legendrian case. Our main result states that the only Special Legendrian submanifolds which are foliated by $(n-1)$-dimensional spheres are the totally geodesic spheres and a one-parameter family of $SO(n)$-equivariant examples. This $SO(n)$-equivariant family appeared for the first time in [CMU]. Here we describe them in terms of some spherical curves and in particular we show that there is a countable family of Special Legendrian submanifolds whose Hopf projection are closed minimal Lagrangian submanifolds in $\mathbb{C}P^n$.

We would like to mention related work: in [CU], it was shown that the only Special Lagrangian submanifolds of $\mathbb{C}^{n+1}$ which are foliated by n-spheres are the Special Lagrangian planes and the so-called *Lagrangian catenoid;* in [ACR] and in [AR], those Lagrangian submanifolds of $\mathbb{C}^{n+1}$ which are foliated by n-spheres have been characterized, recovering as a corollary the result of [CU]. Finally the method used in the present paper to study equivariant Legendrian submanifolds is analogous to the one which is exploited in [An] and [ACR] for the study of equivariant Lagrangian submanifolds.

The remainder of the article is organized as follows: in Section 1 we state some notations and basic definitions about Legendrian submanifolds. In Section 2 we treat the particular case of Legendrian submanifolds which are foliated by great spheres and prove that the only Special Legendrian submanifolds which are foliated by great spheres are themselves great spheres. In Section 3 we treat the general case and prove that the Special Legendrian submanifolds which are foliated by spheres are great spheres or equivariant, which completes the proof of the main theorem. In Section 4 we study in greater detail equivariant special Legendrian submanifolds.

1 Notations and preliminaries

We shall denote by $\langle\langle .,. \rangle\rangle$ the canonical Hermitian product of $\mathbb{C}^{n+1}$, and by J its standard complex structure, *i.e.* scalar multiplication by $i = \sqrt{-1}$. The Hermitian structure yields both a Riemannian and a symplectic structure: in fact, the canonical Euclidean inner product of $\mathbb{R}^{2n+2} \simeq \mathbb{C}^{n+1}$ is nothing but the real part of the Hermitian product: $\langle .,. \rangle = \mathrm{Re}\langle\langle .,. \rangle\rangle$, and the canonical symplectic structure is given by $\omega := \langle ., J. \rangle$.

Legendrian submanifolds of the sphere $\mathbb{S}^{2n+1}$ are defined as follows: at some point p of $\mathbb{S}^{2n+1}$, we consider the hyperplane of $T_p\mathbb{S}^{2n+1}$ orthogonal to Jp. This defines the *contact distribution,* and Legendrian submanifolds are just integral submanifolds of maximal dimension, (namely n) with respect to this distribution. In other words, an n-dimensional submanifold $\mathcal{L}$ of $\mathbb{S}^{2n+1}$ is Legendrian if $\langle v, Jp \rangle = 0$ for any point p of $\mathcal{L}$ and any tangent vector $v \in T_p\mathcal{L}$.

On the other hand, an n-dimensional submanifold of a symplectic manifold of dimension $2n$ is said to be *Lagrangian* if the symplectic form vanishes on it.

It is elementary to check that the cone over a Legendrian submanifold of $\mathbb{S}^{2n+1}$ is a Lagrangian submanifold of $\mathbb{C}^{n+1}$. Another classical fact is that the image by the Hopf projection Π of a Legendrian submanifold of $\mathbb{S}^{2n+1}$ is a Lagrangian submanifold of $\mathbb{C}P^n$.

The *Legendrian angle* of an oriented Legendrian submanifold $\mathcal{L}$ of $\mathbb{S}^{2n+1}$ is defined as follows: at some point p of $\mathcal{L}$ take an (oriented) orthonormal basis $(v_1, ..., v_n)$ of $T_p\mathcal{L}$. Then the Legendrian assumption implies that $(p, v_1, ..., v_n)$ is a Hermitian basis of $\mathbb{C}^{n+1}$. Moreover its determinant does not depend on the choice of the basis $(v_1, ..., v_n)$. We then have $\det_{\mathbb{C}}(p, v_1, ..., v_n) = e^{i\beta}$ for some β which depends only on the point p and the tangent space. This defines the Legendrian angle function $\beta : \mathcal{L} \to \mathbb{R}/2\pi\mathbb{Z}$. A Legendrian submanifold with vanishing Legendrian angle is said to be *Special Legendrian.* The fundamental relation $nH = J\nabla\beta$ between the Legendrian angle and the mean curvature H is proved in [CLU]. A corollary of this formula is that a minimal, connected Legendrian submanifold $\mathcal{L}$ has constant Legendrian angle β_0; moreover the submanifold $e^{-i\beta_0/(n+1)}.\mathcal{L}$ obtained from $\mathcal{L}$ by performing the rotation $p \mapsto e^{-i\beta_0/(n+1)}p$ has vanishing Legendrian angle, and thus is Special Legendrian. So Special Legendrian subamnifolds and minimal Legendrian submanifolds have the same geometry.

In the following, the prime ' will denote derivative with respect to the variable s and the subscript s will stand for the partial derivative with respect to s. We shall denote by $(\epsilon_1, ..., \epsilon_n)$ the canonical basis of $\mathbb{R}^n$.

2 Legendrian submanifolds foliated by great spheres

Proposition 1 *Any Legendrian submanifold of $\mathbb{S}^{2n+1}$ which is foliated by great spheres is locally the image of an immersion of the form*

$$\begin{array}{cccc} X: & I \times \mathbb{S}^{n-1} & \to & \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1} \\ & (s,x) & \mapsto & M(s)(x,0) = \sum_{j=1}^n x_j e_j(s), \end{array}$$

where $M(s) = (e_1(s), ..., e_{n+1}(s)) \in SU(n+1)$ *is a solution of following differential system:*

$$M^{-1}M' = \begin{pmatrix} 0 & \dots & 0 & -\bar{c}_1(s) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -\bar{c}_n(s) \\ c_1(s) & \dots & c_n(s) & 0 \end{pmatrix},$$

where $c_1(s), \dots, c_n(s)$ *are* n *complex-valued functions.*

Proof. Let $\mathcal{L}$ be a submanifold of dimension n in $\mathbb{S}^{2n+1}$ which is foliated by great spheres of dimension $n-1$. Locally, it may be parametrized by the following immersion:

$$\begin{array}{cccc} X: & I \times \mathbb{S}^{n-1} & \longrightarrow & \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1} \\ & (s, x = (x_1, ..., x_n)) & \mapsto & \sum_{j=1}^{n} x_j e_j(s), \end{array}$$

where $(e_1(s), ..., e_n(s))$ is an orthonormal moving frame in $\mathbb{R}^{2n+2}$.

Let $v = (v_1, ..., v_n)$ be a tangent vector to some point $x = (x_1, ..., x_n)$ of $\mathbb{S}^{n-1}$. Then $X_* v = \sum_{j=1}^{n} v_j e_j(s)$ and the Legendrian condition is equivalent to the two following conditions: $\langle X_* v, JX \rangle = 0$ and $\langle X_s, JX \rangle = 0$.

Applying the equation $\langle X_* v, JX \rangle = 0$ to $x = \epsilon_j$ and $v = \epsilon_k$ yields

$$\langle e_j, Je_k \rangle = 0, \quad \forall j, k, \ 1 \leq j, k \leq n.$$

Thus the subspace spanned by $e_1, ..., e_n$ is isotropic in $\mathbb{C}^{n+1}$ (i.e. the symplectic form ω vanishes on it). In particular we may complete this frame by a unit vector e_{n+1} such that the span of $(e_1, ..., e_{n+1})$ is a special Lagrangian space, or in other words the matrix $M := (e_1, ..., e_{n+1})$ belongs to $SU(n+1)$. We shall see that the other condition for $\mathcal{L}$ to be Legendrian (and later the minimality condition) may be expressed in terms of the coefficients c_{jk} of the matrix $M^{-1}M'$. This matrix belongs to $\mathfrak{su}(n+1)$, so its real part is anti-symmetric and its imaginary part is symmetric. The next lemma claims that the gauge invariance of the problem allows us to assume that the coefficients $a_{jk} = \mathrm{Re}\, c_{jk}$ vanish.

Lemma 1 *We may assume, after a possible reparametrization of* $\mathcal{L}$*, that* $\forall\, j, k$, $1 \leq j, k \leq n$, $a_{jk} = \mathrm{Re}\, c_{jk}$ *vanishes, where* c_{jk}*s are the coefficients of the matrix* $M^{-1}M'$ *given in Proposition 1.*

Proof. We reparametrize our submanifold by replacing $(e_1, ..., e_n)$ by $(\bar{e}_1, ...\bar{e}_n)$, with $\bar{e}_j = \sum_{j=1}^n d_{jk} e_k$. In other words, $(\bar{e}_1, ...\bar{e}_n) = N(e_1, ..., e_n)$, where $N = [d_{jk}]_{1 \leq j,k \leq n}$ is a curve of matrices in $SO(n)$.

We derive this expression and we get

$$\bar{e}'_j = \sum_{k=1}^n d'_{jk} e_k + d_{jk} e'_k = \sum_{k=1}^n d'_{jk} e_k + \sum_{k=1}^n \sum_{\alpha=1}^{n+1} d_{jk} a_{k\alpha} e_\alpha$$

Hence

$$\bar{a}_{jl} := \langle \bar{e}'_j, \bar{e}_l \rangle = \sum_{k=1}^n d'_{jk} d_{kl} + \sum_{k=1}^n \sum_{\alpha=1}^n d_{jk} a_{k\alpha} d_{\alpha l} = \sum_{k=1}^n d_{kl} \left(d'_{jk} + \sum_{m=1}^n d_{jm} a_{mk} \right)$$

So $\bar{a}_{jl} = 0 \; \forall j, l, \; 1 \leq j, l \leq n$ if and only if

$$d'_{jk} = -\sum_{m=1}^n d_{jm} a_{mk}$$

The existence of the required d_{jk}s follows now from the existence theorem for differential systems. As the matrix $[a_{jk}]_{1 \leq j,k \leq n}$ belongs to $\mathfrak{so}(n)$, by choosing suitable initial condition we have that $[d_{jk}]_{1 \leq j,k \leq n}$ belongs to $SO(n)$.

To complete the proof of Proposition 1, it remains to examine the second equation $\langle X_s, JX \rangle = 0$. As $X_s = \sum_{j=1}^n x_j e'_j$, we have

$$\sum_{j,k=1}^n x_j x_k \langle e'_j, J e_k \rangle = 0.$$

If we set $x = \epsilon_j$ in the latter, we find that $\operatorname{Im} c_{jj} = \langle e'_j, J e_j \rangle = 0$. Then, writing $x = \frac{\sqrt{2}}{2}(\epsilon_j + \epsilon_k)$ one has

$$\frac{1}{2}(\langle e'_j, J e_k \rangle + \langle e'_k, J e_j \rangle) = 0$$

On the other hand the matrix $M^{-1}M'$ belongs to $\mathfrak{su}(n+1)$, so we also have $\operatorname{Im} c_{jk} - \operatorname{Im} c_{kj} = \langle e'_j, J e_k \rangle - \langle e'_k, J e_j \rangle = 0$, and we deduce that $\operatorname{Im} c_{jk} = \langle e'_j, J e_k \rangle = 0 \; \forall j, k, \; 1 \leq j, k \leq n$. Finally, as $M^{-1}M'$ is traceless, it follows that $c_{n+1,n+1}$ vanishes, which concludes the proof.

Example 1 If all coefficients c_{jk} of the matrix $M^{-1}M'$ are real, we claim that $\mathcal{L}$ is a piece of the totally geodesic sphere $\mathbb{S}^n$.

To see this, we first observe that without loss of generality, we may assume that $(e_1(s_0), ..., e_{n+1}(s_0)) = (\epsilon_1, ..., \epsilon_n)$. Thus, $\forall s \in I$, the coefficients of M are real, so $X(s,x) \in \mathbb{R}^{n+1}$. In the general case, $X(s,x)$ will stay in the (real) span of $(e_1(s_0), ...e_{n+1}(s_0))$.

Theorem 1 *The only Legendrian minimal submanifolds of $\mathbb{S}^{2n+1}$ which are foliated by great $(n-1)$-spheres are pieces of totally geodesic n-spheres.*

Proof. As we have seen in the previous section, we may restrict our proof to the case of Special Legendrian submanifolds. Let $(v_1, ..., v_{n-1})$ be an orthonormal basis of $T_x\mathbb{S}^{n-1}$ at some point x. Thus $(X, X_*v_1, ..., X_*v_{n-1})$ is an orthonormal frame and completing it with e_{n+1} gives a Hermitian basis in $\mathbb{C}^{n+1}$, which we are going to use in order to give a quick computation of the Legendrian angle. In fact, if β vanishes then

$$\det_{\mathbb{C}}(X, X_*v_1, ..., X_*v_{n-1}, X_s) \in \mathbb{R},$$

We recall that $X_s = \sum_{j=1}^{n} x_j e'_j$, so we have

$$\langle\langle X_s, e_{n+1}\rangle\rangle = \sum_{j=1}^{n} x_j \langle\langle e'_j, e_{n+1}\rangle\rangle = \sum_{j=1}^{n} x_j c_{j,n+1}$$

Hence

$$\det_{\mathbb{C}}(X, X_*v_1, ..., X_*v_{n-1}, X_s) = \sum_{j=1}^{n} x_j c_{j,n+1},$$

and β vanishes if and only if the coefficients $c_{j,n+1}$ are real. So we conclude as in the Example 1 that $\mathcal{L}$ is locally to a piece of the sphere $\mathbb{S}^n = \mathbb{R}^{n+1} \cap \mathbb{S}^{2n+1}$

3 Legendrian submanifolds foliated by spheres

Proposition 2 *Any Legendrian submanifold of $\mathbb{S}^{2n+1}$ which is foliated by $(n-1)$-dimensional spheres which are not great spheres is locally the image of an* *immersion of the form*

$$\begin{array}{cccc} Y: & I \times \mathbb{S}^{n-1} & \to & \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1} \\ & (s, x = (x_1, ..., x_n)) & \mapsto & M(s)(\cos\theta(s)x, \sin\theta(s)), \end{array}$$

where $\theta(s)$ *is some function with values in* $(0, \pi/2)$ *and* $M(s) \in U(n+1)$ *is a solution of the following differential system:*

$$M^{-1}M' = \begin{pmatrix} i\lambda(s)\tan^2\theta(s) & ... & 0 & -a_1(s) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & ... & i\lambda(s)\tan^2\theta(s) & -a_n(s) \\ a_1(s) & ... & a_n(s) & -i\lambda(s) \end{pmatrix},$$

where $a_1(s), ..., a_n(s)$ *and* $\lambda(s)$ *are real valued functions. If we denote by* $e_j(s)$, $1 \leq jleqn+1$, *the columns of* $M(s)$, *to any value* s *there corresponds a spherical leaf whose radius is* $\cos\theta(s)$, *whose center is* $\sin\theta(s)e_{n+1}(s)$ *and which lies in the* n*-space spanned by* $(e_1(s), ..., e_n(s))$.

Proof. Let $\mathcal{L}$ be a submanifold of $\mathbb{S}^{2n+1}$ which is foliated by $(n-1)$-dimensional spheres. Locally, it may be parametrized by the following immersion:

$$\begin{array}{cccc} Y: & I \times \mathbb{S}^{n-1} & \to & \mathbb{S}^{2n+1} \subset \mathbb{R}^{2n+2} \\ & (s, x = (x_1, ..., x_n)) & \mapsto & \cos\theta(s)\left(\sum_{j=1}^n x_j e_j(s)\right) + \sin\theta(s)e_{n+1}(s), \end{array}$$

where $(e_1, ..., e_{n+1})$ is an orthonormal frame and $\theta(s)$ is some real function. The vanishing of $\sin\theta$ corresponds to a great circle, and if $\cos\theta$ vanishes, the spherical leaf degenerates to a point, so the submanifold has a singularity.

We introduce $X(s,x) = \sum_{j=1}^n x_j e_j(s)$, so that $Y(s,x) = \cos\theta(s)X(s,x) + \sin\theta(s)e_{n+1}(s)$, and

$$Y_* v = \cos\theta X_* v = \cos\theta \sum_{j=1}^n v_j e_j(s)$$

$$Y_s = -\theta' \sin\theta X + \cos\theta X_s + \theta' \cos\theta e_{n+1} + \sin\theta e'_{n+1}$$

$$= -\theta' \sin\theta(\sum_{j=1}^n x_j e_j) + \cos\theta(\sum_{j=1}^n x_j e'_j) + \theta' \cos\theta e_{n+1} + \sin\theta e'_{n+1}$$

Thus if $\mathcal{L}$ is Legendrian it follows that

$$\langle Y_* v, JY \rangle = \cos^2\theta \langle X_* v, JX \rangle + \cos\theta\sin\theta \langle X_* v, Je_{n+1} \rangle$$

$$= \cos^2\theta \sum_{j,k=1}^{n} v_j x_k \langle e_j, Je_k\rangle + \cos\theta\sin\theta \left(\sum_{j=1}^{n} v_j \langle e_j, Je_{n+1}\rangle \right) = 0 \qquad (1)$$

Next, we have to consider two cases, according to the value of the dimension n:

Case 1: $n > 2$. Here, the argument is based on the following observation: given any vector $v \in \mathbb{R}^n$, there exists a pair (x, y) of distinct elements of $\mathbb{S}^{n-1}$ such that v is tangent to both x and y. Thus, substracting Equation (1) applied to (x, v) and to (y, v), we get

$$\cos^2\theta \sum_{j,k=1}^{n} v_j (x_k - y_k) \langle e_j, Je_k\rangle = 0;$$

in the latter we set $v = \epsilon_j$ and choose x and y such that $x - y = \epsilon_k$; this implies that

$$\langle e_j, Je_k\rangle = 0 \quad \forall j, k, \ 1 \leq j, k \leq n.$$

Going back to Equation (1), we also get the vanishing of $\langle e_j, Je_{n+1}\rangle$.

Case 2: $n = 2$. We write $x = (\cos t, \sin t)$, so that $v = (-\sin t, \cos t)$ is a tangent vector to $\mathbb{S}^1$ at x. Then we compute:

$$\langle Y_* v, JY\rangle = \cos^2\theta \langle X_* v, JX\rangle + \cos\theta\sin\theta \langle X_* v, Je_3\rangle$$

$$= \cos^2\theta \langle e_2, Je_1\rangle + \cos\theta\sin\theta(-\sin t\langle e_1, Je_3\rangle + \cos t\langle e_2, Je_3\rangle) = 0$$

It follows that $\langle e_2, Je_1\rangle$, $\langle e_1, Je_3\rangle$ and $\langle e_2, Je_3\rangle$ all vanish.

In both cases, we deduce that the span of $e_1, ..., e_{n+1}$ must be a Lagrangian plane (but not necessarily special Lagrangian) in $\mathbb{C}^{n+1}$; in other words $(e_1, ..., e_{n+1})$ belongs to $U(n+1)$ but not *a priori* to $SU(n+1)$.

Next we turn our attention to the second equation, that is $\langle Y_s, JY\rangle = 0$. We compute:

$$\langle Y_s, JY\rangle = \langle \cos\theta X_s + \sin\theta e'_{n+1} - \theta' \sin\theta X + \theta' \cos\theta e_{n+1}, \cos\theta JX + \sin\theta Je_{n+1}\rangle$$

$$= \cos^2\theta(\langle X_s, JX\rangle + \theta'\langle e_{n+1}, JX\rangle) + \cos\theta\sin\theta(\langle X_s, Je_{n+1}\rangle + \langle e'_{n+1}, JX\rangle)$$

$$+\sin^2\theta(\langle e'_{n+1}, Je_{n+1}\rangle - \theta'\langle X, Je_{n+1}\rangle)$$

$$= \cos^2\theta\langle X_s, JX\rangle + 2\cos\theta\sin\theta\langle X_s, Je_{n+1}\rangle + \sin^2\theta\langle e'_{n+1}, Je_{n+1}\rangle$$

We have used the fact that X is a linear combination of $e_1, ..., e_n$, so $\langle X, Je_{n+1}\rangle$ vanishes. Finally, we get

$$\langle Y_s, JY\rangle = \cos^2\theta\sum_{j,k=1}^{n} x_j x_k\langle e'_j, Je_k\rangle + 2\cos\theta\sin\theta\sum_{j=1}^{n} x_j\langle e'_j, Je_{n+1}\rangle + \sin^2\theta\,\mathrm{Im}\, c_{n+1,n+1}$$

$$= \cos^2\theta\sum_{j,k=1}^{n} x_j x_k \mathrm{Im}\, c_{jk} + 2\cos\theta\sin\theta\sum_{j=1}^{n} x_j \mathrm{Im}\, c_{j,n+1} + \sin^2\theta\,\mathrm{Im}\, c_{n+1,n+1}.$$

This is a polynomial of degree 2 in the variables $x_1, ..., x_n$ which must vanish on $\mathbb{S}^{n-1}$, that is the zero set of the polynomial $\sum_{i=1}^n x_i^2 - 1$. It follows that the former is a multiple of the latter, which implies the following

- $\mathrm{Im}\, c_{j,n+1}$ vanishes,
- $\mathrm{Im}\, c_{jk}$ vanishes if $j \neq k$,
- $\cos^2\theta\,\mathrm{Im}\, c_{jj} = \sin^2\theta\,\mathrm{Im}\, c_{n+1,n+1}, \forall j,\ 1 \leq j \leq n.$

So the proof is complete.

Example 2 Let $\gamma = (\gamma_1, \gamma_2)$ be a unit speed Legendrian curve in $\mathbb{S}^3$ such that γ_1 never vanishes, then the following immersion is Legendrian

$$\begin{array}{rccc} X: & I \times \mathbb{S}^{n-1} & \to & \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1} \\ & (s, x = (x_1, ..., x_n)) & \mapsto & (\gamma_1(s)x, \gamma_2(s)). \end{array}$$

Moreover, following the notation of Proposition 2, we have

$$M = \mathrm{diag}\left(\frac{\gamma_1}{|\gamma_1|}, ..., \frac{\gamma_1}{|\gamma_1|}, \frac{\gamma_2}{|\gamma_2|}\right)$$

and

$$M^{-1}M' = \mathrm{diag}(i\arg(\gamma_1)', ..., i\arg(\gamma_1)', i\arg(\gamma_2)')$$

$$= \mathrm{diag}\left(i\frac{\langle\gamma_1', i\gamma_1\rangle}{|\gamma_1|^2}, ..., i\frac{\langle\gamma_1', i\gamma_1\rangle}{|\gamma_1|^2}, i\frac{\langle\gamma_2', i\gamma_2\rangle}{|\gamma_2|^2}\right).$$

The image of this immersion if invariant by the action of $SO(n)$ on $\mathbb{S}^{2n+1}$ defined as follows: consider the embedding of $SO(n)$ into $U(n+1)$ given by $A \mapsto \tilde{A} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Then the $SO(n)$-action is defined by $p \mapsto \tilde{A}p$, $\forall p \in \mathbb{S}^{2n+1}$. This family of Legendrians will be studied in greater detail in the next Section.

Theorem 2 *The Legendrian minimal submanifolds of $\mathbb{S}^{2n+1}$ which are foliated by spheres are either locally congruent to a totally geodesic sphere or to a $SO(n)$-equivariant minimal submanifold.*

Proof. Let Y be an immersion as in Proposition 2 and let $(v_1, ..., v_{n-1})$ be an orthonormal basis of $T_x\mathbb{S}^{n-1}$ at some point x. Then $(X, X_*v_1, ..., X_*v_{n-1})$ is an orthonormal basis which spans the same subspace as $(e_1, ..., e_n)$. We are going to work with several Hermitian frames of $\mathbb{C}^{n+1}$: let $\mathcal{B}_0$ be the canonical basis $(\epsilon_1, ..., \epsilon_{n+1})$, $\mathcal{B} = (e_1(s), ..., e_{n+1}(s))$ and $\mathcal{B}' = (X, X_*v_1, ..., X_*v_{n-1}, e_{n+1})$. Moreover, we call β_0 the Lagrangian angle of the Lagrangian subspace spanned by $(e_1, ..., e_{n+1})$, so we have

$$\det_{\mathbb{C}}(\mathcal{B})_{\mathcal{B}_0} = e^{i\beta_0}.$$

Observe also that the following holds: $\det_{\mathbb{C}}(\mathcal{B})_{\mathcal{B}'} = 1$.

Now, Y is special Legendrian if and only if the following determinant is real:

$$\det_{\mathbb{C}}(Y, Y_*v_1, ..., Y_*v_{n-1}, Y_s)_{\mathcal{B}_0}$$

So we compute:

$$\det_{\mathbb{C}}(Y, Y_*v_1, ..., Y_*v_{n-1}, Y_s)_{\mathcal{B}_0} = \det_{\mathbb{C}}(Y, Y_*v_1, ..., Y_*v_{n-1}, Y_s)_{\mathcal{B}} \det_{\mathbb{C}}(\mathcal{B})_{\mathcal{B}_0}$$

$$= \cos^n\theta \det_{\mathbb{C}}(X, X_*v_1, ..., X_*v_{n-1}, Y_s)_{\mathcal{B}} e^{i\beta_0} = \cos^n\theta \det_{\mathbb{C}}(X, X_*v_1, ..., X_*v_{n-1}, Y_s)_{\mathcal{B}'} e^{i\beta}$$

$$= \cos^n\theta(\cos\theta\langle\langle Y_s, e_{n+1}\rangle\rangle - \sin\theta\langle\langle Y_s, X\rangle\rangle)e^{i\beta_0}$$

We now compute $Y_s = -\theta'\sin\theta X + \cos\theta X_s + \theta'\cos\theta e_{n+1} + \sin\theta e'_{n+1}$. It follows that

$$\langle\langle Y_s, e_{n+1}\rangle\rangle = \cos\theta\langle\langle X_s, e_{n+1}\rangle\rangle + \theta'\cos\theta + \sin\theta\langle\langle e'_{n+1}, e_{n+1}\rangle\rangle$$

$$= \cos\theta \sum_{j=1}^{n} x_j c_{j,n+1} + \theta' \cos\theta + \sin\theta c_{n+1,n+1}$$

$$= \cos\theta \sum_{j=1}^{n} x_j a_j + \theta' \cos\theta - i\lambda \sin\theta$$

and that

$$\langle\langle Y_s, X\rangle\rangle = -\theta' \sin\theta + \cos\theta \langle\langle X_s, X\rangle\rangle + \theta' \cos\theta \langle\langle X_s, e_{n+1}\rangle\rangle + \sin\theta \langle\langle e'_{n+1}, X\rangle\rangle$$

$$= -\theta' \sin\theta - \sin\theta \sum_{j=1}^{n} x_j c_{j,n+1} + \cos\theta \sum_{j=1}^{n} x_j^2 c_{jj}$$

$$= -\theta' \sin\theta - \sin\theta \sum_{j=1}^{n} x_j a_j + i \cos\theta \lambda \tan^2\theta$$

it implies that

$$\det_{\mathbb{C}}(Y, Y_* v_1, ..., Y_* v_{n-1}, Y_s)_{\mathcal{B}_0} = e^{i\beta_0} \cos^n\theta \left((\theta' + \sum_{j=1}^{n} x_j a_j) - i\lambda \sin\theta \cos\theta (\tan^2\theta + 1) \right)$$

$$= e^{i\beta_0} \cos^n\theta \left((\theta' + \sum_{j=1}^{n} x_j a_j) - i\lambda \tan\theta \right)$$

Therefore if the latter is real, we have two cases:

a) either λ vanishes and β_0 is constant, so the coefficients of $M^{-1}M'$ are real and hence $\mathcal{L}$ is a (piece of a) totally geodesic sphere (cf Example 1),

b) or all the coefficients a_j vanish (and some condition on β_0 and θ holds). Then in particular the matrix $M^{-1}M'$ is diagonal, so M is as well: thus we are in the equivariant case.

4 Equivariant special Legendrian submanifolds

Let X be a Legendrian immersion as in Example 2:

$$\begin{array}{cccc} X : & I \times \mathbb{S}^{n-1} & \to & \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1} \\ & (s, x = (x_1, ..., x_n)) & \mapsto & (\gamma_1(s)x, \gamma_2(s)). \end{array}$$

It is not difficult to see that X is special Legendrian if and only if $(n-1)\arg\gamma_1 + \beta_\gamma = 0$, where β_γ is the Legendrian angle of the Legendrian curve γ. Deriving this equation, we get

$$(n-1)\frac{\langle\gamma_1', i\gamma_1\rangle}{|\gamma_1|^2} + k_\gamma = 0, \tag{2}$$

where we have used the fact that the derivative of β_γ is the curvature of γ. Now, let $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{S}^2 \simeq \mathbb{C}P^1$ be the image of γ under the Hopf projection, i.e. $\xi_1 + i\xi_2 = 2\gamma_1\bar{\gamma}_2$ and $\xi_3 = |\gamma_1|^2 - |\gamma_2|^2$. According to [CC], the curvature of ξ is half the curvature of γ and we have the following relation $\langle\gamma_1', i\gamma_1\rangle = (\xi \times \xi')_3$. This implies that Equation (2) is equivalent to

$$(n-1)\frac{(\xi \times \xi')_3}{1+\xi_3} + 2k_\xi = 0. \tag{3}$$

In order to solve Equation (3), we shall use spherical coordinates on $\mathbb{S}^2$: we write $\xi(s) = (\cos\phi(s)e^{i\theta(s)}, \sin\phi(s))$, where $e^{i\theta} \in \mathbb{S}^1$ and $\phi \in (-\pi/2, \pi/2)$. This implies that $\xi' = ((-\phi'\sin\phi + i\theta'\cos\phi)e^{i\theta}, \phi'\cos\phi)$. As $\xi(s)$ is parametrized by arclength, there exists some function α such that $\cos\alpha = \phi'$ and $\sin\alpha = \theta'\cos\phi$. Now, the curvature of ξ in $\mathbb{S}^2$ is given by $k_\xi = \det(\xi, \xi', \xi'') = \alpha' - \theta'\sin\phi$ (cf [Ku]).

So finally we are reduced to studying of the system

$$\begin{cases} (n-1)\frac{\sin\alpha\cos\phi}{1+\sin\phi} + 2(\alpha' - \theta'\sin\phi) = 0 \\ \theta' = \frac{\sin\alpha}{\cos\phi} \\ \phi' = \cos\alpha, \end{cases}$$

which reduces to

$$\begin{cases} \alpha' = \sin\alpha(\tan\phi - \frac{n-1}{2}\frac{\cos\phi}{1+\sin\phi}) \\ \phi' = \cos\alpha. \end{cases} \tag{4}$$

This system has an equilibrium point $(\alpha_0, \phi_0) = (\pi/2, \arcsin\frac{n-1}{n+1})$. The corresponding curve ξ is the horizontal circle of height $\sin\phi_0 = \frac{n-1}{n+1}$. Its Legendrian lift is the curve $\gamma(s) = 1/\sqrt{n+1}(\sqrt{n}e^{is/\sqrt{n}}, e^{-i\sqrt{n}s})$ and the corresponding Special Legendrian immersion is

$$\begin{array}{cccc} X: & \mathbb{S}^1 \times \mathbb{S}^{n-1} & \to & \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1} \\ & (e^{is}, x = (x_1, ..., x_n)) & \mapsto & \frac{1}{\sqrt{n+1}}(\sqrt{n}e^{is}x, e^{-ins}). \end{array}$$

According to [CMU], the corresponding minimal Lagrangian immersion $\Pi \circ X$ has been first described by Naitoh in [Na].

Next we observe that System (4) admits a first integral:

$$E(\alpha, \phi) = \sin\alpha\cos\phi(1+\sin\phi)^{(n-1)/2}.$$

The extremal values of the energy E are 0 and $E_0 := \cos\phi_0(1+\sin\phi_0)^{(n-1)/2}$.

We discuss now the closedness properties of the spherical curves ξ. Let $c(E) := (\alpha(s), \phi(s))$ be the integral curve of the system (4) with energy level E. The corresponding spherical curve ξ turns around the vertical axis by an amount equal to the value of the integral $\Theta(E) := \int_{c(E)} \theta' ds$; if this number equals a rational number p/q times 2π, by making q loops on the integral curve $c(E)$, we get a closed curve ξ.

If $\Theta(E)$ is not rationally related to 2π, the full curve ξ makes infinitely many winds and is dense in some subset of the sphere.

In order to get the existence of values of $\Theta(E)$ which are rationally related to 2π, we use the fact that the function $E \mapsto \Theta(E)$ is continuous and we compute its limits:

Lemma 2 *The following limits hold:*

(i) $\lim_{E\to E_0} \Theta(E) = \frac{n+1}{\sqrt{n(n+1)}}\sqrt{2}\pi$,
(ii) $\lim_{E\to 0} \Theta(E) = \frac{n+1}{n}\pi$.

Proof. We shall denote by $\phi_-(E)$ and $\phi_+(E)$ the minimal and maximal values taken by ϕ on the curve $c(E)$; this corresponds to the intersections of $c(E)$ with the straight line $\{\alpha = \pi/2\}$ (cf the phase portrait, Figure 1). We also introduce the function $e(\phi) = \cos\phi(1+\sin\phi)^{(n-1)/2}$, so that $E = e(\phi_-) = e(\phi_+)$. Using the invariance of $c(E)$ by the symmetry $\alpha \mapsto \pi/2 - \alpha$, we get the following expressions for $\Theta(E)$:

$$\begin{aligned}\Theta(E) &= 2\int_{C(E)\cap\{\alpha<\pi/2\}} \frac{\sin\alpha}{\cos\phi} ds = 2\int_{\phi_-}^{\phi_+} \frac{E d\phi}{e(\phi)\cos\phi\sqrt{1-\frac{E^2}{e^2(\phi)}}} \\ &= 2\int_{\phi_-}^{\phi_+} \frac{E d\phi}{\cos\phi\sqrt{e^2(\phi)-E^2}}.\end{aligned}$$

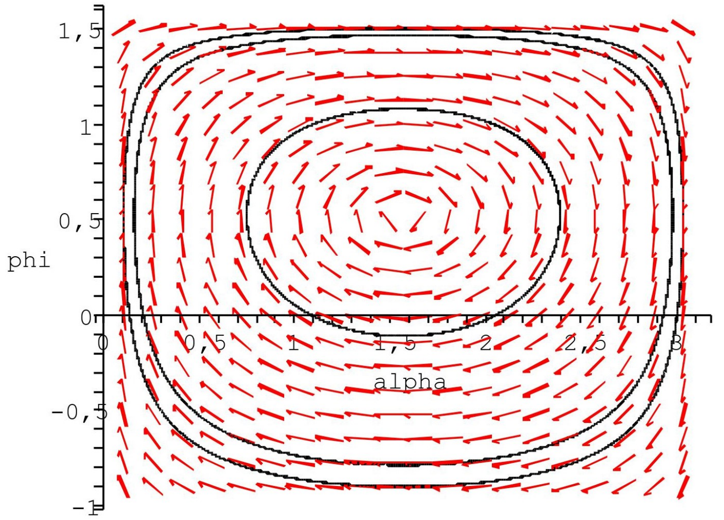

Figure 1: Phase portrait, $n = 3$

In the following we shall split this integral in two parts:

$$\Theta(E) = \Theta_1(E) + \Theta_2(E) := 2\int_{\phi_-}^{\phi_0} + 2\int_{\phi_0}^{\phi_+}$$

Proof of (i). A straightforward computation shows that $e'(\phi_0)$ vanishes and that

$$e''(\phi_0) = e(\phi_0)(-1 - 2\tan^2\phi_0 - \frac{\sin\phi_0}{1+\sin\phi_0}) = -E_0\frac{n+1}{2},$$

so that, writing $\epsilon = \phi - \phi_0$ and $\phi = \phi_0 + \epsilon x$, we get

$$e(\phi) = E_0\left(1 - \frac{n+1}{2}(\epsilon x)^2 + o(\epsilon x)^2\right).$$

We make the change of variable $\phi = \phi_0 + \epsilon x$, $x \in [-1, 0]$:

$$\Theta_1(E) = 2\int_{-1}^{0} \frac{(E_0 + \epsilon^2/2e''(\phi_0) + o(\epsilon^2))\epsilon dx}{(\cos\phi_0 + o(1))\sqrt{(E_0 + (\epsilon x)^2 e''(\phi_0)/2 + o(\epsilon^2))^2 - (E_0 + \epsilon^2 e''(\phi_0)/2 + o(\epsilon^2))^2}}$$

$$= 2\int_{-1}^{0} \frac{E_0 dx}{(\cos\phi_0 + o(1))\sqrt{E_0^2 \frac{n+1}{2}(1 - x^2) + o(\epsilon^2)}} + o(1).$$

Therefore

$$\lim_{E\to E_0} \Theta_2(E) = \lim_{E\to E_0} \frac{2}{\cos\phi_0\sqrt{\frac{n+1}{2}}} \int_{-1}^{0} \frac{dx}{\sqrt{1-x^2}} = \frac{n+1}{\sqrt{n(n+1)}}\frac{\sqrt{2}}{2}\pi.$$

The computation of $\Theta_2(E)$ is analogous and we conclude that $\lim_{E\to E_0}\Theta(E) = \frac{n+1}{\sqrt{n(n+1)}}\sqrt{2}\pi$.

Proof of (ii). We first make the change of variable $\phi = -\pi/2 + \epsilon_- x$, $x \in [1, \epsilon_0/\epsilon_-]$, where $\epsilon_- = \phi_- + \pi/2$ and $\epsilon_0 = \phi_0 + \pi/2$. Therefore

$$\Theta_1(E) = 2\int_{1}^{\epsilon_0/\epsilon_-} \frac{\epsilon_- dx}{\cos(-\pi/2 - \epsilon_- x)} \left(\frac{\cos^2(-\pi/2 + \epsilon_- x)(1 + \sin(-\pi/2 + \epsilon_- x))^{n-1}}{\cos^2(-\pi/2 + \epsilon_-)(1 + \sin(-\pi/2 + \epsilon_-))^{n-1}} - 1 \right)^{-1/2}.$$

As E tends to 0, ϵ_- tends to 0 and the integrand of the latter converges uniformly on any compact interval $[1, M]$ to $\frac{1}{x\sqrt{x^{2n}-1}}$. It follows that

$$\lim_{E\to 0} \Theta_1(E) = 2\int_1^\infty \frac{dx}{x\sqrt{x^{2n}-1}} = \frac{\pi}{n}.$$

In order to compute the limit of Θ_2 when E tends to 0, the method is analogous: we make the change of variable $\phi = \pi/2 - \epsilon_+ x$, $x \in [1, \epsilon_0/\epsilon_+]$, where $\epsilon_+ = \phi_+ - \pi/2$ and $\epsilon_0 = \phi_0 - \pi/2$. Therefore

$$\Theta_1(E) = 2\int_{1}^{\epsilon_0/\epsilon_+} \frac{\epsilon_+ dx}{\cos(\pi/2 - \epsilon_+ x)} \left(\frac{\cos^2(\pi/2 - \epsilon_+ x)(1 + \sin(\pi/2 - \epsilon_+ x))^{n-1}}{\cos^2(\pi/2 - \epsilon_+)(1 + \sin(\pi/2 - \epsilon_+))^{n-1}} - 1 \right)^{-1/2}.$$

As E tends to 0, ϵ_+ tends to 0 and the integrand of the latter converges uniformly on any compact interval $[1, M]$ to $\frac{1}{x\sqrt{x^2-1}}$. It follows that

$$\lim_{E\to 0} \Theta_2(E) = 2\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \pi.$$

Hence we conclude that $\lim_{E\to 0}\Theta(E) = \frac{n+1}{n}\pi$.

This lemma shows that for any number p/q in the interval $(\frac{n+1}{2n}, \frac{n+1}{\sqrt{n(n+1)}}\frac{\sqrt{2}}{2})$ there exists a closed solution ξ to Equation (3). The integers p and q have a geometric meaning: p is the winding number of ξ around the north pole and q is the number of maxima (and minima) of its height ξ_3 (cf Figures 2 and 3). It may be interesting to note that when $E \to 0$ the curve tends to n vertical circles making between them an angle of π/n (cf Figure 4). The Legendrian lift γ of ξ is not

Figure 2: A curve with $n = 5, p = 3, q = 4$, lateral view and view from the top

closed *a priori,* nor the corresponding Special Legendrian submanifold, however the closedness of ξ implies the closedness of the Hopf projection of X: let $s, s' \in \mathbb{R}$ such that $\xi(s) = \xi(s')$; Thus we have $\Pi(\gamma(s)) = \Pi(\gamma(s'))$, i.e. there exists some $\sigma \in \mathbb{R}$ such that $(\gamma_1(s), \gamma_2(s)) = (\gamma_1(s')e^{i\sigma}, \gamma_1(s')e^{i\sigma})$. Since that $X(s,x) = (\gamma_1(s)x, \gamma_2(s))$, we deduce that $X(s,x) = e^{i\sigma}.X(s',x)$, i.e. $\Pi(X(s,x)) = \Pi(X(s',x))$.

Summing up these observations we get the following

Theorem 3 *The equation*

$$(n-1)\frac{(\xi \times \xi')_3}{1+\xi_3} + 2k_\xi = 0,$$

Figure 3: A curve with $n = 8, p = 5, q = 7$, lateral view and view from the top

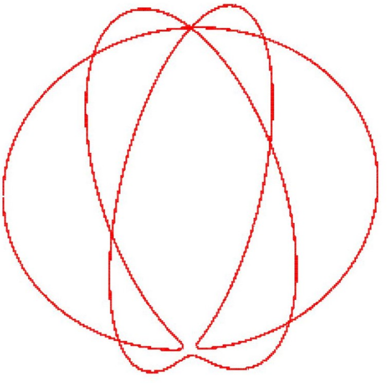

Figure 4: A curve tending to the union of 3 circles, $n = 3$

where $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{S}^2$ *is some spherical curve and* k_ξ *is its curvature, admits a countable family of closed solutions which is parametrized by relatively prime numbers* p *and* q *subject to the condition* $p/q \in (\frac{n+1}{2n}, \frac{n+1}{\sqrt{n(n+1)}}\frac{\sqrt{2}}{2})$*. Let* γ *be the Legendrian lift of such a solution; then the image of the immersion*

$$\begin{array}{rccc} \Pi \circ X : & \mathbb{R} \times \mathbb{S}^{n-1} & \longrightarrow & \mathbb{C}P^n \\ & (s, x = (x_1, ..., x_n)) & \mapsto & \Pi(\gamma_1(s)x, \gamma_2(s)) \end{array}$$

is a closed minimal Lagrangian submanifold of $\mathbb{C}P^n$*.*

It is easy to show that the closedness of the Legendrian lift of ξ amounts to the fact that $\int_{c(E)} \theta' \sin\phi$ is rationally related to 2π. Thus in order to get closed solutions to Equation (2) we must be able to determine if there exist integral curves $c(E)$ such that both integrals $\int_{c(E)} \theta'$ and $\int_{c(E)} \theta' \sin\phi$ are rationally related to 2π, which seems to be a very hard problem.

We end this section by stating briefly the alternative description of the same problem given in [CMU]: writing $|\gamma_1(s)| = \sin r(s)$, any Legendrian curve takes locally the following form:

$$\gamma(s) = (\sin r(s) e^{-i\int_0^s \frac{dt}{\sin^{n+1} r(t)}}, \cos r(s) e^{-i\int_0^s \frac{\tan^2 r(t) dt}{\sin^{n+1} r(t)}})$$

and Equation (2) is then equivalent to

$$r'' \sin r \cos r = (1 - (r')^2)(n \cos^2 r - \sin^2 r), r(0) = \rho, r'(0) = 0.$$

The latter admits a first integral: $E(r, r') = (\sin^{2n} r \cos^2 r)((r')^2 - 1)$; thus the orbits $(r(s), r'(s))$ are closed, but it does not imply *a priori* that the corresponding Legendrian curves or their projections in $\mathbb{S}^2$ are closed.

References

[An] Anciaux, H., *Construction of equivariant self-similar solutions to the mean curvature flow in* $\mathbb{C}^n$, Geom. Dedicata, 120 no. 1,(2006), 37–48.

[ACR] Anciaux, H., Castro, I., Romon, P., *Lagrangian submanifolds foliated by* $(n-1)$*-spheres in* $\mathbb{R}^{2n}$, Acta Math. Sinica, 22 no. 4, (2006), 1197–1214.

[AR] Anciaux, H., Romon, P., *Cyclic and ruled Lagrangian surfaces in complex Euclidean space,* math.DG/0703645.

[BG] Borrelli, V., Gorodski, C., *Minimal Legendrian submanifolds of* $\mathbb{S}^{2n-1}$ *and absolutely area-minimizing cones,* Differential Geom. and its Appl. 21 no. 3, (2004), 337–347.

[CMcI] Carberry, E., McIntosh, I., *Special Lagrangian tori in* $\mathbb{C}P^2$ *come in real families of every dimension,* J. London Math. Soc. (2) 69 no. 2, (2004), 531–544.

[CC] Castro, I., Chen, B.-Y., *Lagrangian surfaces in complex Euclidean plane via spherical and hyperbolic curves,* Tohoku Math. J. 58 no. 4, (2006), 565–579.

[CLU] Castro, I., Li, H., Urbano, F., *Hamiltonian-minimal Lagrangian submanifolds in complex space forms,* Pacific J. Math. 227 no. 1, (2006), 43–63.

[CMU] Castro, I., Montealegre, C. R., Urbano, F., *Minimal Lagrangian submanifolds in the complex hyperbolic space,* Illinois J. Math. 46 no. 3, (2002), 695–721.

[CU] Castro, I., Urbano, F., *On a Minimal Lagrangian Submanifold of* $\mathbb{C}^n$ *Foliated by Spheres,* Michigan Math. J. 46 (1999), 71–82.

[HL] Harvey, R., Lawson, H. B., *Calibrated geometries,* Acta Math. 148 (1982), 47–157.

[Ha] Haskins, M., *Special Lagrangian cones,* Amer. J. of Maths, 126 (2004), 845–871.

[Jo1] Joyce, D., *Singularities of special Lagrangian fibrations and the SYZ conjecture.* Comm. Anal. Geom. 11 no. 5, (2003), 859–907.

[Jo2] Joyce, D., *$U(1)$-invariant special Lagrangian 3-folds in $\mathbb{C}^3$ and special Lagrangian fibrations* Turk. J. Math. 27(2003),99–104.

[Ku] Kühnel, W., *Differential Geometry, Curves—surfaces—manifolds* American Mathematical Society, 2002.

[McI] McIntosh, I., *Special Lagrangian cones in $\mathbb{C}^3$ and primitive harmonic maps* J. London Math. Soc. (2) 67 no. 3, (2003), 769–789.

[Na] Naitoh, H., *Isotropic submanifolds with parallel second fundamental form in $P^m(c)$.* Osaka J. Math. 18 no. 2, (1981), 427–464.

[SYZ] Strominger, A., Yau, S.-T., Zaslow, E., *Mirror symmetry is T-duality.* Nuclear Phys. B 479 no. 1-2, (1996), 243–259.

Departamento de Matemática
Pontifícia Universidade Católica do Rio de Janeiro
22453-041 - Rio de Janeiro - RJ - Brazil
E-mail: henri@mat.puc-rio.br