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ON THE WELL-POSEDNESS AND SCATTERING FOR THE TRANSITIONAL BENJAMIN-ONO EQUATION

Wagner V. L. Nunes *

1 Introduction

In this work we consider the Cauchy problem for the transitional Benjamin-Ono equation (t-BO)

$$\partial_t u + \sigma \partial_x^2 u + f(t) u \partial_x u = 0 \tag{1.1}$$
$$u(t_0, x) = \phi(x) \tag{1.2}$$

where $t_0, t, x \in R$, $\partial_t = \frac{\partial}{\partial_t}$, $\partial_x = \frac{\partial}{\partial_x}$ and σ is the Hilbert transform, i.e.,

$$(\sigma\varphi)(x) = \tfrac{1}{\pi} p.v. \textstyle\int_R \frac{\varphi(y)}{y-x} dy \tag{1.3}$$

where $\varphi \in \mathcal{S}(R)$.

Our purpose is to investigate the local and the global well-posedness of the problem (1.1)-(1.2) in the Sobolev spaces $H^s(R)$, when $s \geq \frac{3}{2}$. The notion of the well-posedness contains: existence, uniqueness, persistence property (i.e., the solution $u(t)$ at any time $t \in [-T, T]$ belongs to the same space X as does the initial data ϕ) and continuity of the solution on the initial data and f (i.e., the continuity of the map $(f, \phi) \in C([-T, T]; R) \times X \to u(t) \in C([-T, T]; X)$). When $T = T(s, f, \|\phi\|_s) < \infty$ it is said that the problem (1.1)-(1.2) is locally

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well-posed in X. If T can be taken arbitrarily large , the problem (1.1)-(1.2) is said globally well-posed in X. We also proof results related on the asymptotic behavior and scattering for (1.1)-(1.2).

The name "transitional Benjamin-Ono" for the equation (1.1) is due to the similarity with the transitional Korteweg-de Vries equation. The term "transitional" is related to the fact that when $f(t)$ goes to ± 1 when t goes to $\pm\infty$ the equation "behaves" like BO, with nonlinearity $-u\partial_x u$, for t "near" $-\infty$ and like BO, with nonlinearity $+u\partial_x u$, for t "near" ∞.

Our proves combines several techniques and ideas including those previously used by J.L.Bona and R.Smith [2], J.C.Saut and R.Teman [11], R.J.Iório Jr. [5], [6], G.Ponce [8], [9] and G.Ponce and L.Vega [10]. The crucial problem is to obtain a priori estimate which allow us to reapply the local theory and to extend the solution to a global one. Usually one considers the problem as a pertubation of a Hamiltonian system. This is not suitable in our case. To solve this problem we construct some functionals, which we call "almost conserved quantities", that play essentially the same role as the conserved quantities of the BO equation (see [4]).

This paper is organized as follows. In section 2 we give a result about the linear equation associated with (1.1). In section 3 we define and obtain some results related on the "almost conserved quantities". The global results are proved in section 4. In the last section we show some results about asymptotic behavior and scattering for the solution to (1.1)-(1.2).

Notation

$H^s(R)$: the real Sobolev space of order s of "L^2 type".

$\|\ \|_s$: the H^s norm.

$(.|.)_s$ = the H^s inner product.

$H^\infty(R) = \bigcap_{k=1}^{\infty} H^k(R)$.

σ : the Hilbert transform.

$\wedge$: the Fourier transform.

$\vee$: the inverse Fourier transform.

$C(.,...,.)$ = constants which are continuous and nondecreasing on theirs arguments.

$' = \frac{d}{dt}$.

$\mathcal{S}(R)$: the Schwartz space.

2 The Linear Equation

The purpose of this section is to state some properties of the group associated to

$$\partial_t u + \sigma \partial_x^2 u = 0 \tag{2.1}$$
$$u(0) = \phi \tag{2.2}$$

proved by G.Ponce and L.Vega in [10].

Consider

$$U(t)\phi = \exp^{-t\sigma\partial_x^2} \phi \tag{2.3}$$

where $\phi \in \mathcal{S}(R)$ and $t \in R$. We have,

Theorem 2.1 : *Let* $0 \le \theta < 1$. *Then*

$$\|U(t)\phi\|_{L^{\frac{2}{1-\theta}}} \le C|t|^{\frac{-\theta}{2}} \|\phi\|_{L^{\frac{2}{1+\theta}}} \tag{2.4}$$

Proof: see [10].

3 The "Almost Conserved Quantities"

To obtain a priori estimate for the t-BO we need the following functionals

Definition 3.1 : *For* $v \in \mathcal{S}(R)$ *and* $\lambda \in R^*$ *we consider*

$$T_i(\lambda, v) = \frac{1}{\lambda^2} \Phi_i(\lambda v) \tag{3.1}$$

where Φ_i *is the* i^{th} *conservation law of the BO (see [4]).*

If $F : R \times \mathcal{S}(R) \to R$ we consider

$$D_2F(\lambda, v)w = \tfrac{d}{d\varepsilon}F(\lambda, v + \varepsilon w)\,\Big|_{\varepsilon=0} \tag{3.2}$$

where $\lambda \in R, v, w \in \mathcal{S}(R)$, when the derivative exists. It is easy to see that for $\lambda \neq 0$, we have

$$D_2T_i(\lambda, v) = \tfrac{1}{\lambda}\Phi_i'(\lambda v) \tag{3.3}$$

We observe that

$$0 = (\Phi_i'(\lambda v)|\partial_x \Phi_j'(\lambda v))_0 = \lambda^2 P_{ij}^v(\lambda) \tag{3.4}$$

where $\lambda \in R$ and $P_{ij}^v(\lambda)$ is a polynomial of degree $k \geq 0$. By(3.3) and (3.4) we obtain

$$(D_2T_i(\lambda, v)|\partial_x D_2T_j(\lambda, v))_0 = \tfrac{1}{\lambda^2}(\Phi_i'(\lambda v)|\partial_x \Phi_j'(\lambda v))_0 = P_{ij}^v(\lambda) \tag{3.5}$$

for $\lambda \in R^*$, $v \in \mathcal{S}(R)$ and so we have,

Proposition 3.2 : *For any $(\lambda, v) \in R \times \mathcal{S}(R)$ and $i, j \in N$ it follows that*

$$(D_2T_i(\lambda, v)|\partial_x D_2T_j(\lambda, v))_0 = 0 \tag{3.6}$$

In particular, for $\lambda = f(t)$.

Definition 3.3 : *For $r \in R, i \in N$ and $v \in \mathcal{S}(R)$ we consider*

$$\Psi_i(r, v) = T_i(f(r), v) \tag{3.7}$$

Remarks 3.4 :

1. *If $f(r)=1$ the $\Psi_i, i\in N$ are the conservation laws of the BO (see [4]).*

2. *We are going to give some of those functionals which we will need in this paper:*

$$\Psi_0(r,v) = \tfrac{1}{2}\int_R v^2\,\mathrm{dx} \tag{3.8}$$
$$\Psi_1(r,v) = \int_R[\tfrac{1}{6}f(r)v^3 - \tfrac{1}{2}v\sigma v_x]\,\mathrm{dx} \tag{3.9}$$
$$\Psi_2(r,v) = \int_R[\tfrac{1}{16}f^2(r)v^4 + \tfrac{3}{8}f(r)v^2\sigma v_x + \tfrac{1}{2}v_x^2]\,\mathrm{dx} \tag{3.10}$$
$$\begin{aligned}\Psi_3(r,v) = \int_R \Big\{&\tfrac{1}{20}f^3(r)v^5 + f^2(r)\left[\tfrac{1}{3}v^3\sigma v_x + v^2\sigma(vv_x)\right] + \\ &+f(r)\left[\tfrac{1}{2}v(\sigma v_x + \tfrac{3}{2}vv_x^2\right] - v_{xx}\sigma v_x\Big\}\,\mathrm{dx}\end{aligned} \tag{3.11}$$
$$\begin{aligned}\Psi_4(r,v) = \int_R \Big\{&\tfrac{1}{24}f^4(r)v^6 + \\ &+f^3(r)\left[\tfrac{5}{16}v^4(\sigma v_x) + \tfrac{5}{12}v^3\sigma(vv_x)\right] + \\ &+f^2(r)\left[\tfrac{25}{8}v^2v_x^2 + \tfrac{5}{8}v^2(\sigma v_x)^2 + \tfrac{5}{4}v(\sigma v_x)\sigma(vv_x)\right] - \\ &-f(r)\left[\tfrac{5}{2}v_x^2\sigma v_x + 5vv_{xx}\sigma v_x\right] - 20v_{xx}^2\Big\}\,\mathrm{dx}\end{aligned} \tag{3.12}$$

It is easy to see that

$$D_2\Psi_0(r,v) = v \tag{3.13}$$
$$D_2\Psi_1(r,v) = \tfrac{1}{2}f(r)v^2 + \sigma v_x \tag{3.14}$$
$$D_2\Psi_2(r,v) = \tfrac{1}{4}f^2(r)v^3 + \tfrac{3}{4}v\sigma v_x + \tfrac{3}{4}f(r)\sigma(vv_x) - v_{xx} \tag{3.15}$$
$$\begin{aligned}D_2\Psi_3(r,v) = &\tfrac{1}{4}f^3(r)v^4 + f^2(r)\left[v^2\sigma v_x = v\sigma(vv_x) + \sigma(v^2v_x)\right] + \\ &+f(r)\left[\tfrac{1}{2}(\sigma v_x)^2 + \sigma(v\sigma v_x)_x - \tfrac{3}{2}v_x^2 - 3vv_{xx}\right] - 2\sigma v_{xxx}\end{aligned} \tag{3.16}$$
$$\begin{aligned}D_2\Psi_4(r,v) = &\tfrac{1}{4}f(r)v^5 + \\ &+\tfrac{1}{4}f^2(r)\left[-25vv_x^2 - 25v^2v_{xx} + 5v(\sigma v_x)^2 + 5\sigma(v^2\sigma v_x)_x\right] + \\ &+\tfrac{1}{4}f(r)\left[-40\sigma(v_xv_{xx} - 20(v_x\sigma v_x)_x - 20v\sigma v_{xxx} - 20\sigma(vv_{xxx})\right] + \\ &+16v_{xxxx}\end{aligned} \tag{3.17}$$

where $v=v(x)$.

4 The Global Nonlinear Problem

In this section we are going to establish the global well-posedness for the Cauchy problem (1.1)-(1.2). We observe that we may suppose that $t_0 = 0$, without loss of generality.

First we will study the regularized equation, i.e.,

$$\partial_t u_\mu + \sigma \partial_x^2 u_\mu + f(t) u_\mu \partial_x u_\mu - \mu \partial_x^2 u_\mu = 0 \tag{4.1}$$

$$u_\mu(0) = \phi \tag{4.2}$$

where $\mu \neq 0$ and later we will treat the case $\mu = 0$.

In [7] we have shown that (4.1)-(4.2) is locally well-posed in $H^s(R)$ for $s > \frac{3}{2}$, $\mu \in R^*$ and $f \in C(R)$. Now, we have,

Theorem 4.1 : *Let $\phi \in H^s(R), s \geq \frac{3}{2}, f \in C(R)$ such that $f' \in L^1_{loc}(R), \mu > 0$ and $u_\mu \in C([0,T]; H^s)$ the local solution of (4.1)-(4.2). Then there exist a continuous non-decreasing function*

$$C : R_+ \times (\tfrac{3}{2}, \infty) \times [0, \infty)^4 \to R_+ \tag{4.3}$$

such that

$$\|u_\mu(t)\|_s \leq C(\mu, s, t, \|f\|_{L^\infty(0,t)}, \|f'\|_{L^1(0,t)}, \|\phi\|_s) \qquad 0 \leq t \leq T \tag{4.4}$$

A similar result holds for $\mu < 0$

Proof: We are going to give the proof for the case $\mu > 0$. The case $\mu < 0$ is similar.

By theorem (4.6) of [7] we have $u_\mu \in C((0,T]; H^\infty)$ for $\mu > 0$ where $T(\mu, s, \|\phi\|_s, f) > 0$. By (3.14) the equation (4.1) can be written as

$$\partial_t u + \partial_x D_2 \Psi_1(t, u) - \mu \partial_x^2 u = 0 \tag{4.5}$$

We claim that (4.4) is true for $s = 0$. In fact, as the solution is real by (3.6), (3.7) $(i = 0, j = 1)$, (4.5) and integration by parts we obtain

$$\begin{aligned}\partial_t \|u_\mu(t)\|_0^2 &= 2\partial_t \Psi_0(t, u_\mu)\partial_t u_\mu \\ &= -2(D_2\Psi_0(t, u_\mu)|\partial_x D_2 \Psi_1(t, u_\mu))_0 + 2\mu(u_\mu|\partial_x^2 u_\mu)_0 \\ &\leq -2\mu\|\partial_x u_\mu(t)\|_0^2 \leq 0 \end{aligned} \tag{4.6}$$

and so

$$\|u_\mu(t)\|_0 \leq \|\phi\|_0 \qquad 0 \leq t \leq T \tag{4.7}$$

Now we claim that (4.4) is true for $s = \frac{3}{2}$. In fact, by (3.11) we have

$$\begin{aligned}\|u_\mu(t)\|_{\frac{3}{2}}^2 + \Psi_3(t, u_\mu(t)) &= \|(1+\xi^2)^{\frac{3}{4}}\hat{u_\mu}\|_0^2 + \Psi_3(t, u_\mu) \\ &\leq \|u_\mu\|_0^2 + \tfrac{1}{20}f^3(u_\mu^3|u_\mu^2)_0 + \tfrac{1}{3}f^2(u_\mu^3|\sigma\partial_x u_\mu)_0 + \tfrac{1}{4}f^2(u_\mu^2|\sigma(u_\mu\partial_x u_\mu))_0 + \\ &\quad + \tfrac{1}{2}f(u_\mu|(\sigma\partial_x u_\mu)^2)_0 + \tfrac{3}{2}f(u_\mu|(\partial_x u_\mu)^2)_0 \end{aligned} \tag{4.8}$$

Using

$$\|\psi\|_{L^\infty} \leq 2\|\psi\|_0^{\frac{1}{2}}\|\partial_x\psi\|_0^{\frac{1}{2}} \tag{4.9}$$

$$x^\alpha y^\beta \leq \eta x^2 + c y^\gamma \tag{4.10}$$

where $\psi \in H^1(R)$, $x, y > 0$, $0 \leq \alpha < 2$, $\beta > 0$, $c = c(\eta) = \frac{2-\alpha}{2}(\frac{\alpha}{2\eta})^{\frac{\alpha}{2-\alpha}}$ and (4.7), Hölder's inequality, Sobolev embeddings and interpolation we have (see [5], [1])

$$|f^3(u_\mu^3|u_\mu^2)_0| \leq c(\eta)|f|^{12}\|\phi\|_0^{10} + \eta\|u_\mu\|_{\frac{3}{2}}^2 \tag{4.11}$$

$$|f^2(u_\mu^3|\sigma\partial_x u_\mu)_0| \leq c(\eta)f^6\|\phi\|_0^8 + \eta\|u_\mu\|_{\frac{3}{2}}^2 \tag{4.12}$$

$$|f^2(u_\mu^2|\sigma(u_\mu\partial_x u_\mu))_0| \leq c(\eta)f^6\|\phi\|_0^8 + \eta\|u_\mu\|_{\frac{3}{2}}^2 \tag{4.13}$$

$$|f(u_\mu|(\sigma\partial_x u_\mu)^2)_0| \leq c(\eta)|f|^3\|\phi\|_0^{\frac{21}{2}} + \eta\|u_\mu\|_{\frac{3}{2}}^2 \tag{4.14}$$

$$|f(u_\mu|(\partial_x u_\mu)^2)_0| \leq c(\eta)|f|^{\frac{21}{2}}\|\phi\|_0^{\frac{21}{2}} + \eta\|u_\mu\|_{\frac{3}{2}}^2 \tag{4.15}$$

So, by (4.11) → (4.15) it follows that

$$\|u_\mu\|^2_{\frac{3}{2}} \leq -\Psi_3(t,u_\mu(t)) + C(c(\eta), \|\dot{f}\|_{L^\infty(0,t)}, \|\phi\|_0) + 5\eta\|u_\mu(t)\|^2_{\frac{3}{2}} \tag{4.16}$$

for $0 \leq t \leq T$.Now, we are going to estimate $-\Psi_3(t,u_\mu(t))$. We observe that (3.6), (3.7) and (4.11) → (4.15) imply

$$\begin{aligned}
-\partial_t\Psi_3(t,u_\mu(t)) &= D_1\Psi_3(t,u_\mu) - D_2\Psi_3(t,u_\mu)\partial_t u_\mu - D_1\Psi_3(t,u_\mu) \\
&\quad +(D_2\Psi_3(t,u_\mu)|\partial_x D_2\Psi_3(t,u_\mu))_0 + \mu(D_2\Psi_3(t,u_\mu)|\partial_x^2 u_\mu)_0 \\
&= -\tfrac{3}{2}f^2(u_\mu^3|u_\mu^2)_0 - \tfrac{2}{3}ff'(u_\mu^3|\sigma\partial_x u_\mu)_0 - \tfrac{1}{2}ff'(u_\mu^2|\sigma(u_\mu\partial_x u_\mu))_0 \\
&\quad -\tfrac{1}{2}f'(u_\mu|(\sigma\partial_x u_\mu)^2)_0 - \tfrac{3}{2}f'(u_\mu|(\partial_x u_\mu)^2)_0 - \mu(D_2\Psi_3(t,u_\mu)|\partial_x^2 u_\mu)_0 \\
&\leq C(\eta)|f'| + \eta C(\|f\|_{L^\infty(0,t)})|f'|\,\|u_\mu\|^2_{\frac{3}{2}} - \mu(D_2\Psi_3(t,u_\mu)|\partial_x^2 u_\mu)_0
\end{aligned} \tag{4.17}$$

Using, Hölder, interpolation, (4.9), (4.10) and (4.7) we can show that (see [5], [1])

$$|f^3(u_\mu^4|\partial_x^2 u_\mu)_0| \leq C(\eta')|f|^{\frac{10}{3}}\|\phi\|_0^{12} + \eta'\|u_\mu\|^2_{\frac{5}{2}} \tag{4.18}$$
$$|f^2(u_\mu^2\sigma\partial_x u_\mu|\partial_x^2 u_\mu)_0| \leq C(\eta')f^{10}\|\phi\|_0^{12} + \eta'\|u_\mu\|^2_{\frac{5}{2}} \tag{4.19}$$
$$|f^2(u_\mu\sigma(u_\mu\partial_x u_\mu)|\partial_x^2 u_\mu)_0| \leq C(\eta')f^{20}\|\phi\|_0^{22} + \eta'\|u_\mu\|^2_{\frac{5}{2}} \tag{4.20}$$
$$|f^2(\sigma(u_\mu^2\partial_x u_\mu)|\partial_x^2 u_\mu)_0| \leq C(\eta')f^{20}\|\phi\|_0^{22} + \eta'\|u_\mu\|^2_{\frac{5}{2}} \tag{4.21}$$
$$|f((\sigma\partial_x u_\mu)^2|\partial_x u_\mu)_0| \leq C(\eta')|f|^{\frac{5}{2}}\|\phi\|_0^{\frac{9}{2}} + \eta'\|u_\mu\|^2_{\frac{5}{2}} \tag{4.22}$$
$$|f(\sigma\partial_x(u_\mu\sigma\partial_x u_\mu)|\partial_x^2 u_\mu)_0| \leq C(\eta')f^{10}\|\phi\|_0^{12} + \eta'\|u_\mu\|^2_{\frac{5}{2}} \tag{4.23}$$
$$|f((\partial_x u_\mu)^2|\partial_x^2 u_\mu)_0| \leq C(\eta')f^{10}\|\phi\|_0^{12} + \eta'\|u_\mu\|^2_{\frac{5}{2}} \tag{4.24}$$
$$|f(u_\mu\partial_x^2 u_\mu|\partial_x^2 u_\mu)_0| \leq C(\eta')f^{10}\|\phi\|_0^{9} + \eta'\|u_\mu\|^2_{\frac{5}{2}} \tag{4.25}$$
$$(\sigma\partial_x^3 u_\mu|\partial_x^2 u_\mu)_0 \leq -c_0\|u_\mu\|^2_{\frac{5}{2}} + \|\phi\|_0^2 \tag{4.26}$$

So, by (4.18) → (4.26) it follows that

$$-\mu(D_2\Psi_3(t,u_\mu(t))|\partial_x^2 u_\mu(t))_0 \leq C(\mu, C(\eta'), \|\phi\|_0, \|f\|_{L^\infty(0,t)}) + \mu(9\eta' - c_0)\|u_\mu\|^2_{\frac{5}{2}} \tag{4.27}$$

By choosing $\eta' = \frac{c_0}{18}$ it follows from (4.17), (4.27) that

$$-\partial_t\Psi_3(t,u_\mu(t)) \leq C_t|f'(t)|(1 + \|u_\mu(t)\|^2_{\frac{3}{2}}) \tag{4.28}$$

for $0 \leq t \leq T$ where $C_t = C(t, \mu, \|f\|_{L^\infty(0,t)}, \|\phi\|_{\frac{3}{2}})$. By integration over $[0, t]$ we obtain

$$\begin{aligned} -\Psi_3(t, u_\mu(t)) &\leq \Psi_3(0, \phi) + \textstyle\int_0^t C(t')|f'(t')|(1 + \|u_\mu(t')\|_{\frac{3}{2}}^2)dt' \\ &\leq |\Psi_3(0, \phi)| + C(t)(1 + \textstyle\int_0^t |f'(t')|\, \|u_\mu(t')\|_{\frac{3}{2}}^2\, dt') \end{aligned} \tag{4.29}$$

where $C(t) = C(t, \mu, \|f\|_{L^\infty(0,t)}, \|f'\|_{L^1(0,t)}, \|\phi\|_{\frac{3}{2}})$.

It is easy to show that

$$|\Psi_3(0, \phi)| \leq C(\|f\|_{L^\infty(0,t)}, \|\phi\|_{\frac{3}{2}}) \tag{4.30}$$

So, (4.16), (4.29) and (4.30) imply

$$\|u_\mu(t)\|_{\frac{3}{2}}^2 \leq C_\eta(t)(1 + \textstyle\int_0^t |f'(t')|\, \|u_\mu(t')\|_{\frac{3}{2}}^2\, dt') + 5\eta\|u_\mu(t)\|_{\frac{3}{2}}^2 \tag{4.31}$$

for $0 \leq t \leq T$, i.e.,

$$(1 - 5\eta)\|u_\mu(t)\|_{\frac{3}{2}}^2 \leq C_\eta(t)(1 + \textstyle\int_0^t |f'(t')|\, \|u_\mu(t')\|_{\frac{3}{2}}^2\, dt') \tag{4.32}$$

By choosing $\eta = \frac{1}{10}$ it follows from (4.32) that

$$\|u_\mu(t)\|_{\frac{3}{2}}^2 \leq C(t)(1 + \textstyle\int_0^t |f'(t')|\, \|u_\mu(t')\|_{\frac{3}{2}}^2\, dt') \tag{4.33}$$

Applying Gronwall's inequality we obtain (4.4) with $s = \frac{3}{2}$.

Now we are going to consider the case $s > \frac{3}{2}$. The main idea is due to G.Ponce in [8] and so what we are going to give is a sketch of the proof. First we observe that

$$\partial_t \|u_\mu\|_s^2 \leq c_s |f(t)|\, \|\partial_x u_\mu(t)\|_{L^\infty} \|u_\mu(t)\|_s^2 \tag{4.34}$$

and so using the estimate, due to Brezis-Gallouet (see [3]), it follows

$$\|v\|_{L^\infty} \leq c_s\{1 + \|v\|_{\frac{1}{2}}\sqrt{\log(1 + \|v\|_r)}\} \tag{4.35}$$

where $r > \frac{1}{2}$ and $v \in H^r(R)$, we can show that

$$\partial_t\|u_\mu(t)\|_s^2 \leq c_s\|f\|_{L^\infty(0,t)}\{1 + C(t)\log(9 + \|u_\mu(t)\|_s^2)\}(9 + \|u_\mu(t)\|_s^2) \tag{4.36}$$

Consider

$$g(t) = 9 + \|u_\mu(t)\|_s^2 \tag{4.37}$$

So, by (4.36) and (4.37) we have

$$g'(t) \leq c_s\|f\|_{L^\infty(0,t)}\{1 + C(t)\log g(t)\}g(t) \tag{4.38}$$

or

$$\log g(t) \leq \log g(0) + c_s\|f\|_{L^\infty(0,t)}t + C(t)\textstyle\int_0^t \log g(t')\,dt' \tag{4.39}$$

Applying Gronwall's inequality it follows that

$$\log g(t) \leq \exp^{C(t)} \qquad 0 \leq t \leq T \tag{4.40}$$

which implies (4.4) and we complete the proof. ■

For the case $\mu \neq 0$ we have

Theorem 4.2 : *Let $\phi \in H^s(R)$, $s \geq \frac{3}{2}$, $f \in C(R)$ such that $f' \in L^1_{loc}(R)$ and $\mu > 0$. Then there exists a unique $u_\mu \in C([0,\infty); H^s)\bigcap C^1([0,\infty); H^{s-2})$, solution of (4.1)-(4.2) which depends continuously on ϕ and f in the following way: let $\phi^n \in H^s(R)$, $f^n \in C(R)$, $f^{n\prime} \in L^1(R)$ such that $\|f^{n\prime}\|_{L^1(0,T)} \leq$ $\leq C(T, \|f'\|_{L^1(0,T)})$, for $T \geq 0$ and $u_\mu^n \in C([0,\infty); H^s)\bigcap C^1([0,\infty); H^{s-2})$ the* *solution of (4.1) such that $u_\mu^n(0) = \phi^n$. Then*

$$\sup_{0\le t\le T} \|u_\mu^n(t) - u_\mu(t)\|_s \le C\{\|\phi^n - \phi\|_s + \|f^n - f\|_{L^\infty(0,T)}\} \tag{4.41}$$

for $n \ge N_0(T)$ where $C = C(s, \frac{1}{\mu}, \|f\|_{L^\infty(0,T)}, \|f'\|_{L^1(0,T)}, \|\phi\|_s) > 0$. Moreover, $u_\mu \in C((0,\infty); H^r)\bigcap C^1((0,\infty); H^q)$ where $r \ge s$ and $q \ge s-2$. A similar result holds for $\mu < 0$.

Proof: The global existence follows from the last theorem and the usual extension argument. The local theory established in [7] implies the uniquiness and continuous dependence. ∎

Now we are going to treat the case $\mu = 0$, i.e., the "transitional" BO equation.

Theorem 4.3 *: Let $\phi \in H^s(R)$, $s \ge \frac{3}{2}$, $f \in C(R)$ such that $f' \in L^1_{loc}(R)$. Then there exists a unique solution $u \in C(R; H^s)\bigcap C^1(R; H^{s-2})$ of (1.1)-(1.2) which depends continuously on ϕ and f in the following way: let $\phi^n \in H^s(R)$, $f^n \in C(R)$ such that f', $f^{n\prime} \in L^1_{loc}(R)$ and $\|f^{n\prime}\|_{L^1(-|t|,|t|)} \le C(|t|, \|f'\|_{L^1(-|t|,|t|)})$ for $t \in R$ and $u^n \in C(R; H^s)\bigcap C^1(R; H^{s-2})$ the solution of (1.1) such that $u^n(0) = \phi^n$. For $T > 0$ if $\phi^n \to \phi$ in H^s and $f^n \to f$ in $L^\infty(-T,T)$ then*

$$\sup_{-T\le t\le T} \|u^n(t) - u(t)\|_s \to 0 \qquad when \quad n \to \infty \tag{4.42}$$

Proof: The global existence for the case $s > \frac{3}{2}$ it follows from the global existence of the solution of the case $\mu \ne 0$ and the fact that $u_\mu(t) \to u(t)$ weakly in H^s uniformly in $[-T,T]$ for all $T > 0$. The local theorem for the case $\mu = 0$ implies the uniqueness and the continuous dependence. The case $s = \frac{3}{2}$ can be treat by using the ideas of [8]. ∎

Remarks 4.4 :

- *We can show that there exists a smoothing effect in $H^s(R)$ and global weak solution in $H^1(R)$ using the ideas of [9].*

5 Asymptotic Behavior and Scattering

In this section we are going to study the asymptotic behavior of the solution of (1.1)-(1.2).

Let $u = u(t)$ the solution of (1.1)-(1.2) in R. It is easy to see that

$$u(t) = U(t)\phi - \tfrac{1}{2}\int_0^t f(t')U(t-t')\partial_x u^2(t')dt' \tag{5.1}$$

for $t \geq 0$. So if we define

$$\phi_+ = \phi - \tfrac{1}{2}\int_0^\infty f(t')U(-t')\partial_x u^2(t')dt' \tag{5.2}$$

then

$$U(-t)u(t) - \phi_+ = \tfrac{1}{2}\int_t^\infty f(t')U(-t')\partial_x u^2(t')dt' \to 0 \tag{5.3}$$

when $t \to \infty$. A similar argument holds for ϕ_-. So, we have to find a space in which the solution is global and the integral (5.2) converges. This space is $H^{\frac{k}{2}}(R)$ where $k = 2, 3, \ldots$. We need some restrictions on f. These restrictions do not allow us to take $f = constant$, i.e., the case of the Benjamin-Ono equation remains unsolved.

To treat the asymptotic behavior and scattering for the solutions we will need the following result:

Theorem 5.1 : *Let $\phi \in H^s(R)$, $s \geq \frac{3}{2}$ $f \in C(R)$ such that $f' \in L^1(R)$ and $u \in C(R; H^s)$ the solution of (1.1)-(1.2) given by theorem (4.3). Then*

$$\|u(t)\|_r \leq C(\|f\|_{L^\infty}, \|f'\|_{L^1}, \|\phi\|_{\frac{m}{2}}) \tag{5.4}$$

for $t \in R$, $r \leq \frac{m}{2}$ where $m \in N$, $\frac{m}{2} \leq s$ and $C = C(.,.,.)$ is continuous not decreasing function in theirs arguments.

Proof: We will consider the case $t \geq 0$ and $k = 3$. The remainder cases will be omitted.

Consider $\phi_n \in H^\infty(R)$ such that $\phi_n \to \phi$ in $H^{\frac{3}{2}}$ when $n \to \infty$ and $\|\phi_n\|_{\frac{3}{2}} = \|\phi\|_{\frac{3}{2}}$. Let $u_n \in C(R; H^\infty)$ the solution of (1.1) such that $u_n(0) = \phi_n$.

(i) case $m = 0$:

As equation (1.1) can be written as

$$\partial_t u + \partial_x D_2 \Psi_1(t, u) = 0 \tag{5.5}$$

from (3.7) and proposition (3.2) it follows that

$$\partial_t \|u_n(t)\|_0^2 = 2\partial_t \Psi_0(t, u_n).\partial_t u_n = 0 \tag{5.6}$$

and so we have

$$\|u_n(t)\|_0 = \|\phi_n\|_0 \tag{5.7}$$

for $t \geq 0$ and $n \in N$.

The continuous dependence implies that

$$u_n(t) \to u(t) \ \ in H^{\frac{3}{2}} \tag{5.8}$$

when $n \to \infty$, in particular in $L^2(R)$. So we have

$$\|u(t)\|_0 = \lim_{n\to\infty} \|u_n(t)\|_0 = \|\phi_n\|_0 \leq C\|\phi\|_0 \tag{5.9}$$

(ii) case $m = 1$:

From (3.9) we have

$$\begin{aligned}\|u_n(t)\|_{\frac{1}{2}}^2 - 2\Psi_1(t, u_n(t)) &= \|(1+.^2)^{\frac{1}{4}}\hat{u_n}\|_0^2 - \tfrac{f}{3}\textstyle\int_R u_n^3\,dx - \|\,|.|^{\frac{1}{2}}\hat{u_n}\|_0^2 \\ &\le \|\hat{u_n}\|_0^2 + \tfrac{|f|}{3}\|u_n\|_{L^3} \le \|u_n\|_0^2 + C|f|\,\|u_n\|_{\frac{1}{6}}^3 \\ &\le C(c(\eta), \|f\|_{L^\infty}, \|\phi\|_0) + \eta\|u_n(t)\|_{\frac{1}{2}}^2 \qquad (5.10)\end{aligned}$$

where we use the immersion $H^{\frac{1}{6}} \hookrightarrow L^3$, interpolation and (4.10). So we need to estimate $\Psi_1(t, u_n(t))$. Observe that

$$\begin{aligned}\partial_t\Psi_1(t, u_n(t)) &= D_1\Psi_1(t, u_n) - D_2\Psi_1(t, u_n)\partial_t u_n + D_1\Psi_1(t, u_n) = \tfrac{1}{6}f'\textstyle\int_R u_n^3\,dx \\ &\le \tfrac{1}{6}|f'|\|u_n\|_{\frac{1}{6}}^2 \le C(\eta)\|\phi\|_0^4|f'(t)| + \eta|f'(t)|\,\|u_n(t)\|_{\frac{1}{2}}^2 \qquad (5.11)\end{aligned}$$

where we use the immersion above, interpolation and (5.9). By integration over $[0, t]$ we obtain

$$\Psi_1(t, u_n(t)) \le \Psi_1(0, \phi) + C(\eta)\|\phi\|_0^4\|f'\|_{L^1(0,t)} + \eta\textstyle\int_0^t |f'(t')|\,\|u_n(t')\|_{\frac{1}{2}}^2\,dt' \qquad (5.12)$$

It is easy to see that

$$|\Psi_1(0, \phi)| \le C(\|f\|_{L^\infty}, \|\phi\|_{\frac{1}{2}}) \qquad (5.13)$$

It follows from (5.10), (5.12) and (5.13) that

$$\begin{aligned}\|u_n(t)\|_{\frac{1}{2}}^2 \le\, &C(C(\eta), \|f\|_{L^\infty}, \|f'\|_{L^1}, \|\phi\|_{\frac{1}{2}}) \\ &+\eta\|u_n(t)\|_{\frac{1}{2}}^2 + \eta\textstyle\int_0^t |f'(t')|\,\|u_n(t')\|_{\frac{1}{2}}^2\,dt' \qquad (5.14)\end{aligned}$$

and so

$$\begin{aligned}\sup_{0\le t\le T}\|u_n(t)\|_{\frac{1}{2}}^2 &\le C + \eta\sup_{0\le t\le T}\|u_n(t)\|_{\frac{1}{2}}^2 + \eta\sup_{0\le t'\le T}\|u_n(t')\|_{\frac{1}{2}}^2\textstyle\int_0^t |f(t')|\,dt' \\ &\le C + \eta\sup_{0\le t\le T}\|u_n(t')\|_{\frac{1}{2}}^2 + \eta\|f'\|_{L^1}\sup_{0\le t\le T}\|u_n(t)\|_{\frac{1}{2}}^2 \quad (5.15)\end{aligned}$$

where $C = C(c(\eta), \|f\|_{L^\infty}, \|f'\|_{L^1}, \|\phi\|_{\frac{1}{2}}) > 0$, which implies

$$(1 - \eta - \eta\|f'\|_{L^1}) \sup_{0\leq t\leq T} \|u_n(t)\|^2_{\frac{1}{2}} \leq C(\tfrac{1}{2})^2 \tag{5.16}$$

Choosing $\eta = \frac{1}{2(1+\|f'\|_{L^1})} > 0$, by (5.16) we obtain

$$\|u_n(t)\|^2_{\frac{1}{2}} \leq \sup_{0\leq t\leq T} \|u_n(t)\|^2_{\frac{1}{2}} \leq C(\tfrac{1}{2})^2 \tag{5.17}$$

where $C(\frac{1}{2}) = C(\|f\|_{L^\infty}, \|f'\|_{L^1}, \|\phi\|_{\frac{1}{2}})$. Therefore

$$\|u_n(t)\|_{\frac{1}{2}} \leq C(\tfrac{1}{2}) \tag{5.18}$$

and the continuous dependence upon initial data implies the statement.
(iii) case $m = 2$:
As

$$\begin{aligned}\|u_n(t)\|_1^2 - 2\Psi_2(t, u_n(t)) &= \|u_n\|_0^2 + \|\partial_x u_n\|_0^2 - \tfrac{f^2}{8}\|u_n\|_{L^4}^4 \\ &\quad -\tfrac{3}{4}f(u_n^2|\sigma\partial_x u_n)_0 - \|\partial_x u_n\|_0^2 \\ &\leq C(c(\eta), \|f\|_{L^\infty}, \|\phi\|_1) + \eta\|u_n\|_1^2\end{aligned} \tag{5.19}$$

where we used (3.10), Hölder, the imersion $H^{\frac{1}{4}} \hookrightarrow L^4$, (5.7) and (4.10). We need estimate $\Psi_2(t, u_n(t))$. By using (3.6) and (3.7) $(i = 1, j = 2)$ we obtain

$$\begin{aligned}\partial_t\Psi_2(t, u_n(t)) &= D_1\Psi_2(t, u_n) + D_2\Psi_2(t, u_n)\partial_t u_n \\ &= \tfrac{1}{8}ff'\|u_n\|_{L^4}^4 + \tfrac{3}{8}f'(u_n^2|\sigma\partial_x u_n)_0 \\ &\leq |f'|[C(c(\eta), \|f\|_{L^\infty}, \|\phi\|_0) + \eta\|u_n\|_1^2]\end{aligned} \tag{5.20}$$

By integration over $[0, t]$ we obtain

$$\begin{aligned}\Psi_2(t, u_n(t)) &\leq \Psi_2(0, \phi) + C(c(\eta), \|f\|_{L^\infty}, \|f'\|_{L^1}, \|\phi\|_1) \\ &\quad +\eta\textstyle\int_0^t |f'(t')|\, \|u_n(t')\|_1^2 dt'\end{aligned} \tag{5.21}$$

It is easy to see that

$$|\Psi_2(0,\phi)| \le C(\|f\|_{L^\infty}, \|\phi\|_1) \tag{5.22}$$

and so, (5.19), (5.20) and (5.21) imply

$$\begin{aligned}\|u_n(t)\|_1^2 &\le C^2(1) + \eta\|u_n(t)\|_1^2 + \eta \textstyle\int_0^t |f(t')|\,\|u_n(t')\|_1^2 dt' \\ &\le C^2(1) + \eta\|u_n(t)\|_1^2 + \eta\|f'\|_{L^1} \sup_{0\le t'\le t} \|u_n(t')\|_1^2 \end{aligned} \tag{5.23}$$

So, taking the supremum over $[0,t]$ in (5.23) we obtain

$$(1-\eta-\eta\|f'\|_{L^1}) \sup_{0\le t'\le t} \|u_n(t')\|_1^2 \le C^2(1) \tag{5.24}$$

where $C(1) = C(c(\eta), \|f\|_{L^\infty}, \|\phi\|_1) > 0$. Choosing $\eta = \frac{1}{2(1+\|f'\|_{L^1})} > 0$ it follows from (5.24) that

$$\|u_n(t)\|_1^2 \le \sup_{0\le t'\le t} \|u_n(t')\| \le C^2(1) \tag{5.25}$$

that is

$$\|u_n(t)\|_1 \le C(1) \tag{5.26}$$

The continuous dependence on initial data ϕ implies the result for $m=2$.
(iii) case $m=3$:
As in (4.8) $\to$ (4.15) we have

$$\|u_n(t)\|_{\frac{3}{2}}^2 - \Psi_3(t, u_n(t)) \le C(1) \tag{5.27}$$

So we have to estimate $\Psi_3(t,u_n(t))$. However

$$\partial_t\Psi_3(t,u_n(t)) = D_1\Psi_3(t,u_n) + D_2\Psi_3(t,u_n)\partial_t u_n = D_1\Psi_3(t,u_n) \le C(1)|f'(t)| \tag{5.28}$$

where we used the same estimates as in (4.18) $\to$ (4.26), (5.5), (3.6) and (3.7) $(i = 3,\ j = 1)$. By integration over $[0,t]$ we have

$$\begin{aligned}\Psi_3(t,u_n(t)) &\le \Psi_3(0,\phi) + C(1)\int_0^t |f(t')|dt' \\ &\le \Psi_3(0,\phi) + C(1) \le C^2(\tfrac{3}{2})\end{aligned} \tag{5.29}$$

where $C(\frac{3}{2}) = C(\|f\|_{L^\infty}, \|f'\|_{L^1}, \|\phi\|_{\frac{3}{2}}) > 0$. Using (5.27) and (5.29) we obtain

$$\|u_n(t)\|^2_{\frac{3}{2}} \le C^2_{\frac{3}{2}} \tag{5.30}$$

that is,

$$\|u_n(t)\|_{\frac{3}{2}} \le C(\tfrac{3}{2}) \tag{5.31}$$

and the continuous dependence implies the case $m = 3$. ∎

Remarks 5.2 :

- *Theorem (5.1) includes the case $f(t) =$ constante, that is, the Benjamin-Ono equation.*

To obtain our seattering result we will need the following theorem about asymptotic behavior of the solution of (1.1)-(1.2).

Theorem 5.3 : *Let $\phi \in H^s(R) \bigcap L^{\frac{4}{3}}(R)$ for $s \ge \frac{3}{2}$, $f \in C(R)$ with $f' \in L^1(R)$ satisfying*

$$f(t) = O((1+t^2)^{-\alpha}) \qquad when \quad t \to \pm\infty \tag{5.32}$$

where $\alpha > \frac{3}{8}$ and $u \in C(R; H^s)$ the solution of (1.1)-(1.2) which is given by theorem (4.3). Then

$$(1+|t|)^{\frac{1}{4}}\|u(t)\|_{L^4} \le C(\alpha, \|f\|_{L^\infty}, \|f'\|_{L^1}, \|\phi\|_1, \|\phi\|_{L^{\frac{4}{3}}}) \tag{5.33}$$

Proof: We are going to treat the case $t \geq 0$. The case $t \leq 0$ is similar. From (5.1) we obtain

$$\begin{aligned} \|u(t)\|_{L^4} &\leq \|U(t)\phi\|_{L^4} + \tfrac{1}{2}\textstyle\int_0^t |f(t')|\,\|U(t')\partial_x u^2(t')\|_{L^4}\,dt' \\ &\leq C(1+t)^{-\frac{1}{4}}\|\phi\|_{L^{\frac{4}{3}}} + C\textstyle\int_0^t |f(t')|\,|t-t'|^{-\frac{1}{4}}\|\partial_x u^2(t')\|_{L^{\frac{4}{3}}}\,dt' \end{aligned} \tag{5.34}$$

where we use (2.4) with $\theta = \frac{1}{2}$. By Hölder, $H^1 \hookrightarrow L^4$, (5.4) (with $m = 2$) and $0 < \delta < 1$ it follows

$$\begin{aligned} \|\partial_x u^2(t')\|_{L^{\frac{4}{3}}} &\leq 2\|u\partial_x u\|_{L^{\frac{4}{3}}} \leq 2\|u\|_{L^4}\|\partial_x u\|_0 = 2\|u\|_{L^4}^{\delta}\|u\|_{L^4}^{1-\delta}\|u\|_1 \\ &\leq C\|u\|_{L^4}^{\delta}\|u\|_1^{2-\delta} \leq C(1)\|u\|_{L^4}^{\delta} \end{aligned} \tag{5.35}$$

which implies

$$\|u(t)\|_{L^4} \leq C(1+t)^{-\frac{1}{4}}\|\phi\|_{L^4} + C\textstyle\int_0^t |f(t')|\,|t-t'|^{-\frac{1}{4}}\|u(t')\|_{L^4}^{\delta}\,dt' \tag{5.36}$$

and so

$$\begin{aligned} &(1+t)^{\frac{1}{4}}\|u(t)\|_{L^4} \leq \\ &C\|\phi\|_{L^4} + C(1+t)^{\frac{1}{4}}\textstyle\int_0^t |f(t')||t-t'|^{-\frac{1}{4}}(1+t')^{-\frac{\delta}{4}}[(1+t')^{\frac{1}{4}}\|u(t')\|_{L^4}]^{\delta}\,dt' \end{aligned} \tag{5.37}$$

For $t > 0$ consider

$$X(t) = \sup_{0 \leq t' \leq t}[(1+t')^{\frac{1}{4}}\|u(t')\|_{L^4}] \tag{5.38}$$

By (5.37) and (5.38) we obtain

$$(1+t)^{\frac{1}{4}}\|u(t)\|_{L^4} \leq C\|\phi\|_{L^{\frac{4}{3}}} + CX(t)^{\delta}(1+t)^{\frac{1}{4}}\textstyle\int_0^t \frac{|f(t')|}{(t-t')^{\frac{1}{4}}(1+t')^{\frac{\delta}{4}}}\,dt' \tag{5.39}$$

We can show that

$$\begin{aligned} &(1+t)^{\frac{1}{4}}\textstyle\int_0^t \frac{|f(t')|}{(t-t')^{\frac{1}{4}}(1+t')^{\frac{\delta}{4}}}\,dt' \leq \\ &\leq (1+t)^{\frac{1}{4}}t^{\frac{4}{3}}\textstyle\int_0^t \frac{1}{(1+(rt)^2)^{\alpha}(1-r)^{\frac{1}{4}}(1+rt)^{\frac{\delta}{4}}}\,dr \leq M \leq \infty \end{aligned} \tag{5.40}$$

where we choose $\delta \in (0,1)$ such that $\alpha > \frac{3}{8} - \frac{\delta}{8}$. So (5.39), (5.40) and (4.10) imply

$$X(t) \leq C\|\phi\|_{L^{\frac{4}{3}}} + CX(t)^{\delta} \leq C(\eta) + \eta X(t) \tag{5.41}$$

and $C(\eta) = C(c(\eta), \alpha, \|f\|_{L^\infty}, \|f'\|_{L^1}, \|\phi\|_1, \|\phi\|_{L^{\frac{4}{3}}} -)$. By choosing $\eta = \frac{1}{2}$ we obtain

$$(1+t)^{\frac{1}{4}}\|u(t)\|_{L^4} \leq X(t) \leq C \tag{5.42}$$

and so we complete the proof. ∎

Now we have the main result of this section,

Theorem 5.4 *: Let $\phi \in H^s(R)$ for $s \geq \frac{3}{2}$, $f \in C(R)$ such that $f' \in L^1(R)$ and satisfying (5.32) for $\alpha \geq 0$ and $u \in C(R; H^s)$ the solution os (1.1)-(1.2) given by theorem (4.3). If*

$$(i)\ \alpha > \tfrac{1}{2} \tag{5.43}$$

or

$$(ii)\ \alpha > \tfrac{3}{8} \ \text{and}\ \phi \in L^{\frac{4}{3}}(R) \tag{5.44}$$

then there exist uniques $\phi_\pm \in H^{\frac{k}{2}}(R)$ where $k \in N$ and $\frac{k}{2} \leq s$ such that

$$\|u(t) - U(t)\phi_\pm\|_{s'} \to 0 \qquad when \quad t \to \pm\infty \tag{5.45}$$

where $0 \leq s' \leq \frac{k}{2}$

Proof: We will consider the case $t \to \infty$, that is, the existence of ϕ_+. The case $t \to -\infty$ is similar.

Suppose that (i) holds. We know that $U(t)$, $t \in R$ is a isometry in $H^s(R)$, so

$$\| \int_0^t f(t')U(t-t')\partial_x u^2(t')dt'\|_0 \le \int_0^t |f(t')|\,\|u(t')\|_1^2 dt' \le C(1)\|f\|_{L^1} < \infty \quad (5.46)$$

where $C(1) = C(\|f\|_{L^\infty}, \|f'\|_{L^1}, \|\phi\|_1)$ is given by theorem (5.1). The integral converges in $L^2(R)$ and we can define

$$\phi_+ = \phi - \tfrac{1}{2}\int_0^\infty f(t')U(t')\partial_x u^2(t')dt' \in L^2(R) \quad (5.47)$$

By (5.3) we have

$$\|U(t)u(t) - \phi_+\|_0 = \tfrac{1}{2}\| \int_t^\infty f(t')U(-t')\partial_x u^2(t')dt'\|_0 \to 0 \qquad when \quad t \to \infty \quad (5.48)$$

It follows from (5.4) (with $r, m = k$) and the fact that $U(t)$ is a isometry that

$$\|U(-n)u(n)\|_{\frac{k}{2}} \le \|u(n)\|_{\frac{k}{2}} \le C(k) \le C(s) \quad (5.49)$$

then there exist $\psi \in H^{\frac{k}{2}}$ and a subsequence $\{U(-n_j)u(n_j)\}_{j\in N}$ such that

$$U(-n_j)u(n_j) \rightharpoonup \psi \qquad when \quad j \to \infty \quad (5.50)$$

in $H^{\frac{k}{2}}$. Then (5.48) and (5.50) imply $\phi_+ = \psi$ and so $\phi \in H^{\frac{k}{2}}$. By interpolation we obtain

$$\begin{aligned}\|u(t) - U(t)\phi_+\|_{s'} &\le C(\|U(-t)u(t)\|_{\frac{k}{2}} + \|\phi_+\|_{\frac{k}{2}})^{\frac{2s}{k}} \|U(-t)u(t) - \phi -_+ \|)_0^{1-\frac{2s}{k}} \\ &\le C(k)\|U(-t)u(t) - \phi_+\|_0^{1-\frac{2s}{k}} \to 0 \quad (5.51)\end{aligned}$$

when $t \to \infty$. The uniqueness is trivial.

Now, suppose that (ii) holds. Then

$$\begin{aligned}\| \int_0^t f(t')U(-t')\partial_x u^2(t')dt'\|_{-1} &\le \int_0^t |f(t')|\,\|u(t')\|_{L^4}^2 dt' \\ &\le C\int_0^t \frac{1}{(1+t'^2)^{\alpha+\frac{1}{4}}} dt' \le C < \infty \quad (5.52)\end{aligned}$$

for $t \geq 0$ where we use the fact that $\alpha > \frac{1}{4}$. Then the integral in (5.2) converges in $H^{-1}(R)$ and so we can define $\phi_+ \in H^{-1}$ as in (5.2). As before we can show (5.45) and the uniqueness. ∎

Remarks 5.5 :

1. *If we have uniqueness of the solution in H^1 we can obtain the same result in H^1.*
2. *The case $\alpha < \frac{3}{8}$ remains open.*

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Instituto de Ciências Matemáticas de São Carlos
ICMSC-USP
Caixa Postal 668 -São Carlos
São Paulo - 13560 - Brasil
e.mail: WVLNUNES@ICMSC.USP.ANSP.BR