

A SURVEY OF THE ANALYSIS OF IRREGULAR SHOCK REFRACTIONS AND ITS APPLICATION TO FRONT TRACKING METHODS

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Abstract

The collision of a shock with a material interface produces a variety of complicated refraction patterns. In this article, transitions from self-similar refractions to more complicated configurations are studied using an approximate scattering analysis. This analysis suggests that there are five different regimes for the transition from a regular self-similar wave to a composite irregular wave. Two of these five cases have been incorporated into a front tracking code to provide enhanced resolution computations of such flows.

1. Introduction

Collisions between highly supersonic shock waves and material interfaces can be identified with Riemann solutions for a supersonic steady-state flow. Waves of this type have the property that they are self-similar in space, and have a well defined translational velocity with respect to an inertial reference frame. Their structure can be computed using shock polar analysis. Such waves, following the terminology of Henderson [9], are called regular refractions. Shock refractions which are not regular are called irregular. It is well established experimentally [12] that the structure of regular shock refractions is in good agreement with the mathematical theory. This theory breaks down for transsonic waves due to the loss of hyperbolicity of the equations of motion for a steady-state flow,

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and the resulting refraction patterns become irregular. The loss of hyperbolicity is associated with a loss of existence to the shock polar equations. For such transsonic interactions the general theory is only poorly understood, but a variety of experiments [1, 2] and computations [3] have begun to shed light on the structures formed by these waves. This article will focus on one aspect of this problem, namely the transition of regular to irregular refractions along a smoothly curved interface. Here, the loss of existence of a solution to the steady flow equations corresponds to a loss of time independence in the refraction pattern. A single self-similar wave scatters into a collection of individual wavelets that propagate away from each other at separate speeds. These ideas can be used in numerical methods such as front tracking to provide enhanced resolution computations of these interactions. An important application is the simulation of the acceleration of a fluid interface by a shock wave. This problem, known as the Richtmyer-Meshkov problem, studies the unstable mixing between two fluids that occurs when the boundary surface between two fluids is hit by a shock wave.

2. Equations of Motion

The continuum motion of an inviscid non-heat conducting gas is described by the classical Euler equations that express the laws of conservation of mass, momentum, and energy [4]:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{q}) = 0 \quad (2.1a)$$

$$\partial_t (\rho \mathbf{q}) + \nabla \cdot (\rho \mathbf{q} \otimes \mathbf{q}) + \nabla P = \rho \mathbf{g}, \quad (2.1b)$$

$$\partial_t (\rho (\frac{1}{2} q^2 + e)) + \nabla \cdot (\rho (\frac{1}{2} q^2 + H) \mathbf{q}) = \rho \mathbf{q} \cdot \mathbf{g}. \quad (2.1c)$$

Here ρ is the mass density, P is the thermodynamic pressure, \mathbf{q} is the fluid velocity, e is the specific internal energy, $H = e + P/\rho$ is the specific enthalpy, and \mathbf{g} is the gravitational acceleration.

In the case of the refraction of a planar shock at a planar interface, we obtain an initial value problem for system (2.1) with scale invariant initial data. In the

absence of gravity ($g = 0$) the Euler equations are scale invariant, and system (2.1) reduces to the pseudo-steady Euler equations:

$$\nabla \cdot (\rho \tilde{\mathbf{q}}) + n\rho = 0, \quad (2.2a)$$

$$\nabla \cdot (\rho \tilde{\mathbf{q}} \otimes \tilde{\mathbf{q}}) + \nabla P + (n+1)\rho \tilde{\mathbf{q}} = 0, \quad (2.2b)$$

$$\nabla \cdot (\rho(\frac{1}{2}\tilde{\mathbf{q}}^2 + H)\tilde{\mathbf{q}}) + n\rho(\frac{1}{2}\tilde{\mathbf{q}}^2 + H) + \rho \tilde{\mathbf{q}} = 0. \quad (2.2c)$$

Here $\tilde{\mathbf{q}} = \mathbf{q} - \mathbf{x}/t$ is the self-similar fluid velocity, and n is the space dimension of the flow.

The equations for regular refractions are further simplified since such waves are both scale invariant and time independent (or more precisely are Galilean equivalent to a scale invariant steady-state flow). If the flow upstream of the point of interaction is supersonic, regular refractions correspond to Riemann solutions for the supersonic steady-state Euler equations. For a planar flow this system is a set of four conservation laws, with timelike direction given by the stream direction. The standard Riemann analysis [7] shows that this system has two genuinely nonlinear wave families in the sense of Lax [13], and a doubly linearly degenerate wave family consisting of contact discontinuities and slip lines. The pressure and flow direction are common Riemann invariants for the linearly degenerate family. The Riemann problem is solved by finding the intersection of the projections of the wave curves of the two nonlinear families in the pressure-flow direction phase space. This is exactly the classical method of shock polars [4]. The solution is stable with respect to small perturbations in the flow parameters provided both the upstream and downstream flows are supersonic. However, the formation of transsonic waves in the Riemann solution can lead to the breaking up of the self-similar wave.

3. Two Dimensional Riemann Problems and Elementary Waves

An elementary wave (or node) [6] is a self-similar solution to the Euler equations that is Galilean equivalent to a steady-state flow. Such waves are building

blocks out of which more complicated flows are constructed. The elementary waves for two dimensional gas dynamics with a polytropic equation of state were classified in [5]. However this classification was incomplete due to an overly restrictive genericity assumption on the flow. For planar flows, the elementary waves can be divided into two classes depending on whether the flow around the wave is supersonic or transsonic. The supersonic elementary waves correspond to regular refractions of shocks at fluid interfaces, and the crossing or overtaking of two shocks. The transsonic waves consist of regular Mach reflections, shock transmissions at a fluid interface, and total internal reflection. Shock transmission is distinguished from a regular shock refraction in that the flow behind the incident shock wave is subsonic, thus precluding the existence of reflected waves. Total internal reflection is a shock refraction where there is no transmitted wave, the incident wave being entirely reflected as a Prandtl-Meyer wave. The total internal reflection node was originally dismissed in [5] as unphysical, however subsequent experiments and computations [2, Fig. 9] and [3, Figs. 13], clearly indicate its occurrence in certain situations. Furthermore, total internal reflection is a well known phenomena in the refraction of waves in acoustics between materials with a suitable impedance mismatch [14, pp. 196]. None of the transsonic elementary waves are commonly found in isolation, but instead occur in composites of elementary waves produced by the scattering of a regular wave.

We are interested in describing the behavior of irregular refractions that are perturbations of regular refractions. For this purpose we can consider the problem of a shock propagating through a slightly curved interface. For sufficiently short periods of time this curvature can be neglected in computing the shock refraction. Other variables being fixed, the shock refraction configuration will be a function of the angle between the incident shock and the fluid interface. There is a loss of existence to the shock polar equations for a stationary flow for sufficiently large incident angles and hence such refractions must be irregular. On the other hand, regular refractions always occur for sufficiently small incident angles. Thus there must be some smallest angle where a transition from

a regular to irregular refraction will occur. When a regular refraction reaches this irregular transition point, its spatial self-similarity breaks down and the solution near the refraction becomes fully time dependent. This transition can be interpreted as a scattering of the original shock refraction into a set of nodes that propagate away from the initial point of interaction. Each of these scattered nodes in turn corresponds to the interaction of a pair of waves, that is interactions between pairs of shock waves or contact discontinuities. Although the flow in the regions between the scattered waves may not be constant, the leading order behavior of each scattered node will approximate that of a pure elementary wave. This is a consequence of the fact that the elementary waves are just Riemann solutions to a steady flow Riemann problem and the piecewise smoothness of the flow near the node. Interactions that involve rarefaction waves, or other more complex unsteady waves are also possible. These are more problematic since the local wave configurations are not self-similar. In the numerical simulations, the leading edge interactions of such configurations may be tracked as degenerate (zero strength) wave interactions with the remaining structure captured by a finite difference method. An important observation is that each scattered node is in relative motion with respect to the others, and it is not possible to apply shock polar analysis to the wave ensemble. However if the velocity of an individual node is known, its local wave configuration can be computed using shock polars. Thus the problem of computing the scattered wave configuration can be thought of as a set of shock polar equations that are coupled by their relative node velocities.

4. Regular to Irregular Refraction Transitions

Regular refractions are classified according to whether the reflected wave is a shock or a rarefaction, and whether the interaction is fast-slow or slow-fast [1, 2, 10]. Let M_i , M_r , and M_t denote the Mach numbers of the flow behind the incident, reflected and transmitted shocks respectively. It should be noted that for most equations of state, and for the polytropic equation of state in particular,

transmitted rarefaction waves are not possible for regular shock refractions since they would turn the flow in an incompatible direction. Indeed the transmitted wave must be of the same family as the incident wave, while the reflected wave is of the opposite wave family. Furthermore the base state of the reflected wave lies on a wave curve of the same family as the transmitted wave, and at a higher pressure. Thus if the equation of state is such that the supersonic portions of the wave curves are monotonic in the pressure-flow direction phase space, and the solution to the supersonic steady state Riemann problem between the state behind the incident shock and the state on the transmitted side of the fluid interface exists and is supersonic, then the wave on the transmitted side of the interface must be a shock wave. A regular refraction is said to be fast-slow or slow-fast according to whether the Mach numbers increase or decrease across the fluid interface.

$$\text{(Fast-Slow Refraction)} \quad \max(M_i, M_r) \leq M_t \quad (4.1)$$

$$\text{(Slow-Fast Refraction)} \quad \max(M_i, M_r) \geq M_t \quad (4.2)$$

Consider a shock of given strength that is incident on an interface between two fluids. The steady state Mach number ahead of the incident shock is related to the angle β_i between the incident shock and the fluid interface by the relation $M_i \sin |\beta_i| = m/\rho_0 c_0$, where $m^2 = -\Delta P/\Delta V$ is the mass flux across the incident shock, and $V = 1/\rho$ is the specific volume. If β_i small, the interaction will be highly supersonic and regular. The limiting case as $\beta_i \rightarrow 0$ is that of a head on collision of a shock with a contact discontinuity [7]. Transitions from regular to irregular refractions occur at three possible points,

$$\text{(Fast-Slow Refraction)} \quad \min(M_i, M_r) = 1, \quad M_t > 1, \quad (4.3)$$

$$\text{(Slow-Fast Refraction)} \quad \min(M_i, M_r) > 1, \quad M_t = 1, \quad (4.4)$$

and the mechanical equilibrium point, which is defined by the condition that the intersection between the reflected and transmitted wave shock polars coincides with an intersection between the incident and transmitted wave shock polars. Experimental and computational evidence suggests that the transition from regular to irregular refraction occurs at the smallest angle β_i for which one of

the above conditions holds. Figure 1 shows representative shock polar diagrams for the five transition cases.

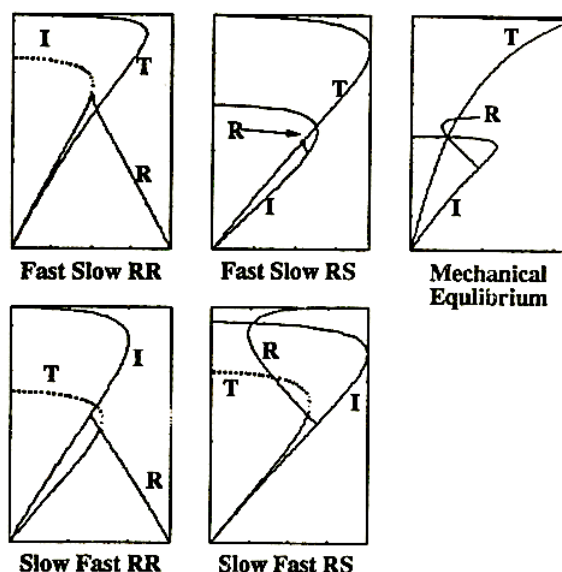


Figure 1: Shock polar diagrams for transitions from regular to irregular refraction. Transition occurs when one of the waves experiences a sonic transition, or the flow reaches the mechanical equilibrium point. The subsonic portion of the shock polar for the transitional wave is shown as a dotted line. The figures show the log of the pressure verses the flow direction.

The scattering behavior of regular refractions at the transition points can be divided into five classes depending on whether at transition, the interaction is fast-slow, slow-fast, or mechanical equilibrium, and on whether the reflected wave is a shock or rarefaction wave. In the following, we will describe five potential configurations for the scattered waves produced for a shock refraction near, but just past, transition. This list is exhaustive only in the sense that all regular refractions are of either the fast-slow or slow-fast variety, with either

reflected shocks or rarefaction waves, and thus fall into one of the first four basic categories. The descriptions should be generally interpreted as being qualitative. Furthermore, the structure of the waves can be complicated by additional bifurcations. For example, shock crossing may occur as pairs of Mach reflections instead of a single cross node. The inclusion of the mechanical equilibrium condition as a transition case is based primarily on experimental evidence [11]. Addition weight is given to the choice of the mechanical equilibrium condition as a transition criterion by the fact that the fluid flow can change continuously from a regular to irregular refraction with a reflected Mach wave at that point. However the exact description of the transition point for this case is complicated, and depends on the interaction of the waves with the viscous boundary layer near the contact, as well as (in unsteady flows) the boundary conditions downstream from the wave.

4.1 Fast-slow transition with reflected rarefaction

This interaction, discussed in [8], occurs for a regular refraction with a reflected rarefaction wave when $M_i = 1$ and $M_t > 1$. Here the leading edge of the reflected rarefaction wave is moving faster than the incident shock wave. Self-similarity is broken as the rarefaction wave overtakes the incident shock. The flow immediately behind the incident shock consists of the nonlinear superposition of a set of weak overtake interactions as the rarefaction overtakes the shock. The composite wave appears as two sections, a region of interaction between the reflected rarefaction and the incident shock, and a nearly regular refraction behind this region. This wave typically occurs when the incident shock is progressing from a dense fluid into a lighter more compressible fluid. Figure 2 shows a detail from a front tracking computation that produced such an interaction.

4.2 Slow-fast transition with reflected rarefaction

Slow-fast transitions occur when the transmitted shock becomes transsonic. At this point the refraction can be influenced by perturbations in the flow that are

downstream from the point of interaction on the transmitted side of the fluid interface.

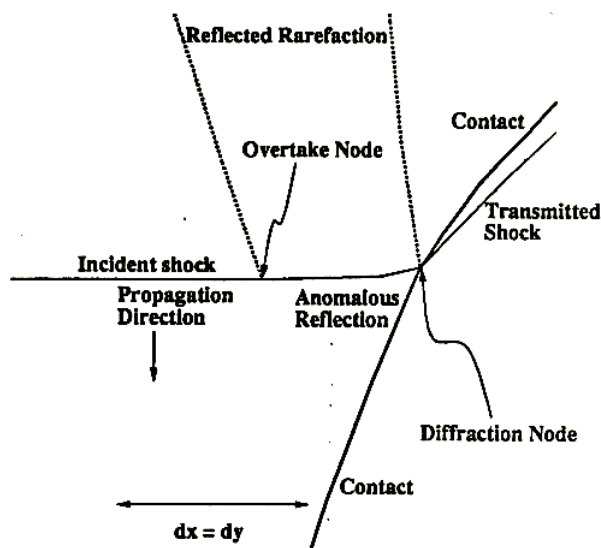


Figure 2: Detail from a computation showing the fast-slow transition from regular to anomalous reflection. The picture is taken from a full simulation of a shock colliding with a bubble. Note that front tracking allows the interaction to be resolved on the order of one grid block. Prior to transition, the wave was self-similar in space, with the incident, transmitted and reflected waves meeting at a point on the fluid interface.

It is conjectured that in this case the solution near the point of refraction is unstable with respect to the disturbances from behind the incident shock. In any case, the point where the transmitted shock becomes transsonic is close to the point of maximum extension of the transmitted shock polar beyond which the steady state Riemann solution will cease to exist. After transition, the transmitted shock moves ahead of the incident shock becoming a precursor that is itself refracted by the interface. Since the transmitted shock is transsonic, the re-

fraction pattern produced by the precursor shock must be a shock transmission node. The transmitted precursor wave in turn collides with the original transmitted shock, leading to a cascade of wave interactions. One of the reflected waves from the shock crossing is directed towards the fluid interface where it experiences a total internal reflection. The rarefaction

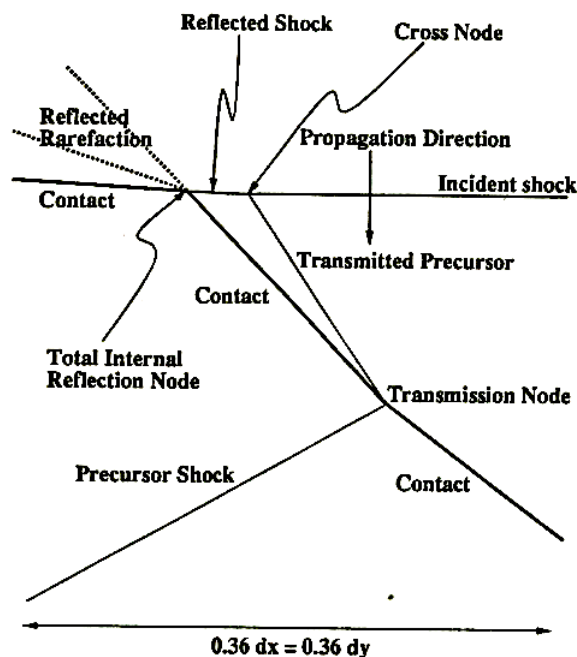


Figure 3: Detail showing the implementation of the slow-fast reflected rarefaction case. The figure is taken from a full simulation of the Richtmyer-Meshkov problem. Note that at the shock crossing, only the reflected wave directed toward the fluid interface is tracked.

from this total internal reflection overtakes the other reflected shock from the crossing, and is dampened back to nearly the strength of the pretransition rarefaction. Figure 3 shows an example of the sort of configuration produced by this interaction.

4.3 Fast-slow transition with reflected shock

This interaction typically occurs for weaker interactions that produce reflected shocks. The transsonic reflected shock breaks away from the fluid interface to produce a Mach type reflection. The slip line from the Mach reflection will be swept up by the material interface. This regime is probably the most complicated from the point of view of interpreting its structure using node analysis. In many cases the shock waves appear to break up into bands of compression waves on either side of the interface, as shown for example in [12, Fig. 14d], and the scattered nodes are more correctly interpreted as interactions between leading edges of compression and rarefaction waves, rather than between discrete waves.

4.4 Slow-fast transition with reflected shock

As in case 4.2, the transmitted wave becomes a precursor, and in this case the precursor propagates through the interface as an anomalous reflection. Behind this wave, the pressure is returned to approximately that behind the original shock. The transmitted precursor collides with the incident shock producing a shock crossing interaction. One of the reflected waves produced by the shock crossing is directed towards the fluid interface and is reflected in a total internal reflection node. The rarefaction from this total internal reflection overtakes the other reflected wave from the shock crossing downstream, and is dampened back to a neighborhood of the pretransition reflected shock. Nodes of this type have been observed in both computations and experiments, for example see Figure 9 of [2] and Figures 13 of [3]. The configuration shown there is somewhat more complicated than the above description, since the shock crossing node is now replaced by a pair of Mach reflections.

4.5 Mechanical equilibrium point

The situation here is almost identical with a transition from regular to Mach reflection. A Mach reflection is formed between the incident and reflected waves. The Mach stem produced by the Mach reflection is transmitted through the interface as a transmission node. The slip line produced by the Mach reflection is directed towards the fluid interface, where it is asymptotically swept up by the interface.

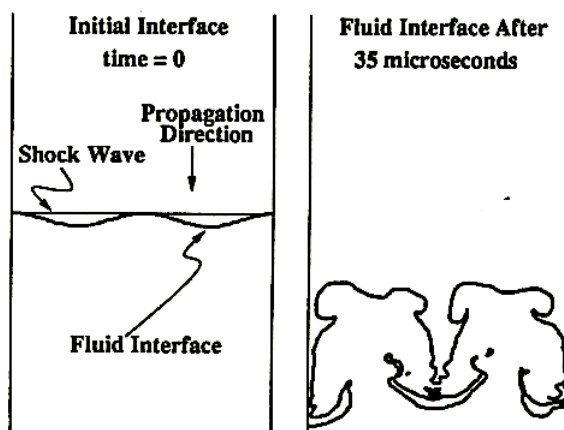


Figure 4: A Richtmyer-Meshkov unstable interface. A fluid accelerated by a shock wave experiences an unstable growth. An analysis for wave refraction is a key element of the application of front tracking to such problems. The horizontal resolution is 60 cells across the computational domain.

5. Conclusion

The qualitative description of the wave scattering given above can be incorporated into numerical methods that use front tracking. The tracked fronts allow a precise computation of the local reference frame of each node, and allow the inclusion of the information obtained from the shock polar analysis into the numerical solution near that node. Transitions are detected by checking the results of the shock polar analysis for one of the five transition cases. When a transition is detected, the single node is split into its constituents corresponding to the given transition. This is done on a case by case manner, but the small number of cases make the process feasible. Currently two of the five cases have been implemented in our front tracking code. Figures 2 and 3 show the local structure of the wave scattering. Figure 4 shows the result of the acceleration of a fluid interface by a shock wave. As can be seen, front tracking provides a high degree of resolution relative to a given computational mesh.

References

- [1] A. M. Abd-el-Fattah and L. F. Henderson, *Shock Waves at a Fast-Slow Gas Interface*, J. Fluid Mech. 86 (1978), 15-32.
- [2] ———, *Shock Waves at a Slow-Fast Gas Interface*, J. Fluid Mech. 89 (1978) 79-95.
- [3] P. Colella, L. F. Henderson and E. G. Puckett, *On the Refraction of Shock Waves at a Slow-Fast Gas Interface*, J. Fluid Mech. 224 (1991), 1-27.
- [4] R. Courant and K. Friedrichs, *Supersonic Flow and Shock Waves*, John Wiley & Sons, New York, NY (1948).
- [5] J. Glimm, C. Klingenberg, O. McBryan, B. Plohr, D. Sharp and S. Yaniv, *Front Tracking and Two Dimensional Riemann Problems*, Adv. Appl. Math. 6 (1985), 259-290.

- [6] J. Glimm and D. Sharp, *An S-matrix Theory for Classical Nonlinear Physics*, Found. Phys. 16 (1986), 125-141.
- [7] J. W. Grove, *The Interaction of Shock Waves with Fluid Interfaces*, Adv. Appl. Math. 10 (1989), 201-227.
- [8] J. W. Grove and R. Menikoff, *The Anomalous Reflection of a Shock Wave at a Material Interface*, J. Fluid Mech. 219 (1990), 313-336.
- [9] L. F. Henderson, *The Refraction of a Plane Shock Wave at a Gas Interface*, J. Fluid Mech. 26 (1966), 607-637.
- [10] ———, *On the Refraction of Longitudinal Waves in Compressible Media*, LLNL Report UCRL-53853, (1988).
- [11] ———, *Private communication*, (1991).
- [12] R. G. Jahn, *The Refraction of Shock Waves at a Gaseous Interface*, J. Fluid Mech. 1 (1956,) 457-489.
- [13] P. Lax, *Hyperbolic Systems of Conservation Laws II*, Comm. Pure Appl. Math. 10 (1957,) 537-566.
- [14] P. Thompson, *Compressible-Fluid Dynamics*, McGraw-Hill, New York, NY, (1972).

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