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NONLOCAL UNILATERAL PROBLEM FOR A NONLINEAR HYPERBOLIC EQUATION OF THE THEORY OF ELASTICITY

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Abstract

Assume that $u_o \in H_o^1(\Omega) \cap H^2(\Omega)$, $u_1 \in H_o^1(\Omega)$, $f \in L^2(0,T; H_o^1(\Omega))$, $\frac{df}{dt} \in L^2(0,T; L^2(\Omega))$, $|\text{ grad } u_1|^2 < 1$, a.e. in Ω; $M(\lambda)$, $0 \leq \lambda < \infty$ is a real function, continuously differentiable and $M(\lambda) \geq m_o > 0$. We prove, in the present work, that if Ω is an square of $\mathbf{R}^2$, then there exists only one function $u = u(x,t)$, $x \in \Omega$, $0 \leq t < T$, satisfying the inequality

$$u'' - M\left(\int_\Omega |\nabla u|^2 \, dx\right) \Delta u \geq f \text{ in } Q, \quad u(x,0) = u_o(x), u'(x,0) = u_1(x) \text{ in } \Omega.$$

Introduction

The Carrier-Kirchoff model for the vertical vibrations of an elastic streched string, fixed at the ends, when the tension in each point has only vertical components, is given by:

$$\frac{\partial^2 u}{\partial t^2} - \left(P_o + P \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right) \frac{\partial^2 u}{\partial x^2} = 0. \qquad (*)$$

In (*), $P_o = \frac{\tau_o}{m}$, τ_o the initial tension, m the mass of the string of length L, $P = \frac{k}{2Lm}$, $k = Ea$, E the Young's modulus of the material, a the area of the vertical section of the string at $t = 0$, supposed constant, cf. Carrier [4].

The model (*) is the motivation for a large field of research on a nonlinear mixed problem, formulated as follows. Let $M(\lambda)$ be a real function on $[0, \infty[$, Ω a bounded subset of $\mathbf{R}^n$, $T > 0$, $Q = \Omega \times]0, T[$ and Σ is the lateral boundary

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of Q. The question is find a function $u: Q \to \mathbf{R}$, $u = u(x,t)$, $x \in \Omega$, $t \in]0,T[$, satisfying the conditions:

$$\left| \begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} - M\left(\displaystyle\int_\Omega |\nabla u|^2\, dx\right) \Delta u = f \quad \text{in} \quad Q \\ u = 0 \quad \text{on} \quad \Sigma \\ u(x,0) = u_o(x), \ \dfrac{\partial u}{\partial t}(x,0) = u_1(x) \quad \text{on} \quad \Omega. \end{array} \right. \tag{1}$$

The functions f, u_o, u_1 are known.

The mixed problem (1) was studied by the first time by Bernstein [3] and Dickey [7] for the case $n = 1$. Pohozhaev [24] studied problem (1) for a bounded open set Ω of $\mathbf{R}^n$ and obtained a regular solutions for a particular class of initial data. In Lions [16] we find a formulation of the problem under a transparent abstract framework, which permits to propose several mathematical questions about it. For the solutions of those questions look the works listed in the references. We only observe that to obtain global solution on t, when $u_o \in H_o^1(\Omega) \cap H^2(\Omega)$, $u_1 \in H_o^1(\Omega)$, Ω bounded, the results was obtained with strong boundedness on the data u_o, u_1. The same type of results are obtained when Ω is not bounded. If the data u_o, u_1 are sufficiently regular, then it was proved global existence.

In the present work, we look the problem under another point of view. We let u_o free on $H_o^1(\Omega) \cap H^2(\Omega)$, but we consider $u_o \in H_o^1(\Omega)$ such that $|\operatorname{grad} u_o|^2 < 1$ on Ω. We prove that if Ω is an square of $\mathbf{R}^2$, $M(\lambda) \geq m_o > 0$, continuously differentiable on $0 \leq \lambda < +\infty$, then the variational inequality $\frac{\partial^2 u}{\partial t^2} - M\left(\int_\Omega |\nabla u|^2\, dx\right)\Delta u \geq f$ has a global solution in t. In Medeiros-Milla Miranda [21] it was proved only local solutions for a general bounded open set Ω of $\mathbf{R}^n$.

The results contained in the present work was summarized in Larkin-Medeiros [15]. It is opportune to call the atention, of the reader, that the present results was extended by Cicero Frota [12] to the case where Ω is a cube of $\mathbf{R}^n$.

1. Notations

First of all let us formulate the problem we want to solve. In fact, taking $u_1 \in H_o^1(\Omega)$, such that $|\operatorname{grad} u_1|^2 < 1$ a.e. in Ω, we want to solve the problem:

$$\left|\begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} - M\left(\int_\Omega |\nabla u(x,t)|^2\,dx\right)\Delta u \geq f \quad \text{on} \quad Q \\ u = 0 \quad \text{on} \quad \Sigma \\ u(x,0) = u_o(x), \ \dfrac{\partial u}{\partial t}(x,0) = u_1(x) \quad \text{on} \quad \Omega. \end{array}\right. \tag{2}$$

The following considerations are true for a bounded open set of $\mathbf{R}^n$, in general, cf. Lions [17] for proofs. Let us define:

$$K = \{u \in H_o^1(\Omega); \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|^2 \leq 1. \quad \text{a.e. in} \quad \Omega\}. \tag{3}$$

Whence K is a convex, closet subset of $W_o^{1,4}(\Omega)$.

By v^- we represent the function $v^-(x) = \max\{-v(x), 0\}$ and then for $u, v \in W_o^{1,4}(\Omega)$ we have $\left[\left(1 - \left|\frac{\partial u}{\partial x_i}\right|^2\right)^- \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\right] \in L^1(\Omega)$, $i = 1, 2, \ldots, n$. It follows that for each $u \in W_o^{1,4}(\Omega)$ is well defined the mappings

$$\beta_i(v) = \int_\Omega \left(1 - \left|\frac{\partial u}{\partial x_i}\right|^2\right)^- \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx,$$

from $W_o^{1,4}(\Omega)$ into $\mathbf{R}$, for $i = 1, 2, \ldots, n$, linear and continuous. Then $\beta_i(u)$ is an object of $W_o^{-1,\frac{4}{3}}(\Omega)$, the dual of $W_o^{1,4}(\Omega)$. Then if we consider $\beta(u) = \sum_{i=1}^n \beta_i(u)$, β is a mapping from $W_o^{1,4}(\Omega)$ into $W^{-1,\frac{4}{3}}(\Omega)$. We have:

$$\langle \beta(u), v\rangle = \sum_{i=1}^n \int_\Omega \left(1 - \left|\frac{\partial u}{\partial x_i}\right|^2\right)^- \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx. \tag{4}$$

The operator β is monotonous, hemicontinuous and takes bounded sets of $W_o^{1,4}(\Omega)$ in bounded sets of $W_o^{-1,\frac{4}{3}}(\Omega)$ and $u \in K$ is equivalent to $\beta(u) = 0$. The operator β is a penalty operator, cf. Lions op.cit. We have

$$\beta(u) = -\sum_{i=1}^n \frac{\partial}{\partial x_i}\left[\left(1 - \left|\frac{\partial u}{\partial x_i}\right|^2\right)^- \frac{\partial u}{\partial x_i}\right] \tag{5}$$

in the sense of $\mathcal{D}'(\Omega)$.

By $(\ ,\ )$, $|\cdot|$ we represent the inner product and norm in $L^2(\Omega)$ and $((\ ,\ ))$, $\|\cdot\|$, inner product and norm for $H_o^1(\Omega)$.

2. The main result

Let us consider Ω a square of $\mathbf{R}^2$, i.e:, $\Omega =]0,1[\times]0,1[$, then we have the main result:

Theorem 1. *Given:*

$$u_o \in H_o^1(\Omega) \cap H^2(\Omega); u_1 \in H_o^1(\Omega); |\operatorname{grad} u_1| < 1 \quad \text{a.e. in} \quad \Omega; \tag{6}$$
$$f \in L^2(0,T;H_o^1(\Omega)),\ \frac{\partial f}{\partial t} \in L^2(0,T;L^2(\Omega)),$$

$$M(\lambda) \geq m_o > 0, \text{ is continuously differentiable on } [0,\infty[, \tag{7}$$
then there exists a function $u: Q = \Omega\times]0,T[\to \mathbf{R}$,

satisfying:

$$u \in L^2(0,T;H_o^1(\Omega) \cap H^2(\Omega)) \tag{8}$$

$$u' \in L^2(0,T;H_o^1(\Omega)),\ u'(t) \in K \quad \text{a.e.} \tag{9}$$

$$u'' \in L^2(0,T;L^2(\Omega)) \tag{10}$$

$$\int_0^T (u''(t), v - u'(t))dt + \int_0^T M(a(u(t)))a(u(t), v - u'(t))dt \tag{11}$$
$$\geq \int_0^T (f(t), v - u'(t))dt$$

for all $v \in L^4(0,T;W_o^{1,4}(\Omega))$, $v \in K$.

$$u(x,0) = u_o(x); \quad \frac{\partial u}{\partial t}(x,0) = u_1(x). \tag{12}$$

We represent by $a(u)$ *the quadratic form associated to the bilinear form* $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$, *defined for* $u, v \in H_o^1(\Omega)$.

The proof of the Theorem 1 will be given in Section 3 by the penalty method. In fact, we consider a penalized mixed problem obtained from (2), which solutions u_ϵ depends of a parameter $\epsilon > 0$. We solve this mixed problem, and take the limit of u_ε when $\varepsilon \to 0$, in the weak topology, obtaining a function u which is the solution of the unilateral problem (2).

The penalized problem asociated to (2) consists in find $u_\varepsilon : Q \to \mathbf{R}$, $\varepsilon > 0$, solution of the mixed problem:

$$\left| \begin{array}{l} \dfrac{\partial^2 u_\varepsilon}{\partial t^2} - M(a(u_\varepsilon(t))\Delta u_\varepsilon + \dfrac{1}{\varepsilon}\beta(u'_\varepsilon) = f \quad \text{on} \quad Q \\ u_\varepsilon = 0 \quad \text{on} \quad \Sigma \\ u_\varepsilon(x,0) = u_o(x), \quad \dfrac{\partial u_\varepsilon}{\partial t}(x,0) = u_1(x) \end{array} \right. \tag{13}$$

We need to prove the following:

Theorem 2. *If the functions u_o, u_1, f satisfies the condition (6), then, for every $0 < \varepsilon < 1$, exists a function u_ε such that*

$$u_\varepsilon \in L^\infty(0,T; H_o^1(\Omega) \cap H^2(\Omega)) \tag{14}$$

$$u'_\varepsilon \in L^\infty(0,T; H_o^1(\Omega)) \tag{15}$$

$$u''_\varepsilon \in L^\infty(0,T; L^2(\Omega)) \tag{16}$$

$$(u''_\varepsilon(t), v) + M(a(u_\varepsilon))a(u_\varepsilon(t), v) + \frac{1}{\varepsilon}\langle \beta(u'_\varepsilon(t)), v\rangle = (f(t), v) \tag{17}$$

in $L^{4/3}(0,T)$, for all $v \in W_o^{1,4}(\Omega)$.

$$u_\varepsilon(x,0) = u_o(x), \quad \frac{\partial u_\varepsilon}{\partial t}(x,0) = u_1(x). \tag{18}$$

3. Proofs of the theorems

Suppose we have the proof of Theorem 2, and let us prove Theorem 1. Then u_ε is solution of:

$$\int_0^T (u''_\varepsilon, w)dt + \int_0^T M(a(u_\varepsilon))a(u_\varepsilon, w)dt + \frac{1}{\varepsilon}\int_0^T \langle \beta(u'_\varepsilon), w\rangle \, dt \geq \int_0^T (f(t), w)dt \tag{19}$$

for all $w \in L^4(0,T; W_o^{1,4}(\Omega))$. Im particular let us consider $w = v - u'_\varepsilon$ where $v \in L^4(0,T; W_o^{1,4}(\Omega))$ with $v(t) \in K$, a.e. in $[0,T]$. Then, from (19) we get:

$$\begin{aligned} \int_0^T (u''_\varepsilon(t), v(t) - u'_\varepsilon(t))dt + \int_0^T M(a(u_\varepsilon(t))a(u_\varepsilon(t), v(t) - u'_\varepsilon(t))dt + \\ + \frac{1}{\varepsilon}\int_0^T \langle \beta(u'_\varepsilon(t), v(t) - u'_\varepsilon(t)\rangle \, dt = \int_0^T (f(t), v(t) - u'_\varepsilon(t))dt. \end{aligned} \tag{20}$$

Since $v(t) \in K$, $\beta(v(t)) = 0$, then it follows that

$$\frac{1}{\varepsilon}\int_0^T \langle \beta(u'_\varepsilon), v - u'_\varepsilon\rangle\, dt = \frac{1}{\varepsilon}\int_0^T \langle \beta(u'_\varepsilon) - \beta(v), v - u'_\varepsilon\rangle\, dt \leq 0. \tag{21}$$

By (21), we transform (20) in the following inequality:

$$\int_0^T (u''_\varepsilon, v - u'_\varepsilon)dt + \int_0^T M(a(u_\varepsilon))a(u_\varepsilon, v - u'_\varepsilon)dt \geq \int_0^T (f, v - u'_\varepsilon)dt.$$

Whence,

$$\begin{aligned} & \int_0^T (u''_\varepsilon, v)dt + \int_0^T M(a(u_\varepsilon))a(u_\varepsilon, v)dt - \int_0^T (f(t), v - u'_\varepsilon)dt \qquad (22)\\ \geq \; & \int_0^T (u''_\varepsilon, u'_\varepsilon)dt + \int_0^T M(a(u_\varepsilon))a(u_\varepsilon, u'_\varepsilon)dt \\ = \; & \int_0^T \frac{1}{2}\frac{d}{dt}|u'_\varepsilon(t)|^2\, dt + \int_0^T \frac{1}{2}\frac{d}{dt}\,\hat{M}(a(u_\varepsilon)) \\ = \; & \frac{1}{2}|u'_\varepsilon(T)|^2 - \frac{1}{2}|u'_\varepsilon(0)|^2 + \frac{1}{2}\,\hat{M}(a(u_\varepsilon(T)) - \frac{1}{2}\,\hat{M}a(u_\varepsilon(0)). \end{aligned}$$

Taking the limit when $\varepsilon \to 0$, in (22), we get:

$$\begin{aligned} \int_0^T (u''(t), v(t))dt + \int_0^T M(a(u(t))a(u(t), v(t))dt - \int_0^T (f(t), v(t) - u'(t))dt \; (23) \\ \geq \frac{1}{2}|u'(T)|^2 - \frac{1}{2}|u'(0)|^2 \frac{1}{2}\,\hat{M}(a(u(T))) - \frac{1}{2}\,\hat{M}(a(u(0))). \end{aligned}$$

Note that $|u'(T)|^2 \leq \liminf_{\varepsilon\to 0} |u'_\varepsilon(T)|^2$ and the same argument for the others terms.

From (23) we obtain:

$$\begin{aligned} \int_0^T (u''(t), v(t))dt \; & + \; \int_0^T M(a(u(t)))a(u(t), v(t))dt - \int_0^T (f(t), v(t) - u'(t))dt \geq \\ & \geq \; \int_0^T \frac{1}{2}\frac{d}{dt}|u'(t)|^2\, dt + \int_0^T \frac{1}{2}\frac{d}{dt}\,\hat{M}(a(u(t)))dt = \\ & = \; \int_0^T (u''(t), u'(t))dt + \int_0^T M(a(u(t))a(u(t), u'(t))dt, \end{aligned}$$

and it follows that:

$$\begin{aligned} \int_0^T (u''(t), v(t) - u'(t))dt \; & + \; \int_0^T M(a(u(t))a(u(t), v(t) - u'(t))dt \geq \\ & \geq \; \int_0^T (f(t), v(t) - u'(t))dt \end{aligned}$$

for all $v \in L^4(0,T;W_o^{1,4}(\Omega))$ with $v(t) \in K$, a.e. in $[0,T]$.

Let us prove that $u'(t) \in K$ a.e. in $[0,T]$. In fact we have

$$\frac{1}{\varepsilon}|\langle\beta(u'_\varepsilon),v\rangle| \leq |\langle f(t),v\rangle| + |(u''_\varepsilon(t),v)| + M(a(u_\varepsilon(t)))|(\Delta u_\varepsilon(t),v)|.$$

Then:

$$|\langle\beta(u'_\varepsilon),v\rangle| \leq \varepsilon\left(|u''_\varepsilon(t)| + M(a(u_\varepsilon(t)))|\Delta u_\varepsilon(t)| + |f(t)|\right)\|v\|_{W_o^{1,4}(\Omega)}$$

whence:

$$\|\beta(u'_\varepsilon)\|_{L^\infty(0,T;W^{-1,\frac{4}{3}}(\Omega))} \leq c\,\varepsilon$$

what implies that

$$\lim_{\varepsilon\to 0}\beta(u'_\varepsilon) = 0 \quad \text{in} \quad L^\infty(0,T;W^{-1,\frac{4}{3}})\Omega)).$$

We already proved that:

$$\lim_{\varepsilon\to 0}\beta(u'_\varepsilon) = \beta(u') \quad \text{in} \quad L^\infty(0,T;W^{-1,\frac{4}{3}}(\Omega)).$$

Then $\beta(u'(t)) = 0$, or $u'(t) \in K$, a.e. in $[0,T]$.

The uniqueness and the initial data follows as usual.

Corollary 1. *The solution $u = u(x,t)$ obtained in Theorem 1, satisfies:*

$$(u''(t) - M(a(u(t))\Delta u(t), v - u(t)) \geq (f(t), v - u(t)),$$

a.e. in $[0,T]$.

Proof: In fact, let u the solution obtained in Theorem 1. Let s be a Lebesgue's point of the function:

$$G(t) = (u''(t) - M(a(u(t))\Delta u(t) - f(t), v - u'(t))).$$

Consider $]s-\frac{1}{h},\ s+\frac{1}{h}[$, $h>0$, a neighborhood of the Lebesgue's point of $G(t)$. The function

$$v_h(t) = \left|\begin{array}{ll} v & \text{on} \quad \left]s-\frac{1}{h},\ s+\frac{1}{h}\right[\\ & \\ u''(t) & \text{on the complement on }]0,T[, \end{array}\right.$$

belongs to $L^4(0,T;W_o^{1,4}(\Omega))$. Then

$$\frac{1}{2h}\int_{s-\frac{1}{h}}^{s+\frac{1}{h}}(u''(t)-M(a(u(t))\Delta u(t)-f(t),v-u'(t))dt \geq 0.$$

Taking the limit when $h\to 0$, since s is Lebesgue's point, we obtain:

$$(u''(s)-M(a(u(s))\Delta u(s)-f(t),v-u'(s))\geq 0.$$

Since $G(t)$ is $L^1(0,T)$, it follows that almost all points of $]0,T[$ are Lebesgue's points of $G(t)$, which proves Corollary 1.

Proof of the Theorem 2

In this step we will prove that the mixed problem (13) has a global solution u_ε on $[0,T]$, $0<T<\infty$, $0<\varepsilon<1$. We consider the spectral problem for $-\Delta$ in $H_o^1(\Omega)$, $\Omega=]0,1[\times]0,1[$. We obtain the functions $w_{rs}(x_1,x_2)=\sin r\pi x_1\cdot \sin s\pi x_2$, $r,s\in \mathbf{N}$. We construct then a sequence $(w_\nu)\_\nu\in\mathbf{N}$ of eigenfunctions of $-\Delta$. By V_m we represent the subspace $[w_1,w_2,\ldots,w_m]$ generated by the m first eigenfunctions.

Approximated penalized problem

$$\left|\begin{array}{ll} u_m(t)\in V_m & \\ (u''_{\varepsilon m}(t),v)+M(a(u_\varepsilon(t)))a(u_\varepsilon(t),v)+\frac{1}{\varepsilon}\left(\beta(u'_{\varepsilon m}),v\right)=(f(t),v) & \\ \text{for each}\quad v\in V_m & \qquad (24)\\ u_{\varepsilon m}(0)=u_{o\varepsilon m}\to u_o \quad \text{in}\quad H_o^1(\Omega)\cap H^2(\Omega),\ \text{as}\ m\to\infty & \\ u'_{\varepsilon m}(0)=u_{1\varepsilon m}\to u_1 \quad \text{in}\quad H_o^1(\Omega)\ \text{as}\ m\to\infty & \end{array}\right.$$

There exists local solution of the system (24) of ordinary differential equations. The estimates obtained in the following steps, permits to extend the solution globally to $[0,T]$. The weak limit u_ε of the solution $u_{\varepsilon m}$ satisfies the same estimates independent of ε and is solution of (13). Then we can pass the limit when $\varepsilon \to 0$ or for a subnet, which limit u is the solution of the Theorem 1, as we have seen in its prove.

Step 1 - In this step we obtain an estimate for $u'_{\varepsilon m}$ in $L^4(0,T;W_o^{1,4}(\Omega))$ which is a consequence of the choice of the penalty term $\beta(u)$. This estimate is crucial to obtain global solutions. When we used different penalty term, cf. Medeiros-Milla Miranda [21], it was not possible to obtain global solutions. Another remark useful, is that in this step the method works for a bounded open set of $\mathbf{R}^2$, not necessarily a square, even $\mathbf{R}^n$.

Take $v = 2u'_{\varepsilon m}(t)$ in $(24)_2$. We obtain:

$$\frac{d}{dt}|u'_{\varepsilon m}(t)|^2 + M(a(u_{\varepsilon m}))\frac{d}{dt}\,a(u_{\varepsilon m}) + \tag{25}$$
$$+\frac{2}{\varepsilon}\sum_{i=1}^{2}\int_\Omega (1-u'^2_{\varepsilon m x_i})^-\,u'_{\varepsilon m x_i}u'_{\varepsilon m x_i}\,dx = 2(f,u'_{\varepsilon m}).$$

If we define

$$\hat{M}(\lambda) = \int_0^\lambda M(s)ds,$$

then, from (25):

$$\frac{d}{dt}[|u'_{\varepsilon m}(t)|^2 + \hat{M}(a(u_{\varepsilon m}(t)))] + \tag{26}$$
$$+\frac{2}{\varepsilon}\sum_{i=1}^{2}\int_\Omega (1-u'^2_{\varepsilon m x_i})^-(u'_{\varepsilon m x_i})^2\,dx = 2(f,u'_{\varepsilon m}).$$

Note that integral in (26) is non negative. Integrating (26) we obtain:

$$|u'_{\varepsilon m}(t)|^2 + \hat{M}(a(u_{\varepsilon m}(t))) + \frac{\varepsilon}{2}\sum_{i=1}^{2}\int_0^t\int_\Omega (1-u'^2_{\varepsilon m x_i})^-(u'_{\varepsilon m x_i})^2\,dx = \tag{27}$$
$$= |u'_{\varepsilon m}(0)|^2 + \hat{M}(a(u_{\varepsilon m}(0))) + 2\int_0^t (f(s),u'_{\varepsilon m}(s))ds.$$

By $(24)_2$, $(24)_3$ we modify (27) and obtain:

$$|u'_{\varepsilon m}(t)|^2 + \hat{M}(a(u_{\varepsilon m}(t)) + \frac{2}{\varepsilon}\sum_{i=1}^{2}\int_0^t\int_\Omega (u'^2_{\varepsilon m x_i}-1)u'^2_{\varepsilon m x_i}\,dx \leq \tag{28}$$
$$\leq C + \int_0^t |u'_{\varepsilon m}(s)|^2\,ds,$$

with $u'^2_{\varepsilon m x_i} > 1$. Note that when $u'^2_{\varepsilon m x_i} \leq 1$, the integral in (27) is zero.

By Gronwall's inequality and (28), we get:

$$|u'_{\varepsilon m}(t)|^2 + m_o \|u_{\varepsilon m}(t)\|^2 + \frac{2}{\varepsilon}\sum_{i=1}^{2}\int_0^t \int_\Omega (u'^2_{\varepsilon m x_i} - 1)u'^2_{\varepsilon m x_i}\,dx < C, \qquad (29)$$

for $u'_{\varepsilon m x_i} \geq 1$, independently of ε, m.

By (29) we can extend $u_{\varepsilon m}$ to $[0,T]$. Then (29) is true for $0 \leq t \leq T$, $0 < T < \infty$, independent of ε, m, for $u'_{\varepsilon m x_i} > 1$. Note that (29) implies also that $u_{\varepsilon m x_i}$ is bounded in $L^2(\Omega)$.

From (29) we obtain:

$$\int_Q u'^4_{\varepsilon m x_i}(x,t)dxdt - \int_Q u'^2_{\varepsilon m x_i}(x,t)dxdt < \frac{\varepsilon c}{2}, \quad 0 < \varepsilon < 1,$$

or

$$\int_Q u'^4_{\varepsilon m x_i}(x,t)dxdt \leq C + \int_Q u'^2_{\varepsilon m x_i}(x,t)dxdt. \qquad (30)$$

Applying Schwarz inequality in the second integral of (30), we obtain:

$$\int_Q u'^4_{\varepsilon m x_i}(x,t)dxdt < C, \qquad i = 1,2. \qquad (31)$$

From (31) we have:

$$\int_Q |\nabla u'_{\varepsilon m}(x,t)|^4\,dxdt < C, \qquad (32)$$

independent of $0 < \varepsilon < 1$ and m.

By (29), (32) we obtain a unique inequality:

$$|u'_{\varepsilon m}(t)|^2 + \|u_{\varepsilon m}(t)\|^2 + \|u'_{\varepsilon m}(t)\|^4_{L^4(0,T;W^{1,4}_o(\Omega))} \leq C_o, \qquad (33)$$

independent of $0 < \varepsilon < 1$ and m.

By Schwarz inequality applied to second integral in (30) and by (31), we obtain $u'_{\varepsilon m}$ is bounded in $L^2(0,T;H^1_o(\Omega))$.

Step 2 – In this step we obtain the estimate of the $\Delta u_{\varepsilon m}(t)$ in $L^2(\Omega)$. In all the results obtained, this estimate is local, that is, exist $0 < T_o < T$ and the estimate is obtained for $0 \leq t \leq T_o$. To avoid this difficulty some extra conditions are added to the problem, for instance, a damping or certains growth conditions for the initial data. This estimate depends on the global solution of an inequality of the type:

$$\varphi'(t) \leq C + \int_0^t \theta(s)\varphi(s)ds$$

and we don't know if $\theta \in L^1(0,T)$. In our case, the penalty term, gives by (31), an estimate that implies $\theta \in L^1(0,T)$. Therefore, we need that a certain integral that contains the penalty term is non negative. This is obtained, at the present, if we do an strong restriction on the geometry of Ω, that is, if Ω is an square of $\mathbf{R}^2$.

Take $v = -2\Delta u'_{\epsilon m}(t)$ in $(24)_2$. Note that $-2\Delta u'_{\epsilon m}(t) \in V_m$, because we use eigenfunctions of $-\Delta$ in $H^1_0(\Omega)$, Ω the square. We obtain:

$$\begin{aligned} &2(u''_{\epsilon m}, -\Delta u'_{\epsilon m}) + 2M(a(u_{\epsilon m}))a(u_{\epsilon m}, -\Delta u'_{\epsilon m}) - \\ &-\frac{2}{\epsilon}\sum_{i=1}^{2}\int_\Omega \frac{\partial}{\partial x_i}\Big[\Big(1 - u'^2_{\epsilon m x_i}\Big)^- \ u'_{\epsilon m x_i}\Big](-\Delta u'_{\epsilon m})dx = \\ &= 2(f, -\Delta u'_{\epsilon m}). \end{aligned} \tag{34}$$

The two first term of (34) are controlable.

Let us prove that

$$\sum_{i=1}^{2}\int_\Omega \frac{\partial}{\partial x_i}\Big[\Big(1 - u'^2_{\epsilon m x_i}\Big)^- \ u'_{\epsilon m x_i}\Big](-\Delta u'_{\epsilon m})dx \geq 0. \tag{35}$$

Let us write in the calculus, u instead of $u_{\epsilon m}$. Then, (34) is decomposed in the following four integrals:

$$I_i = \int_\Omega \frac{\partial}{\partial x_i}\Big[\Big(1 - u'^2_{x_i}\Big)^- \ u'_{x_i}\Big]u'_{x_i x_i}\, dx,$$

$i = 1, 2$, and

$$I_{ij} = \int_\Omega \frac{\partial}{\partial x_i}\Big[\Big(1 - u'^2_{x_i}\Big)^- \ u_{x_i}\Big]u'_{x_j x_j}\, dx$$

for the terms I_{12}, I_{21}.

Calculus of I_i – Note that if $u'^2_{x_i} \geq 1$, $(1 - u'^2_{x_i})^- = 0$, then $I_i = 0$. Suppose $u'^2_{x_i} > 1$ and taking derivatives, we obtain:

$$I_i = \int_\Omega (3u'^2_{x_i} - 1)u'^2_{x_i x_i}\, dx \geq 0, \qquad i = 1, 2.$$

Calculus of I_{ij} – It is sufficient to consider the case $u'^2_{x_i} > 1$. Then

$$I_{ij} = \int_\Omega \frac{\partial}{\partial x_i}\Big[\left(u'^2_{x_i} - 1\right) u'_{x_i}\Big] u'_{x_j x_j}\, dx.$$

By Gauss' Lemma, we obtain:

$$\begin{aligned} I_{ij} &= -\int_\Omega \Big[\left(u'^2_{x_i} - 1\right) u'_{x_i}\Big] \frac{\partial}{\partial x_i} u'_{x_j x_j} + \\ &+ \int_\Gamma \Big[\left(u'^2_{x_i} - 1\right) u'_{x_i}\Big] u'_{x_j x_j} \cos(\nu, x_i) d\Gamma. \end{aligned}$$

Note that Γ is the boundary of the square $]0,1[\times]0,1[$. The integral on Γ is zero because $u'_{x_j x_j} = 0$ on Γ.

Therefore,

$$I_{ij} = -\int_\Omega \Big[\left(u'^2_{x_i} - 1\right) u'_{x_i}\Big] \frac{\partial}{\partial x_i} u'_{x_j x_j}\, dx. \tag{36}$$

We first modify (36) changing the order of derivation, obtaining:

$$I_{ij} = -\int_\Omega \Big[\left(u'^2_{x_i} - 1\right) u'_{x_i}\Big] \frac{\partial}{\partial x_j} u'_{x_i x_j}\, dx.$$

Applying Gauss' Lemma, we obtain:

$$I_{ij} = \int_\Omega \frac{\partial}{\partial x_j}\Big[\left(u'^2_{x_i} - 1\right) u'_{x_i}\Big] u'_{x_i x_j}\, dx - -\int_\Gamma (u'^2_{x_i} - 1) u'_{x_i} u'_{x_i x_j} \cos(\nu, x_j) d\Gamma. \tag{37}$$

Let us prove that the integral on Γ is zero. In fact, decompose Γ in $\Gamma_1 =]0,1[$ on x coordinate, the following side is Γ_2, then Γ_3 and $\Gamma_4 =]0,1[$ on y coordinates. We have two types of terms: $u'_{x_1} \cos(\nu, x_2)$ or $u'_{x_2} \cos(\nu, x_1)$ where ν is the normals to Γ_1, Γ_2, Γ_3, Γ_4. To fix ideas, let us consider the first of the two terms. Then, it is sum of terms of type $h(t)(\cos r\pi x_1 \sin s\pi x_2)\cos(\nu, x_2)$. In Γ_1 and Γ_2, $\sin s\pi x_2$ is zero, because $x_2 = 0$ or $x_2 = 1$ respectively. On Γ_3, Γ_4, $\cos(\nu, x_2) = 0$ because the normal is perpendicular to the faces. Consequently the integral on the boundary Γ is zero and from (37) we have:

$$I_{ij} = \int_\Omega \frac{\partial}{\partial x_j}\Big[\left(u'^2_{x_i} - 1\right) u'_{x_i}\Big] u'_{x_i x_j}\, dx$$

which is non negative because $u'^2_{x_i} > 1$. It follows that in fact (35) is non negative. Then from (34) we obtain:

$$\begin{aligned} &\tfrac{d}{dt} \quad [|\nabla u'_{\epsilon m}(t)|^2 + M(a(u_{\epsilon m}(t)))|\nabla u_{\epsilon m}(t)|^2] \le \\ &\le \quad [\frac{d}{dt} M(a(u_{\epsilon m}(t)))]|\Delta u_{\epsilon m}(t)|^2 + |\nabla f(t)|^2 + |\nabla u'_{\epsilon m}(t)|^2. \end{aligned} \tag{38}$$

Integrating (38) on $]0,t[$, we obtain:

$$\begin{aligned} & |\nabla u'_{\epsilon m}(t)|^2 + M(a(u_{\epsilon m}(t)))|\Delta u_{\epsilon m})|^2 \leq \\ \leq \ & |\nabla u'_{\epsilon m}(0)|^2 + M(a(u_{\epsilon m}(0)))|\Delta u_{\epsilon m}(0)|^2 + \int_0^T |\nabla f(s)|^2\, ds + \\ + \ & \int_0^t |\nabla u'_{\epsilon m}(s)|^2\, ds + \int_0^t [\frac{d}{dt} M(a(u_{\epsilon m}(t)))]|\Delta u_{\epsilon m}(s)|^2\, ds. \end{aligned} \tag{39}$$

Let us estimate the last integral on the right hand side of (39). We have:

$$\left|\int_0^t [\frac{d}{dt} M(a(u_{\epsilon m}))]|\Delta u_{\epsilon m}(s)|^2\, ds\right| \leq \int_0^t |M'(a(u_{\epsilon m}))|2|a(u_{\epsilon m}, u'_{\epsilon m})||\Delta u_{\epsilon m}|^2\, ds.$$

Since $0 \leq a(u_{\epsilon m}(t)) \leq C_o$, for $0 \leq t \leq T$, by (33), we obtain

$$\max_{0 \leq a(u_{\epsilon m}(t)) \leq C_o} |M'(a(u_{\epsilon m}(t))| < C.$$

Also from (33) we find:

$$|a(u_{\epsilon m}, u'_{\epsilon m})| \leq C_o |\nabla u'_{\epsilon m}|.$$

Whence,

$$\int_0^t [\frac{d}{dt} M(a(u_{\epsilon m}))]|\Delta u_{\epsilon m}(s)|^2\, ds \leq C_o \int_0^t |\nabla u'_{\epsilon m}(s)||\Delta u_{\epsilon m}(s)|^2\, ds. \tag{40}$$

By (39) and (40), follows:

$$\begin{aligned} & |\nabla u'_{\epsilon m}(t)|^2 + m_o|\Delta u_{\epsilon m}(t)|^2 \\ \leq \ & C + \int_0^t |\nabla u'_{\epsilon m}(s)|^2\, ds + C_o \int_0^t |\nabla u'_{\epsilon m}(s)||\Delta u_{\epsilon m}(s)|^2\, ds. \end{aligned} \tag{41}$$

The main contribution given by the penalty term $\beta(u)$ is $\|u'_{\epsilon m}\|^4_{L^4(0,T;W_o^{1,4}(\Omega))} < C_o$, which implies that $|\nabla u'_{\epsilon m}(t)| \in L^1(0,T)$. Also, as observed in Step 1, it implies, $u'_{\epsilon m}$ bounded in $L^2(0,T;H_o^1(\Omega))$. From this remark and (40), we have:

$$|\Delta u_{\epsilon m}(t)|^2 \leq a + b\int_0^t |\nabla u'_{\epsilon m}(s)||\Delta u_{\epsilon m}(s)|ds. \tag{42}$$

Note that (42) is an inequality of the type $\varphi(t) = a + b\int_0^t \theta(s)\varphi(s)ds$, $\varphi \geq 0$, $\theta \geq 0$, $\theta \in L^1(0,T)$. Then by Gronwall's Lemma we obtain $\int_0^t \theta(s)\varphi(s)ds < C$ on $[0,T]$.

From the above remark, (41) and (42), obtain the estimate of Step 2:

$$|\nabla u'_{\varepsilon m}(t)|^2 + |\Delta u_{\varepsilon m}(t)|^2 < C \tag{43}$$

on $[0,T]$, independent of m and $0<\varepsilon<1$.

Step 3 – In this step we obtain estimates for $u''_{\varepsilon m}(t)$. The method to obtain is standard and we do only a summary, cf. Lions [17].

First of all we bound $u''_{\varepsilon m}(0)$. In fact, take $t=0$ in the equation (24). We obtain:

$$(u''_{\varepsilon m}(0),v) = (f(0),v) + M(a(u_{\varepsilon m}(0)))(-\Delta u_{\varepsilon m}(0),v) + \frac{1}{\varepsilon}\langle \beta(u_{1\varepsilon m}),v\rangle . \tag{44}$$

Note that $|\text{ grad } u_1|^2 < 1$ by hypothesis. Since $u_{1\varepsilon m} \to u_1$ in $H^1_o(\Omega)$ and $|\text{ grad } u_1| < 1$, we can obtain a sub of $u_{1\varepsilon m}$, still represented by $u_{1\varepsilon m}$, such that $|\text{ grad } u_{1\varepsilon m}| < 1$ and $u_{1\varepsilon m} \to u_1$. Then this subsequence $u_{1\varepsilon m} \in K$ and $\beta(u_{1\varepsilon m}) = 0$. Here we have a key point of the proof. From here on we work with this sequence. The penalty term in (44) is zero and if $v = u''_{\varepsilon m}(0)$, we obtain $u''_\varepsilon(0)$ bounded in $L^2(\Omega)$.

The correct method is to consider the difference equation in $t+h$ and t, divided by h and take the limits when $h\to 0$. This justify the formal procedure of take the derivative with respect to t, of the both sides of (14) and take $v = 2u''_{\varepsilon m}(t)$. As $u''_{\varepsilon m}(0)$ is bounded we obtain the estimate:

$$|u''_{\varepsilon m}(t)|^2 + \|u'_{\varepsilon m}(t)\|^2 < C \quad \text{on} \quad [0,T]. \tag{45}$$

From the estimates (33), (43) and (45), we obtain subsequence, still represented by $u_{\varepsilon m}$ such that

$$\left|\begin{array}{lll} u_{\varepsilon m} \rightharpoonup u_\varepsilon & \text{weak star in} & L^\infty(0,T;H^1_o(\Omega)) \\ u_{\varepsilon m} \rightharpoonup u_\varepsilon & \text{weak star in} & L^\infty(0,T;H^1_o(\Omega)\cap H^2(\Omega)) \\ u'_{\varepsilon m} \rightharpoonup u'_\varepsilon & \text{weak star in} & L^\infty(0,T;L^2(\Omega)) \\ u'_{\varepsilon m} \rightharpoonup u'_\varepsilon & \text{weak star in} & L^\infty(0,T;H^1_o(\Omega)) \\ u''_{\varepsilon m} \rightharpoonup u''_\varepsilon & \text{weak star in} & L^\infty(0,T;L^2(\Omega)) \\ u'_{\varepsilon m} \rightharpoonup u'_\varepsilon & \text{weak star in} & L^2(0,T;W^{1,4}_o(\Omega)) \end{array}\right. \tag{46}$$

To obtain the limit in the nonlinear term, we need strong convergence. We have $u_{\varepsilon m}$ bounded in $L^2(0,T;H_o^1(\Omega)\cap H^2(\Omega))$ and $u'_{\varepsilon m}$ bounded in $L^2(0,T;L^2(\Omega))$. The imbeding of $H_o^1(\Omega)\cap H^2(\Omega)$ in $H_o^1(\Omega)$ is compact. Then there exists subsequence, still represented by $u_{\varepsilon m}$ such that:

$$u_{\varepsilon m} \to u_\varepsilon \quad \text{strongly in} \quad L^2(0,T;H_o^1(\Omega)). \tag{47}$$

With the convergences (46), (47) we can take the limit in the equation (24) when $m \to \infty$, and we obtain u_ε of the Theorem 2. Note that we have the estimates (33), (43) and (45) for u_ε. It follows that we obtain subnet still represented by u_ε, $0 < \varepsilon < 1$, with the convergences (46), (47). The function u_ε satisfies the condition of Theorem 2. The proof is like in Lions [17]. With u_ε and convergences (46), (47) we prove that the limit u of u_ε, is the solution of Theorem 1, as have been done.

We have uniqueness in the Theorem 1 and the solution u satisfies the initial data. The proofs are like in Medeiros-Milla Miranda [21].

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